SIMPLE STATISTICAL MODELS OF HEAVY ION COLLISIONS; APPLICATIONS TO LIQUID/GAS PHASE TRANSITION

COLLABORATORS

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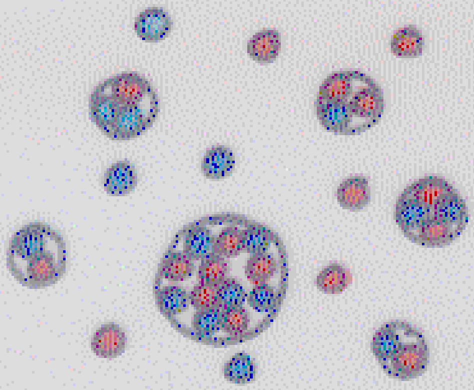
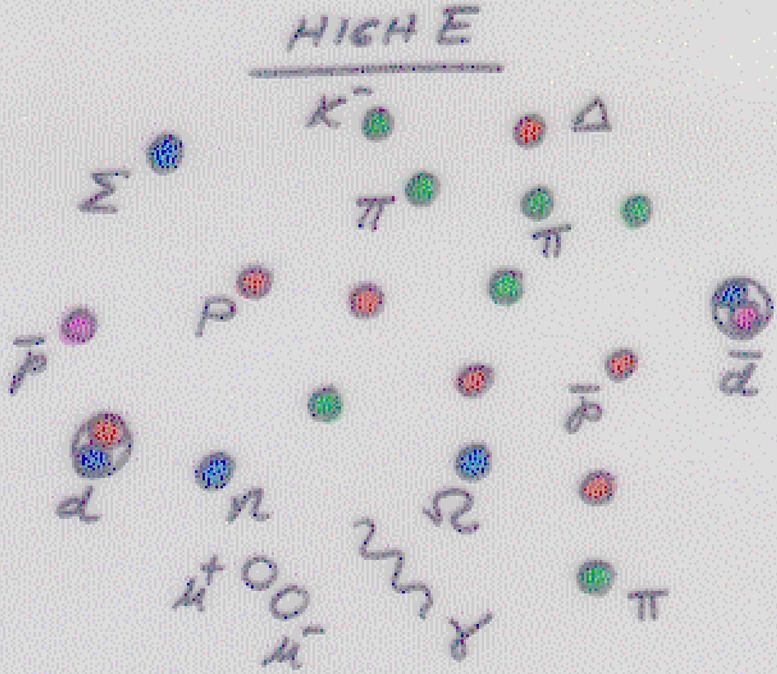
2 min

$\vec{A} + \overleftarrow{A}$ collision

LOW E



HIGH E



COMPLEX SET OF REACTIONS

$A + B + \cdots \rightleftarrows C + D + \cdots$

$p + p \rightleftarrows p + p$

$p + p + n \rightleftarrows d + p$

$d + d \rightleftarrows t + p$

$\qquad\qquad {}^3He + n$

$\vdots$

$N + N \rightleftarrows N + N$

$N + N \rightleftarrows N + N + n\pi's$

pions

$N + \pi \rightleftarrows \Delta$

$\pi + p \rightleftarrows \Lambda^0 + K$

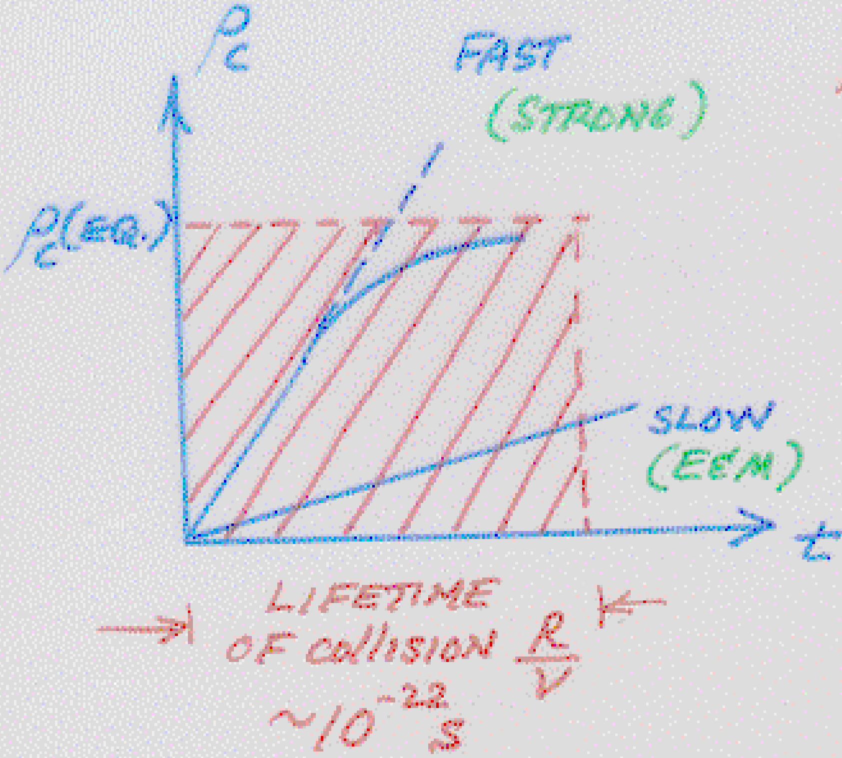
$\vdots$

STATISTICAL MODELS ASSUME RATES FAST $\longrightarrow$ EQUILIBRIUM IN SOME VOLUME V AT SOME TEMPERATURE T

REACTION RATES

EX $A + B \rightleftharpoons C + D$

FOR C $\frac{d\rho_C}{dt} = K_+ \rho_A \rho_B - K_- \rho_C \rho_D \qquad K_\pm \sim \langle \sigma_\pm v \rangle$



$$\rho_C = (1 - e^{-\lambda_R t})\, \rho_C(\text{EQ})$$

$\lambda_R \sim \rho \sigma v$

$\rho = \rho_0$
$\sigma = 10\,\text{mb}$
$v \sim \frac{1}{3} c$

$\lambda_R \sim 10^{22}/\text{s}.$

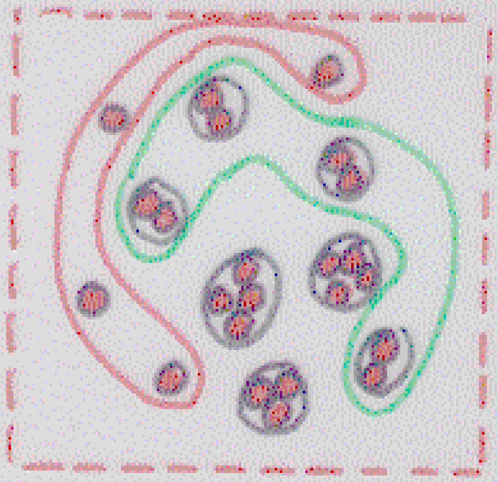
FOR FAST PROCESS USE STATISTICAL MODEL

INTERACTION ARE PRESENT
PRODUCE THE EQUILIBRIUM (THERMAL & CHEMICAL)
IF YOU CHANGE V OR T DISTRIBUTIONS CHANGE
BECAUSE OF THE UNDERLYING REACTIONS

CLUSTERS

$V, T \rightarrow F = U - TS$

"NON-INTERACTING" MEANS $F = \sum F_k n_k$



n_k = # OF CLUSTERS OF SIZE k

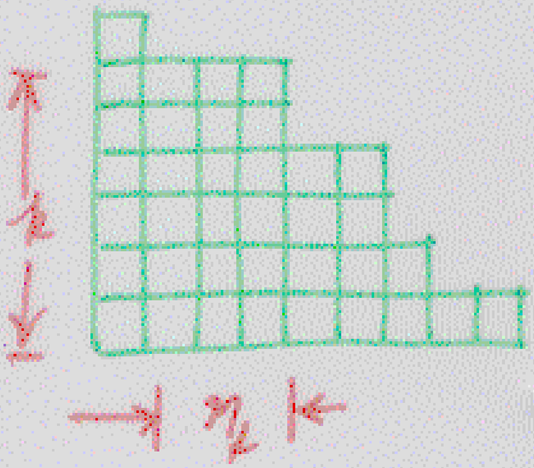
EACH EVENT HAS

$\vec{n} = (n_1, n_2, n_3, \ldots)$

SUBJECT TO

$A = \sum k\, n_k$

$\vec{n}$ AS:



WEIGHT FOR EACH $\vec{n}$

$$W_A(\vec{n}) \sim e^{-\beta F} = e^{-\beta \sum F_k n_k}$$

$$\rightarrow \prod_k X_k^{n_k} \qquad X_k = e^{-\beta F_k}$$

$$P_A(\vec{n}, \vec{x}) = \frac{W_A(\vec{n}, \vec{x})}{Z_A(\vec{x})} = \frac{\prod_{k=1}^{A} X_k^{n_k} / n_k!}{Z_A(\vec{x})}$$

$$Z_A(\vec{x}) = \sum \prod_k X_k^{n_k} / n_k!$$

All BLOCK CONFIGURATION

= All PARTITIONS OF A

= CANONICAL PARTITION FUNCTION (CPF)

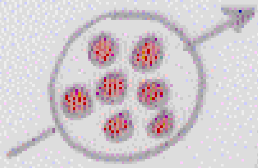
EX X_k CLUSTERS IN SSM – STANDARD STATISTICAL MODEL

STATISTICAL MODELS INVOLVE:

PHASE SPACE

$$V\,d^3p / h^3$$

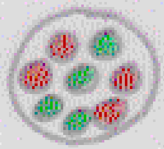
ENERGY CONSIDERATION $e^{-E/T}$



KE: $e^{-p^2/2M_kT}$

$$\int \frac{V\,d^3p}{h^3}\, e^{-p^2/2M_kT} = \frac{V\,k^{3/2}}{\lambda_T^3} \qquad \lambda_T = \frac{h}{(2\pi M_p T)^{1/2}}$$

BINDING ENERGY



i p's j n's $k = i + j$

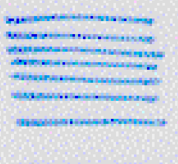
$$a_v k - a_{surf}\, k^{2/3} - a_{sym}\frac{(i-j)^2}{k} - a_c \frac{Z^2 e^2}{R}\left(1 - \left(\frac{V_0}{V}\right)^{1/3}\right)$$

ONE COMPONENT

TWO COMPONENTS

WIGNER SEITZ TERM

EXCITATION ENERGY.



G.S.

$$\rho \sim e^{2\sqrt{aE^*}} \qquad k\frac{T^2}{\epsilon_0} \sim \langle E^* \rangle$$

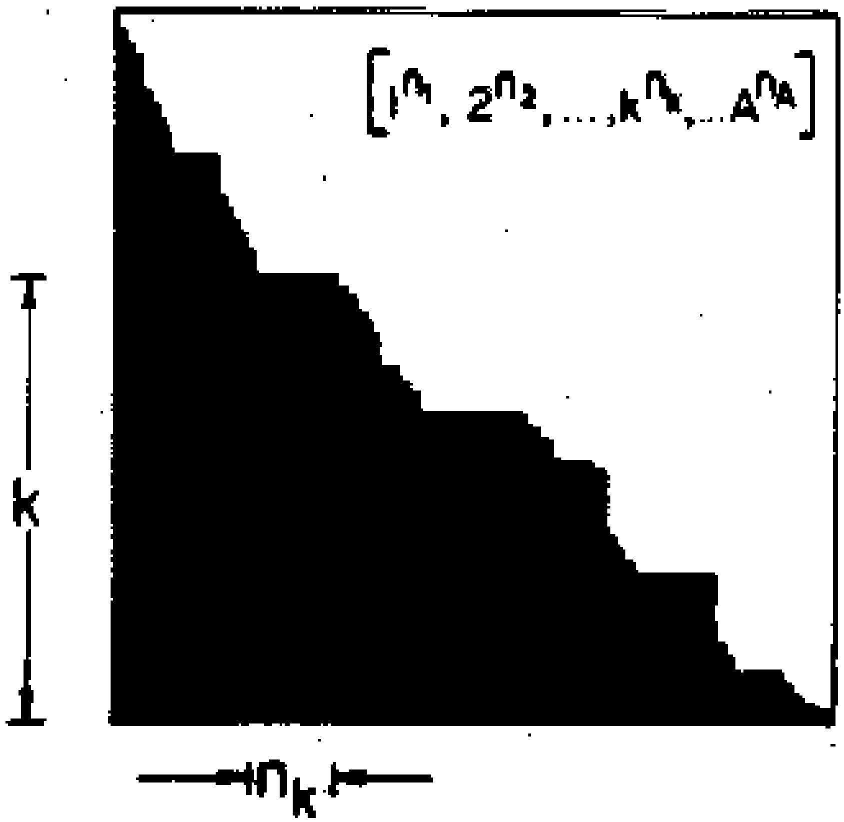
EXCLUDED VOLUME $V \to (V - Ab) = V'_{\text{RELAXEL}} \to V$

VAN DER WAALS APPROX $V = A/\rho$ $\rho = \rho_0/2.7$

$$\boxed{X_k = \frac{V}{\lambda_T^3}\, k^{3/2}\, e^{\left[\left(a_v + \frac{T^2}{\epsilon_0}\right)k - a_{surf}\, k^{2/3}\right]/T}\; e^{-E_{sym}/T - E_c/T}}$$

$X_{i,j}$ TWO COMPONENTS

OF PARTITIONS OF A



$P(200) = 3.9\times 10^{12}$

$P(2000) = 4.7\times 10^{45}$

$P(5000) = 1.7\times 10^{74}$

$P(50{,}000) = 3\times 10^{243}$

$$P(A) \cong \frac{1}{4\sqrt{3}A}\, e^{\sqrt{\pi^2 \frac{2}{3} A}}$$

$$\sim e^{\sqrt{A}}$$

TO OBTAIN Z_A NEED A WAY TO $\sum_{\pi_A(\vec{n})}$ QUICKLY

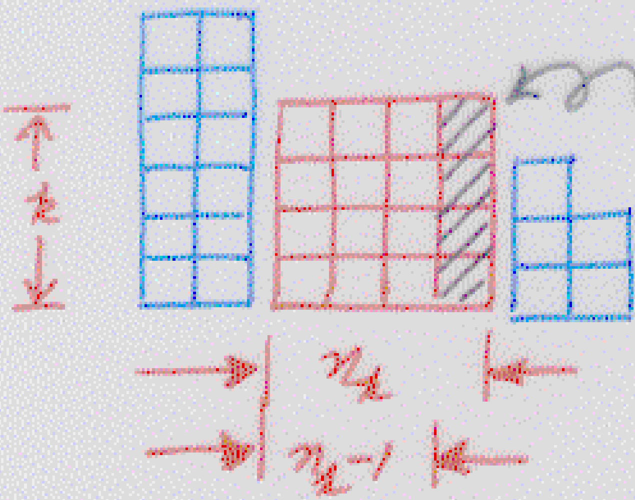
$e^{-\beta F(\vec{n})}$ HAS 1.7×10^{74} VALUES @ $A = 5000$.

HOW TO OBTAIN: EXPECTATION VALUES, CANONICAL PARTITION FUNCTION, ...

GIVEN: $$\frac{W_A(\vec{n},\vec{x})}{Z_A} = \prod_k \frac{x_k^{n_k}}{n_k!}\,\frac{1}{Z_A(\vec{x})} \quad ; \quad A = \sum_{k=1}^{A} k\, n_k$$

$$\langle n_k \rangle = \sum_{\pi_A(\vec{n})} n_k \frac{W_A(\vec{n},\vec{x})}{Z_A} = \sum \frac{\frac{x_1^{n_1}}{n_1!} \cdots \frac{x_k^{n_k-1+1}}{(n_k-1)!}}{Z_A}$$

$$\left\{ \frac{n_k}{n_k!} = \frac{1}{(n_k-1)!} \right\}$$



$n_k \to n_k - 1 \Rightarrow$
REMOVED ONE COLUMN
OF k BLOCKS
$A \to A-k$.

1. $$\langle n_k \rangle = x_k \frac{Z_{A-k}}{Z_A} \qquad A = \sum_{k=1}^{A} k\, n_k = \sum_{k=1}^{A} k \langle n_k \rangle = \sum_{k=1}^{A} k\, x_k \frac{Z_{A-k}}{Z_A}$$

2. $$Z_A(\vec{x}) = \frac{1}{A} \sum_{k=1}^{A} k\, x_k Z_{A-k}(\vec{x}) \qquad Z_0 = 1.$$

$$\sum_{\pi_A(\vec{n})} \prod_{k=1}^{A} x^{n_k}/n_k = Z_A(\vec{x}) = \frac{1}{A} \sum k\, x_k Z_{A-k}$$

$1 + A(A+1)/2$ STEPS

$A = 40$	3.7×10^4 STEPS	NP PROBLEM	780 STEPS	POLYNOMIAL PROBLEM.
$A = 200$	3.9×10^{12} STEPS		2×10^4 STEPS	
$A = 2000$	4.7×10^{45} STEPS		2×10^6 STEPS	

EVOLUTIONARY NETWORK & RECURRENCE RELATIONS

$$Z_A = \frac{1}{A}\sum k X_k Z_{A-k}$$

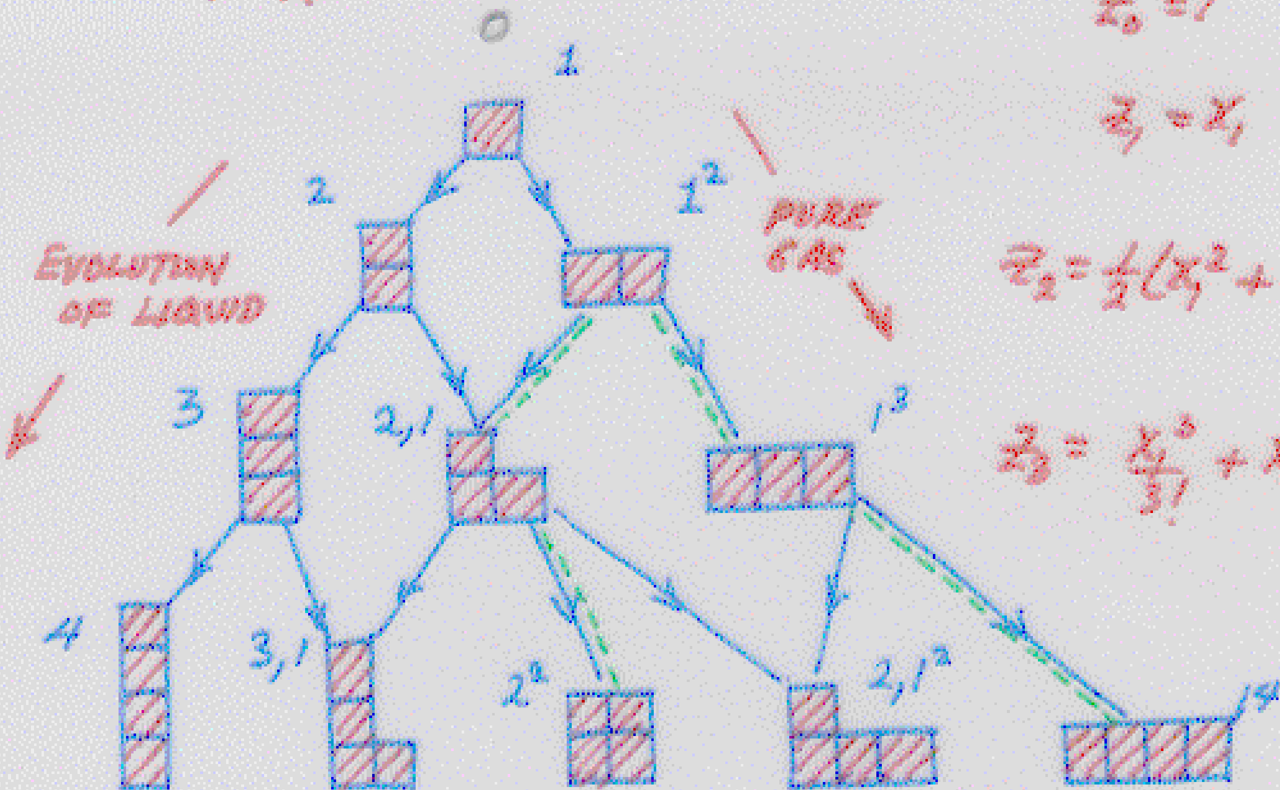
$$Z_4 = \frac{1}{4}\left(X_1 Z_3 + 2X_2 Z_2 + 3X_3 Z_1 + 4X_4 Z_0\right)$$

$$Z_0 = 1$$

$$Z_1 = X_1$$

$$Z_2 = \frac{1}{2}\left(X_1^2 + 2X_2\right)$$

$$Z_3 = \frac{X_1^3}{3!} + X_1 X_2 + X_3$$



$$X_4 + X_3 X_1 + X_2^2/2! + X_2 X_1^2/2! + X_1^4/4! = Z_4$$

BOILING IS SUDDEN DISAPPEARANCE OF A VERY LARGE CLUSTER AS T IS RAISED. STUDY THIS USING:

PROBABILITY THAT k IS THE LARGEST CLUSTER:
(K. CHASE & A.Z.M.)

$$P_k \equiv \frac{1}{Z_A(\vec{X})}\left[Z_A[X_1, X_2, \dots X_k, 0, 0, \dots] - Z_A[X_1, X_2, \dots X_k = 0, 0, 0 \dots] \right]$$

NO CLUSTER IS BIGGER THAN k

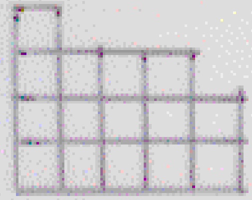
REMOVES PARTITIONS IN WHICH k ITSELF IS NOT PRESENT AND NO CLUSTER IS BIGGER THAN k.

EX: FOR $A=3$ PROB THAT 2 IS LARGEST CLUSTER

IS $$\frac{X_1 X_2}{Z_3} = \frac{1}{Z_3}\left[Z_3(X_1, X_2, 0) - Z_3(X_1, 0, 0) \right]$$

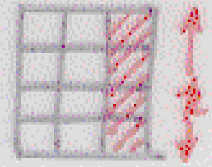
$$= \frac{1}{Z_3}\left[X_1 X_2 + \frac{X_1^3}{3!} - \frac{X_1^3}{3!} \right]$$

$\frac{W_A(\vec{x},\vec{n})}{Z_A(\vec{x})}$ CONTAINS EVENT-BY-EVENT PROBABILITIES FOR EACH $\vec{n}$

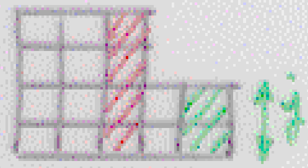


$Z_A(\vec{x})$ CONTAINS:

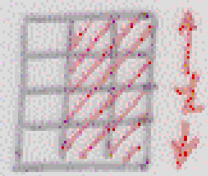
1) CLUSTER YIELDS $\langle n_k \rangle = x_k \frac{Z_{A-k}}{Z_A}$



2) CORRELATIONS $\langle n_k n_j \rangle = x_k x_j \frac{Z_{A-k-j}}{Z_A}$



$\langle n_k (n_k - 1) \rangle = x_k^2 \frac{Z_{A-2k}}{Z_A}$



$\vdots$

3) $Z_A(\vec{x})$ ALSO CONTAINS ALL THERMODYNAMIC PROPERTIES

$F_A = -T \ln Z_A$ $\quad dF = -S\,dT - p\,dV + \mu\,dA$

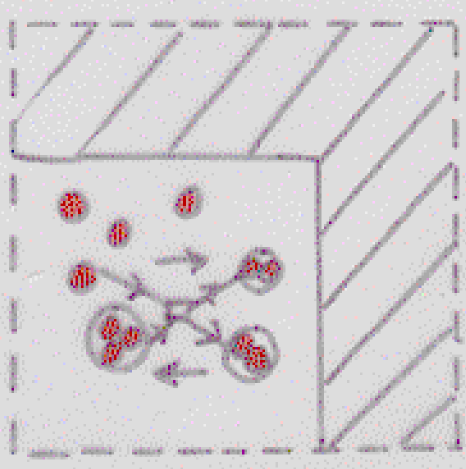
$S = -\frac{\partial F}{\partial T}$ $\quad$ EOS $p = -\frac{\partial F}{\partial V}$ $\quad \mu = \frac{\partial F}{\partial A}$

$C_V = T \frac{\partial S}{\partial T}$ $\quad C_p = C_V + VT\frac{\alpha^2}{K}$ $\quad \alpha = \frac{1}{V}\frac{\partial V}{\partial T}\Big|_p, \; K_T = \frac{-1}{V}\frac{\partial V}{\partial p}\Big|_T$

↖ TALK BY DAS

4) BOILING FEATURE $P_k \equiv \left(Z_A(x_1 \ldots x_k, \vec{0}) - Z_A(x_1 \ldots x_{k-1}, \vec{0}) \right) / Z_A$

5) INTERACTIONS



CHANGE V OR T

CHANGE Z,

$\langle n_k \rangle$:

$A + B \rightleftarrows C + D$

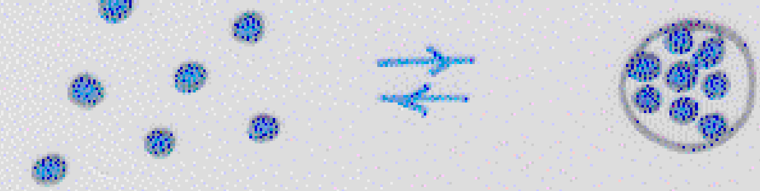
GCPF – GRAND CANONICAL PARTITION FUNCTION

$$\mathcal{Z}(\vec{x},u) = \sum_{A=0}^{\infty} Z_A(\vec{x})\, u^A$$

$u = e^{\beta\mu_\bullet}$ $\quad\mu_\bullet$ = CHEMICAL POTENTIAL OF A NUCLEON

CHEMICAL EQUILIBRIUM IN GCPF IS

$$k\mu_\bullet = \mu_k$$



Follows that $\left(Z_A = \sum_{\{n_k(\vec{x})\}} \prod \frac{X_k^{n_k}}{n_k!} \right)$

$$\mathcal{Z}(\vec{x},u) = \exp\left[\sum_k X_k u_\bullet^k\right] = \exp\left[X_1 u_\bullet^1 + X_2 u_\bullet^2 + \dots\right]$$

ALSO

$$\langle n_k \rangle_{GC} = \lim_{A\to\infty} X_k \frac{Z_{A-k}}{Z_A} = X_k e^{\beta\mu k} = X_k e^{\beta\mu_k}$$

$$e^{\beta\mu} = \lim_{A\to\infty} \frac{Z_{A-1}}{Z_A} \qquad \beta\mu = \lim\left(\ln Z_{A-1} - \ln Z_A\right)$$

SINCE $F_A = -k_B T \ln Z_A$; $\mu = \lim_{A\to\infty} F_A - F_{A-1}$

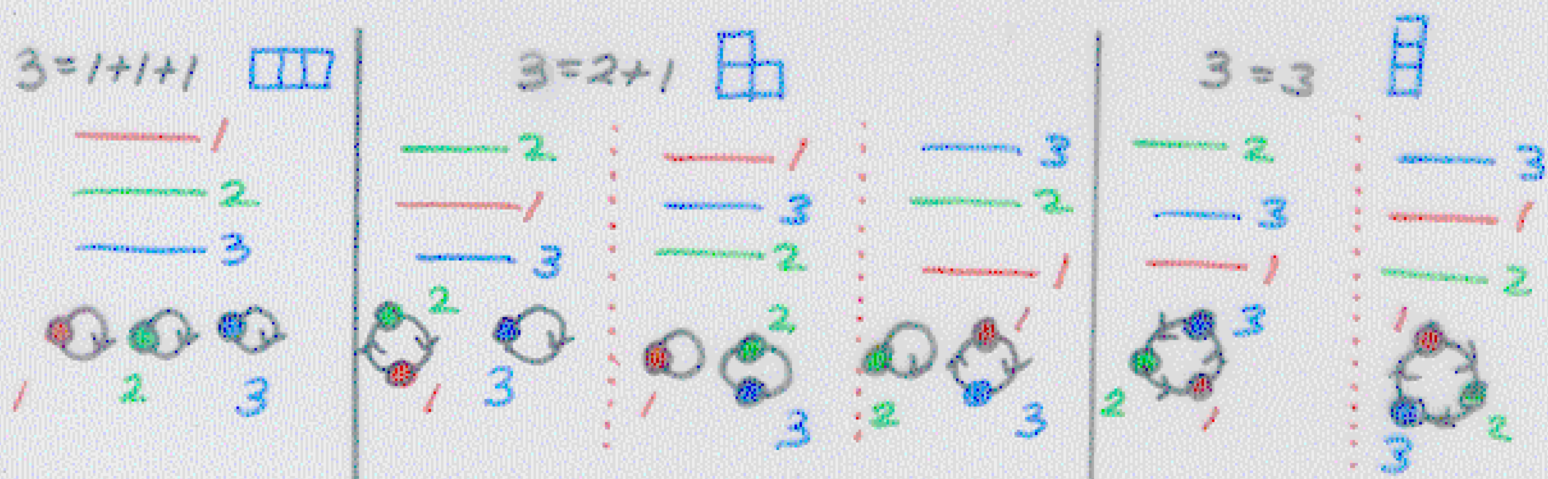
EXAMPLE PERMUTATIONS; FERMI-DIRAC & BOSE-EINSTEIN PROBLEMS - ATOMS IN A LASER TRAP; PROTONS AND NEUTRONS IN A HARMONIC OSCILLATOR TRAP - SHELL MODEL QUANTUM OPTICS, PION MULTIPLICITY DISTRIBUTIONS PION LASER (PRATT), CARD SHUFFLES & RANDOMIZED GAMES.

PERMUTATION: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \underbrace{2 \;\; 1} & & 3 & 4 & \underbrace{7 \;\; 8 \;\; 6 \;\; 5} & & & & \underbrace{10 \;\; 9} & \end{pmatrix}$

$$\vec{n} = (n_1 = 2,\ n_2 = 2,\ n_3 = 0,\ n_4 = 1,\ 0, 0, \ldots 0)$$

CYCLE OF LENGTH k  $\longleftrightarrow$  CLUSTER OF SIZE k

DECK OF 3 CARDS $A! = 3! = 6$



$M_2 = 1$ $M_2 = 3$ $M_2 = 2$

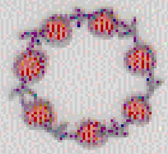
$$M_2 = \text{\# OF MICROSTATES FOR A GIVEN } \vec{n} = \frac{A!}{\prod_k n_k!\, k^{n_k}}$$

A PROBLEM WITH $X_k = \frac{1}{k}$
$\langle n_k \rangle = 1/k$

ENCOUNTER MANY CASES ABOVE WITH $X_k = X t^k / k^\tau$

BE CONDENSATION – Parallel WITH L/G TRANSITION

ATOMS IN A BOX

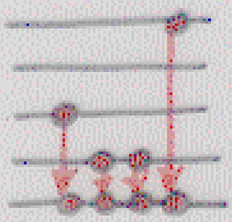


$$X_k = \frac{1}{k}\sum_j e^{-kE_j/T} \to \frac{V}{\lambda_T^3}\frac{1}{k^{5/2}}$$

FERMIONS $\quad X_k = (-1)^{k+1}\frac{V}{\lambda_T^3}\frac{1}{k^{5/2}}$

– SIGN

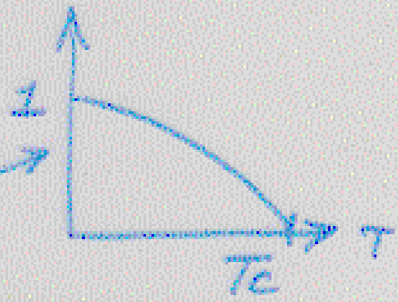
CONDENSATION OCCURS (GC LIMIT) WHEN $\mu = E_{G.S.} = 0$



G.S.

$$\langle n_j\rangle = \sum_k e^{-kE_j/T}\frac{Z_{A-k}}{Z_A}$$

$$\sim f_{G.S.} = \frac{N_{GS}}{A} = 1-\left(\frac{T}{T_c}\right)^{3/2}$$

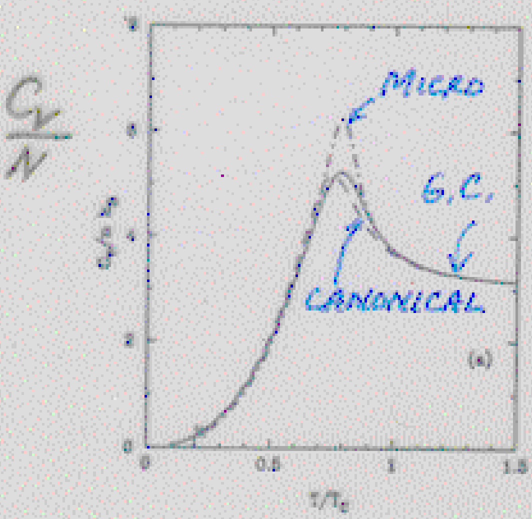


SIMILARLY – ATOMS IN LASER TRAP → HARMONIC OSCILLATOR

CHASE, ZAMICK, A.Z.M.

d = # OF DIMENSIONS

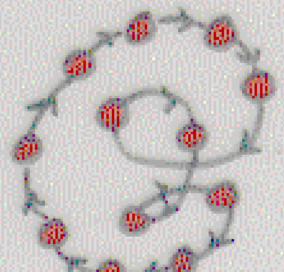
$$X_k = \frac{1}{k}\sum_j e^{-kE_j/T} = \frac{1}{k}\frac{q^{kd/2}}{(1-q^k)^d} \qquad q = e^{-\hbar\omega/T}$$



HAS A PEAK.

L/G	B.E.
GO FROM G → L "INFINITE" CLUSTER APPEARS	CONDENSATION INTO G.S. & VERY LONG CYCLES APPEAR

HAS SURFACE ENERGY.

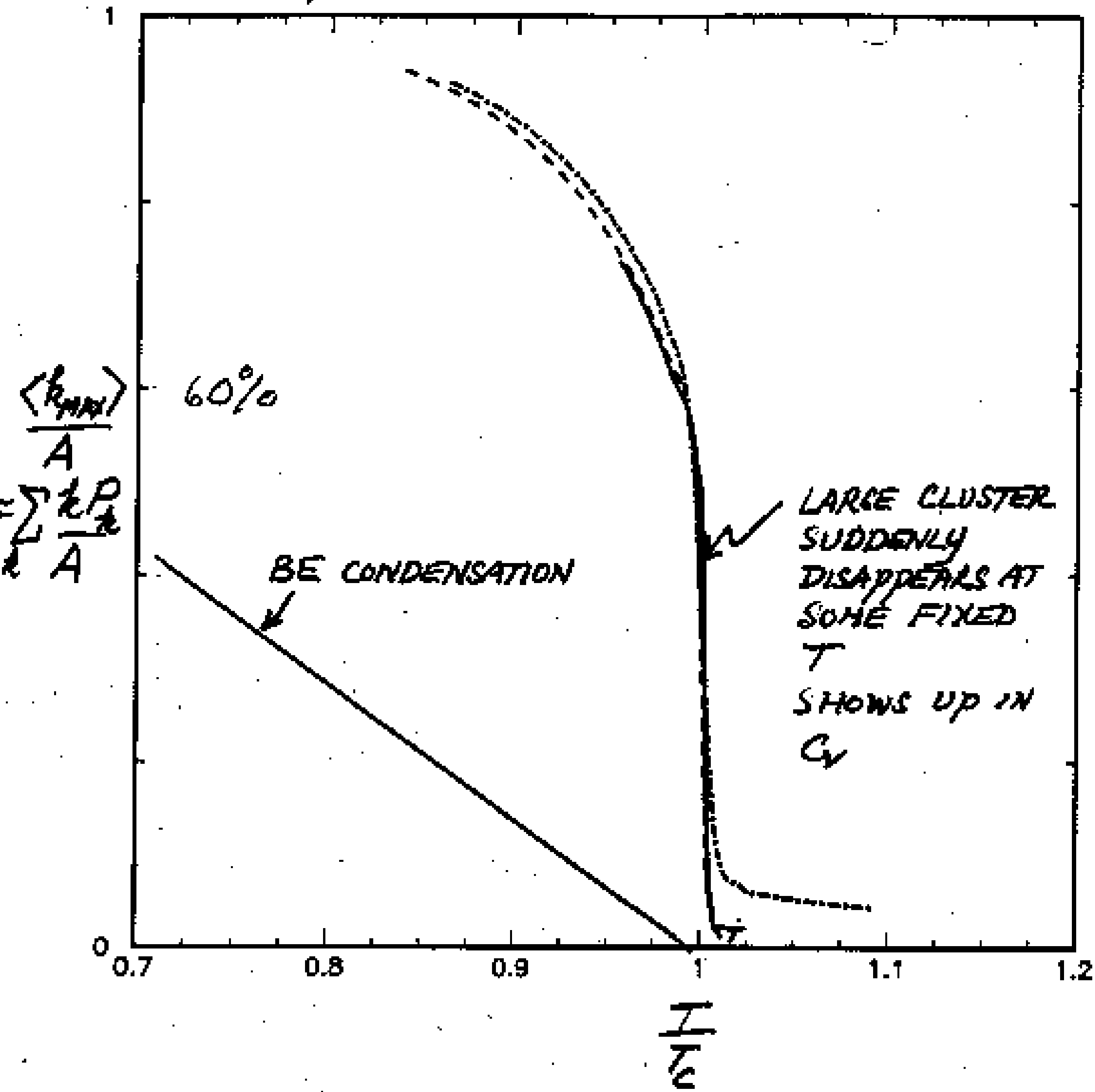


STATISTICAL MODEL & NUCLEAR BOILING

$$X_k = \frac{V}{\lambda_T^3} k^{3/2} e^{\left[\left(a_v + \frac{T^2}{\epsilon_0}\right)k - a_s k^{2/3}/T\right]}$$

$a_v = 16$
$\epsilon_0 = 16$
$a_s = 18\left(\frac{T_c^2 - T^2}{T^2 + T_c^2}\right)^{5/4}$

DAS GUPTA & A.Z.M.



IN GC (GRAND CANONICAL) LIMIT
"INFINITE" CLUSTER APPEARS WHEN

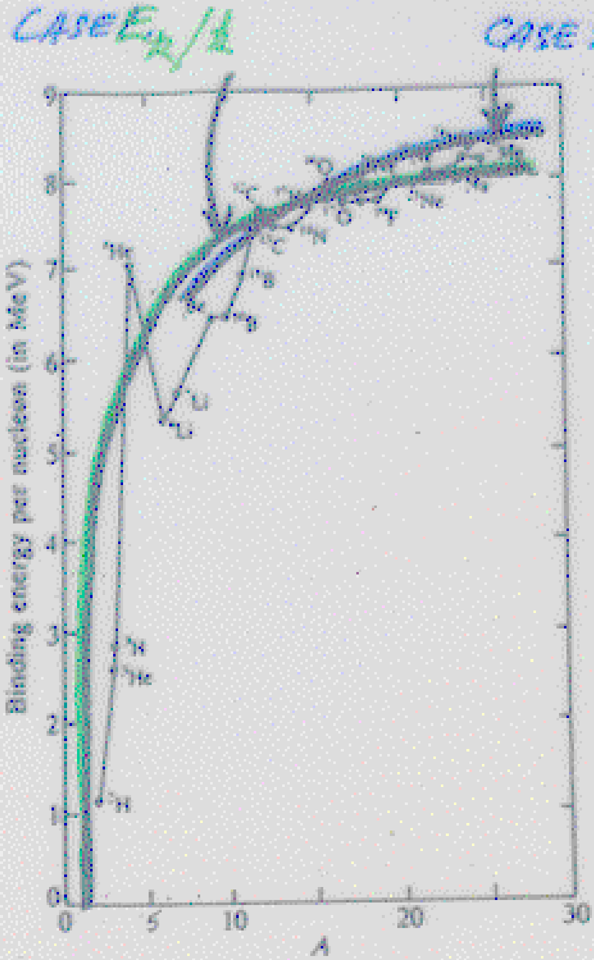
$\mu = -(a_v + T^2/\epsilon_0)$ GIVES BOILING T

$\mu = E_0$ (G.S. E) FOR B.E. CONDENSATION

SURFACE E & BIRTH OF THE "INFINITE" (LARGEST) CLUSTER.

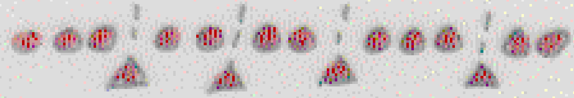
CASE 1 $E_k = a_v(k-1)$ FITS B.E DATA WITH $a_v = 8$ MeV

CASE 2. $E_k = a_v k - a_s k^{2/3} - a_c Z^2/k - \ldots$

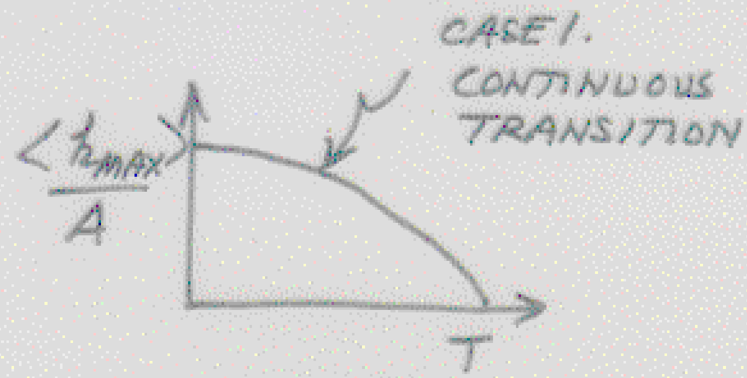


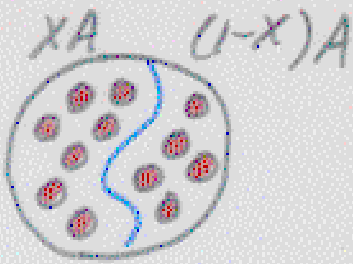
FOR 1.

BREAK A INTO 2 PIECES COSTS a_v INDEPENDENT OF k, A



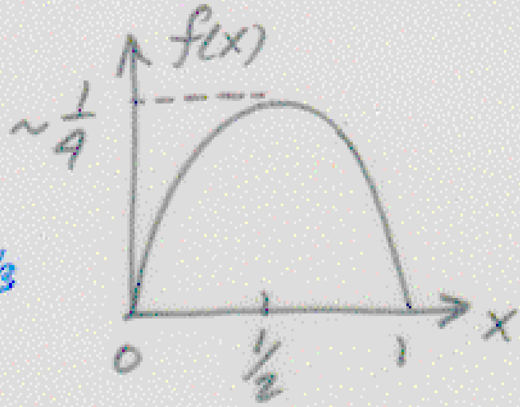
m PIECES COSTS $(m-1)\,a_v$

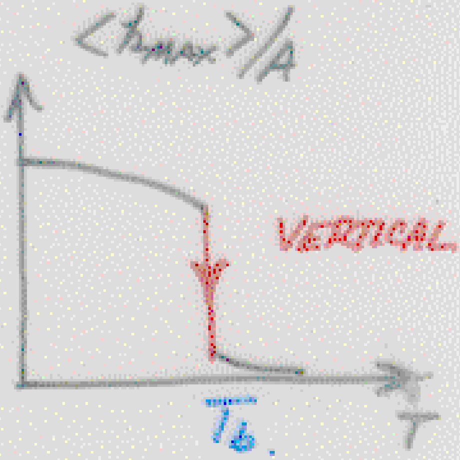




CUT ACROSS A GROUP $\delta E_S \sim A^{2/3}$

$$\delta E_S = f(x)\, A^{2/3}$$





LARGE $\delta E_S \sim A^{2/3} \Rightarrow$ LARGE PROB THAT A BIG CLUSTER IS PRESENT UNTIL T_b.

SSM – SPECIFIC HEAT

$$\langle E \rangle = \frac{3}{2} T \sum_{k=1}^{A} \langle n_k \rangle + \left(-a_v + \frac{T^2}{\epsilon_0}\right)\left(A - \langle n_1 \rangle\right) + \sum_{k=2}^{A} a_s k^{2/3} \langle n_k \rangle$$

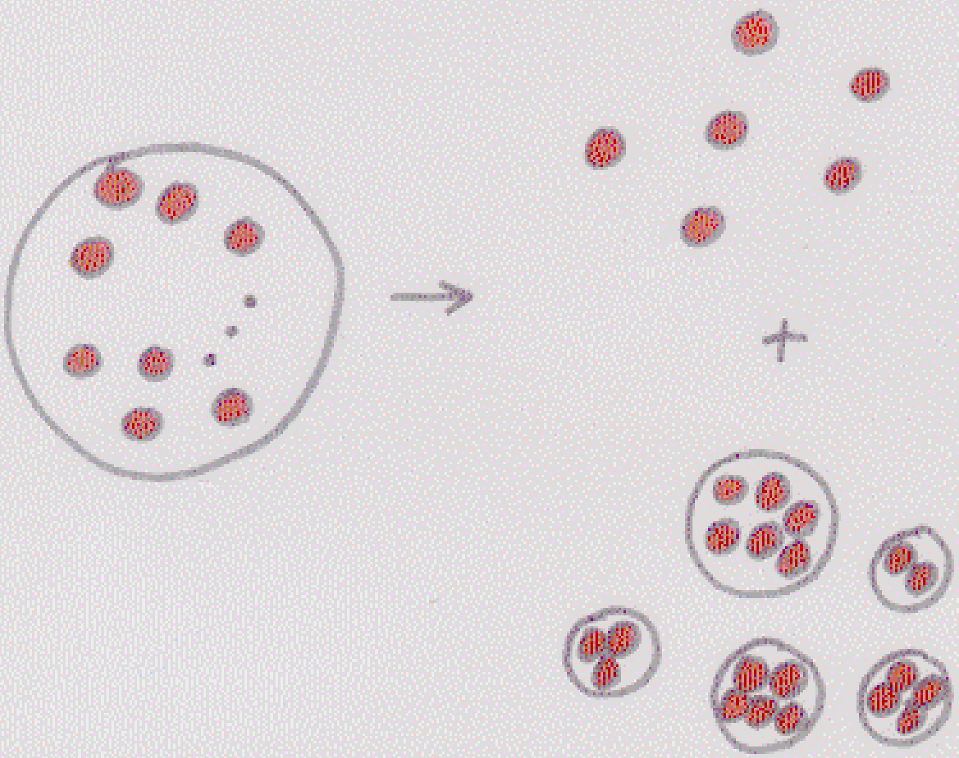
K.E. = EACH DEGREE OF FREEDOM HAS $\frac{1}{2} k_B T$ PER PARTICLE OR CLUSTER.

B.E. & E^* EFFECTS

NUCLEONS HAVE NO B.E. & E^*

SURFACE ENERGY CONTRIBUTION

ONE COMPONENT – NO COULOMB OR SYMMETRY.
IDEALIZED SYSTEM.



COST B.E. & THERMAL E TO GO INTO NUCLEONS

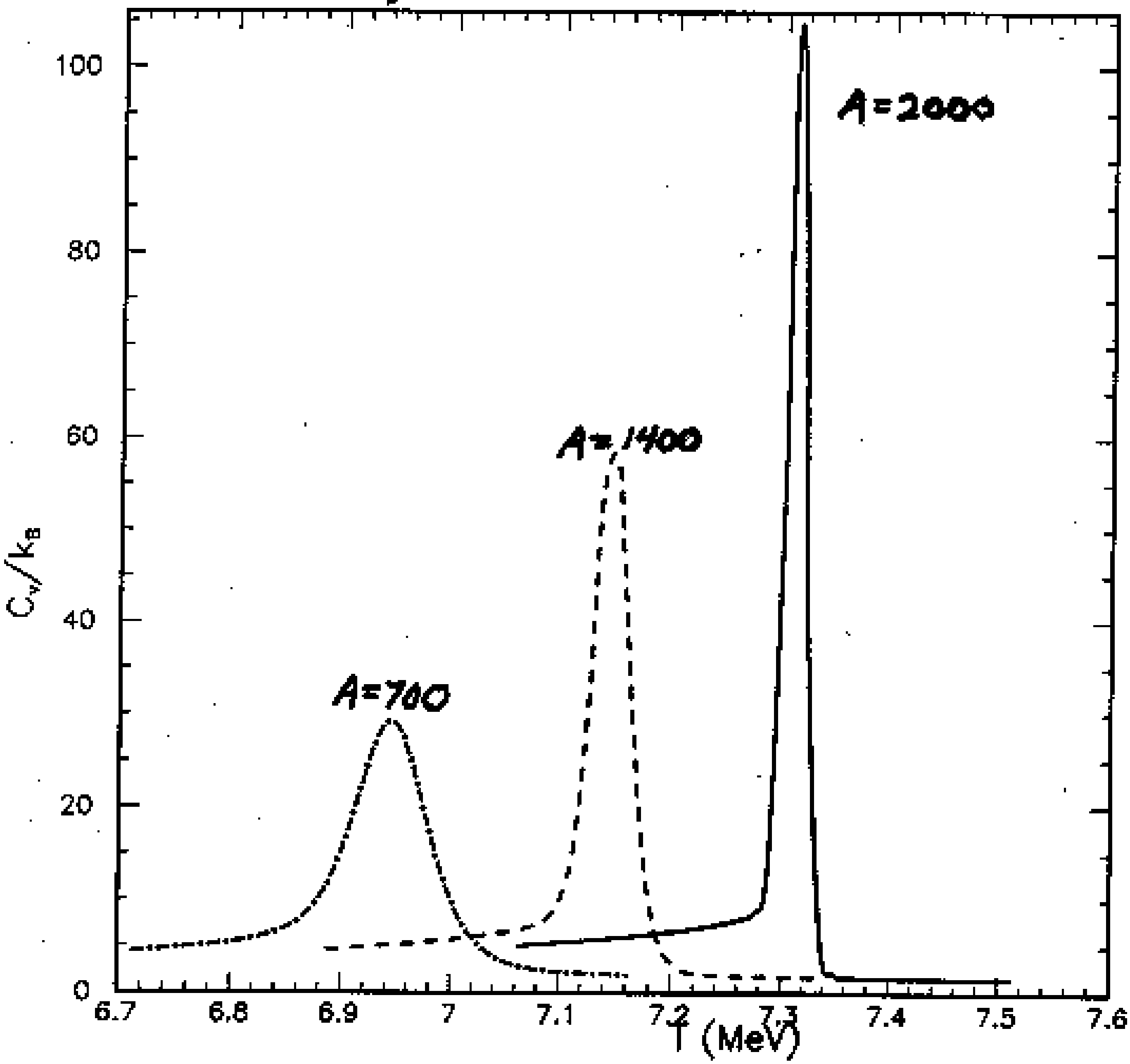
COSTS SURFACE ENERGY & THERMAL E TO GO INTO CLUSTERS.

$$C_V = \frac{\partial \langle E \rangle}{\partial T}$$

BEHAVIOR OF C_V WITH SURFACE TERMS

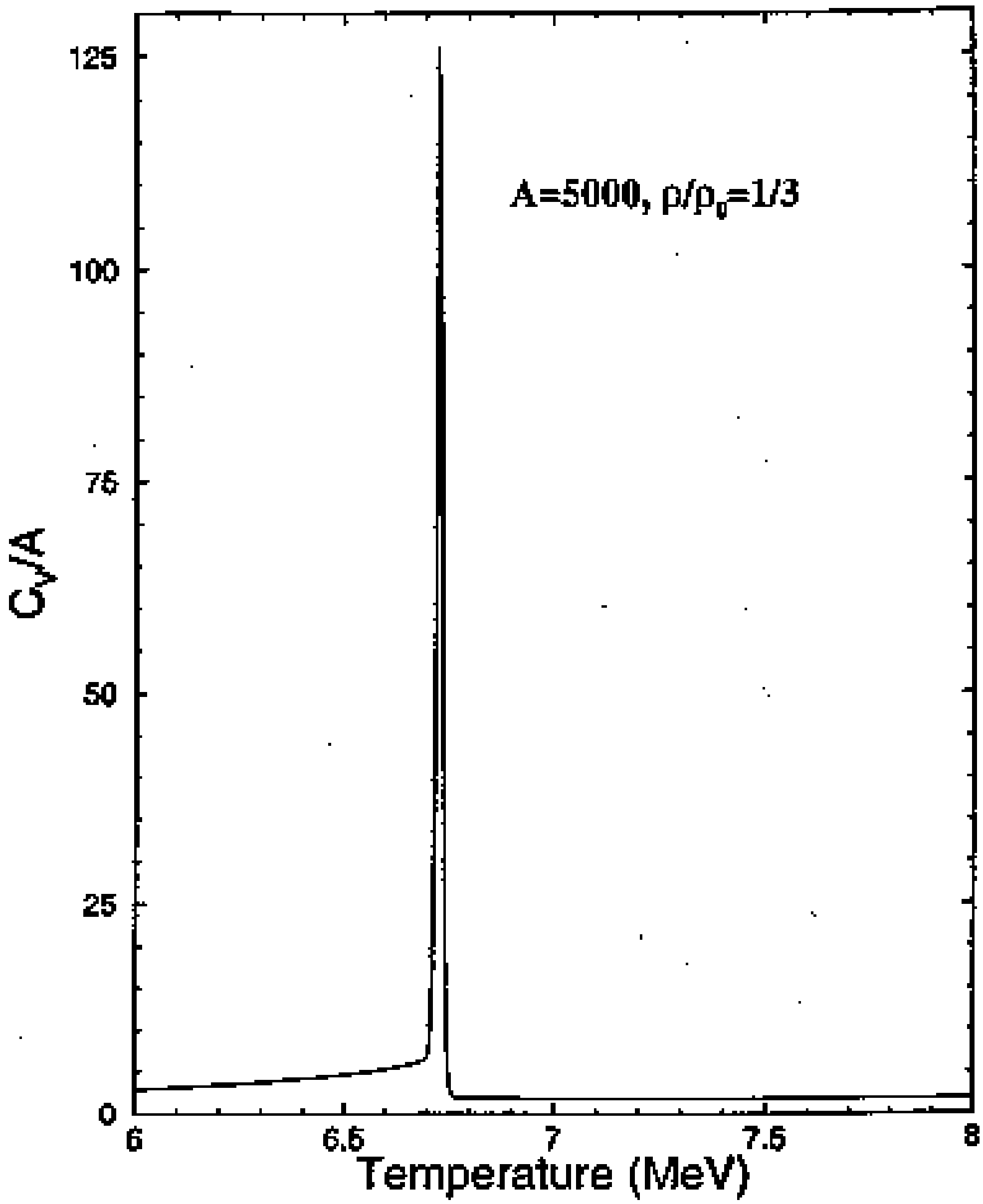
OBSERVE: C_V BECOMES NARROW WITH INCREASING A
SPIKE STRUCTURE AT VERY LARGE A

DAS GUPTA, A.Z.M.



PROOF OF 1ST ORDER PHASE TRANSITION
IN GRAND CANONICAL ENSEMBLE GIVEN
BY BUGAEV, GORENSTEIN, MISHUSTIN, GREINER

MODEL ALSO STUDIED BY ELLIOT & HIRSCH.



C. DAS, S. DAS GUPA, A.Z.M.

BREAK IN FREE ENERGY

$$F_A = -T \ln Z_A$$

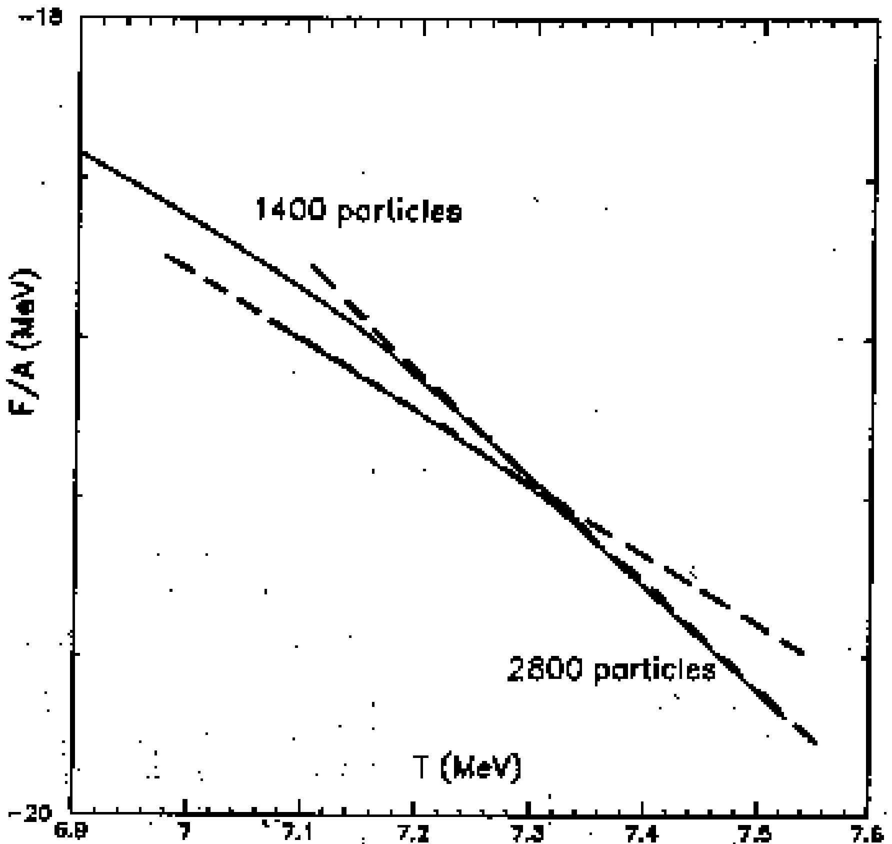


FIG. 1. The free energy per particle for (a) a system of 2800 particles and (b) a system of 1400 particles plotted as a function of temperature. The curves suggest that a break in the first derivative occurs at 7.3 MeV and at 7.15 MeV, respectively.

$dF = -S\,dT \ldots$ $\qquad \frac{S}{A} = -\frac{\partial E/A}{\partial T}$

$S_{LIQ} \sim 1.8$

AT $T = T_b$

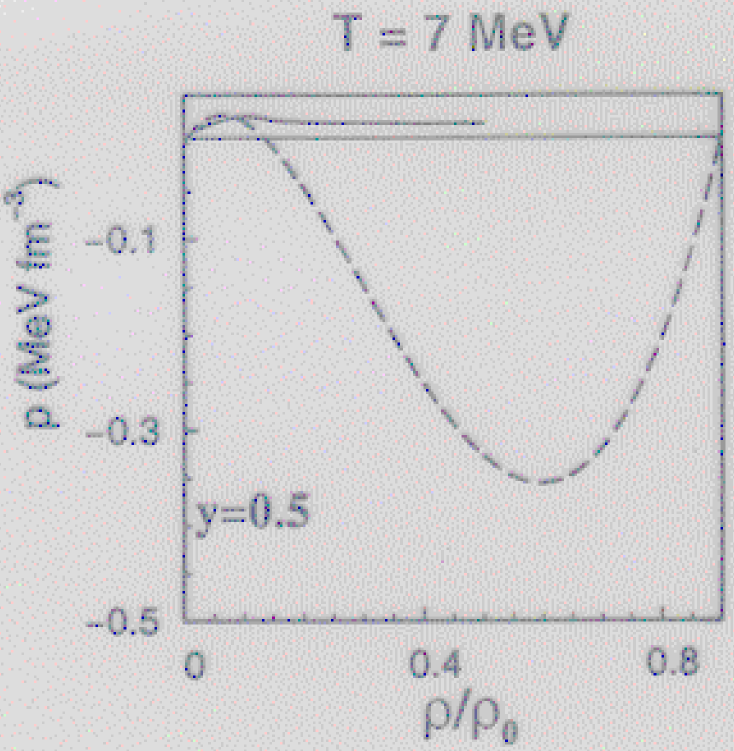
$S_{GAS} \sim 2.4$

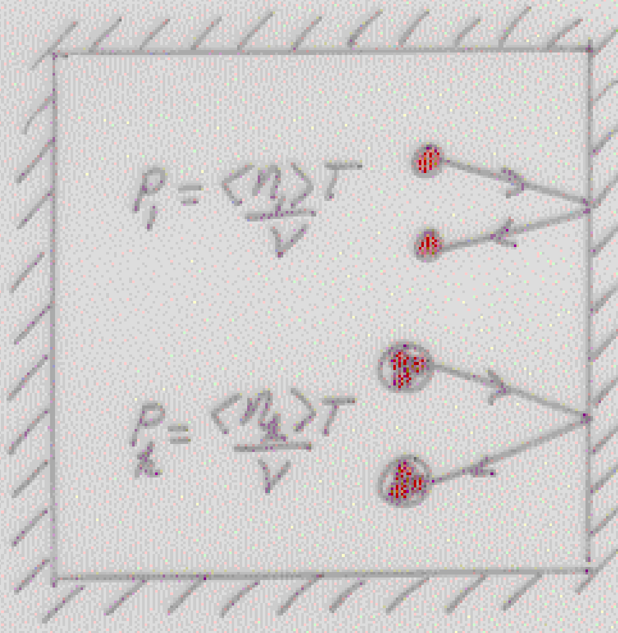
$\Rightarrow$ LATENT HEAT $\quad \frac{Q_L}{A} = T \frac{\Delta S}{A}$

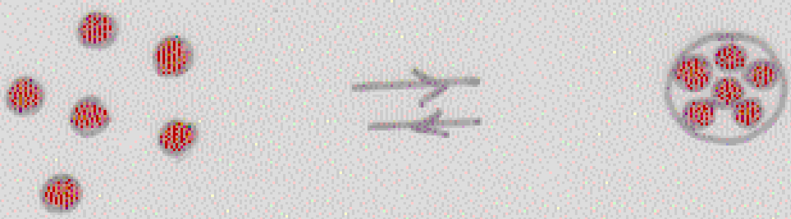
EOS: $PV = \sum_k \langle n_k \rangle T \qquad P = \sum_k P_k \qquad P_k = \langle n_k \rangle T/V$

FROM DAS, DASGUPTA, A.Z.M

LAW OF PARTIAL P'S







GCE

$k\mu_1 = \mu_k$

$$\frac{\langle n_k \rangle}{V} = \left(\frac{\langle n_1 \rangle}{V}\right)^k \left(\lambda_T^3\right)^{k-1} k^{3/2} Z_{INT}(k)$$

$$\langle A \rangle = \sum k \langle n_k \rangle$$

LOW ρ:

$$P \sim \rho T \left(1 + B(T)\rho + C(T)\rho^2 + \dots\right)$$

$$B(T) \propto -\langle n_2 \rangle$$

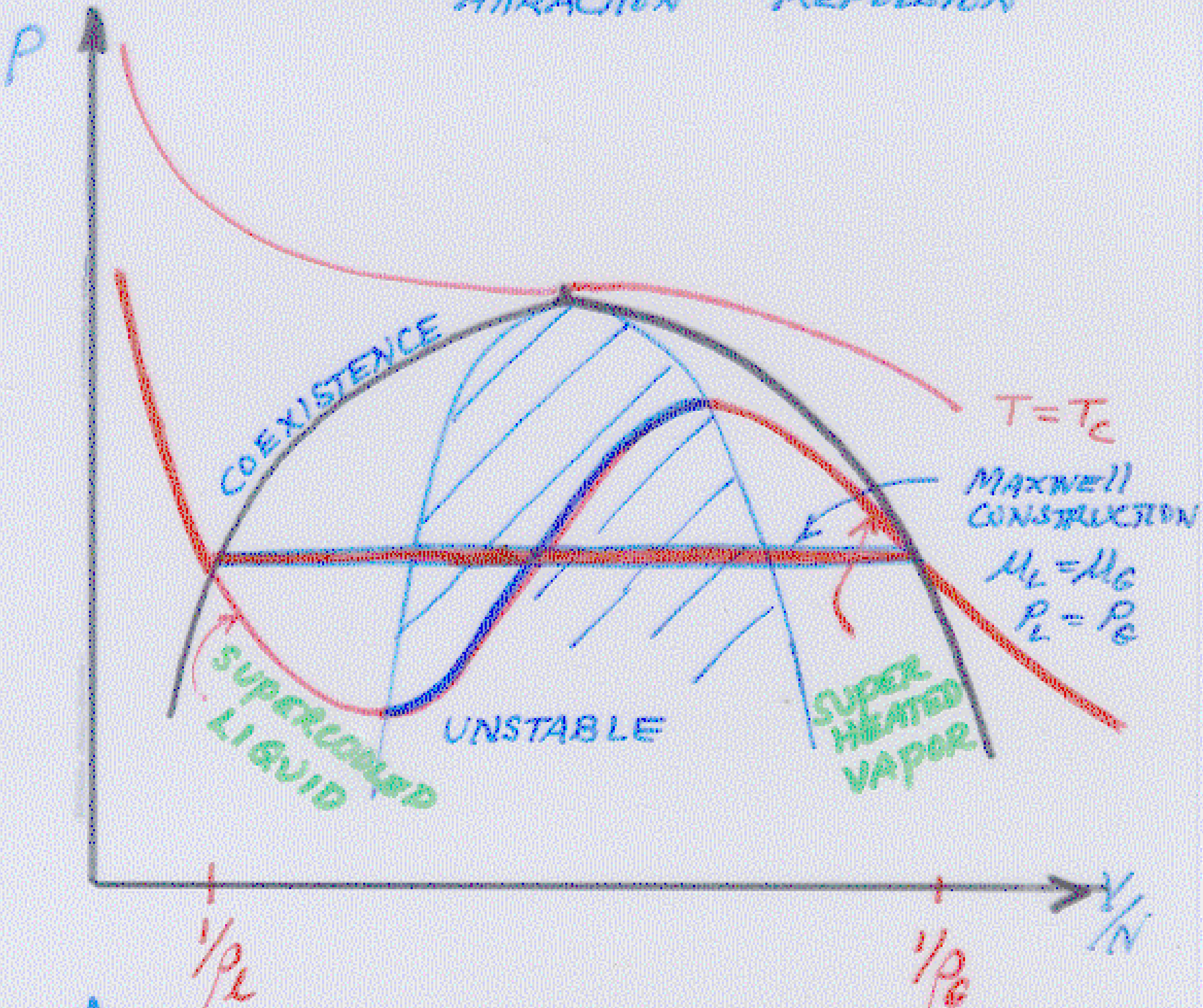
$$C(T) \propto -2\langle n_3 \rangle$$

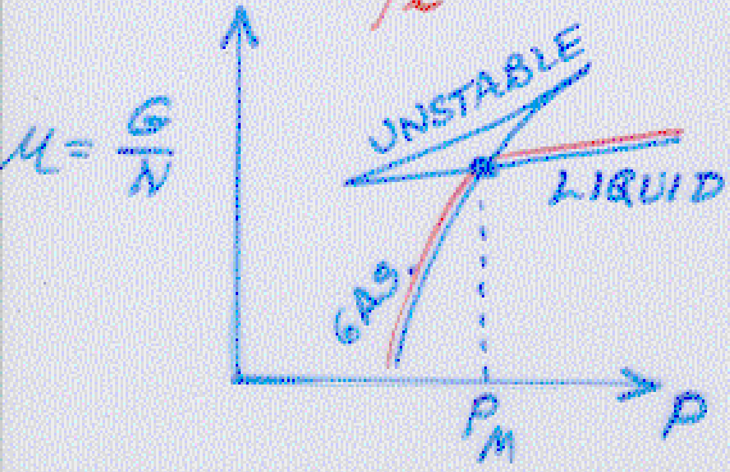
MEAN FIELD THEORY OF L/G. – ONE COMPONENT

SKYRME + SIMPLIFIED KINETIC ENERGY

$$p = \rho T - a_0 \rho^2 + 2 a_3 \rho^3$$

LONG RANGE ATTRACTION — SHORT RANGE REPULSION

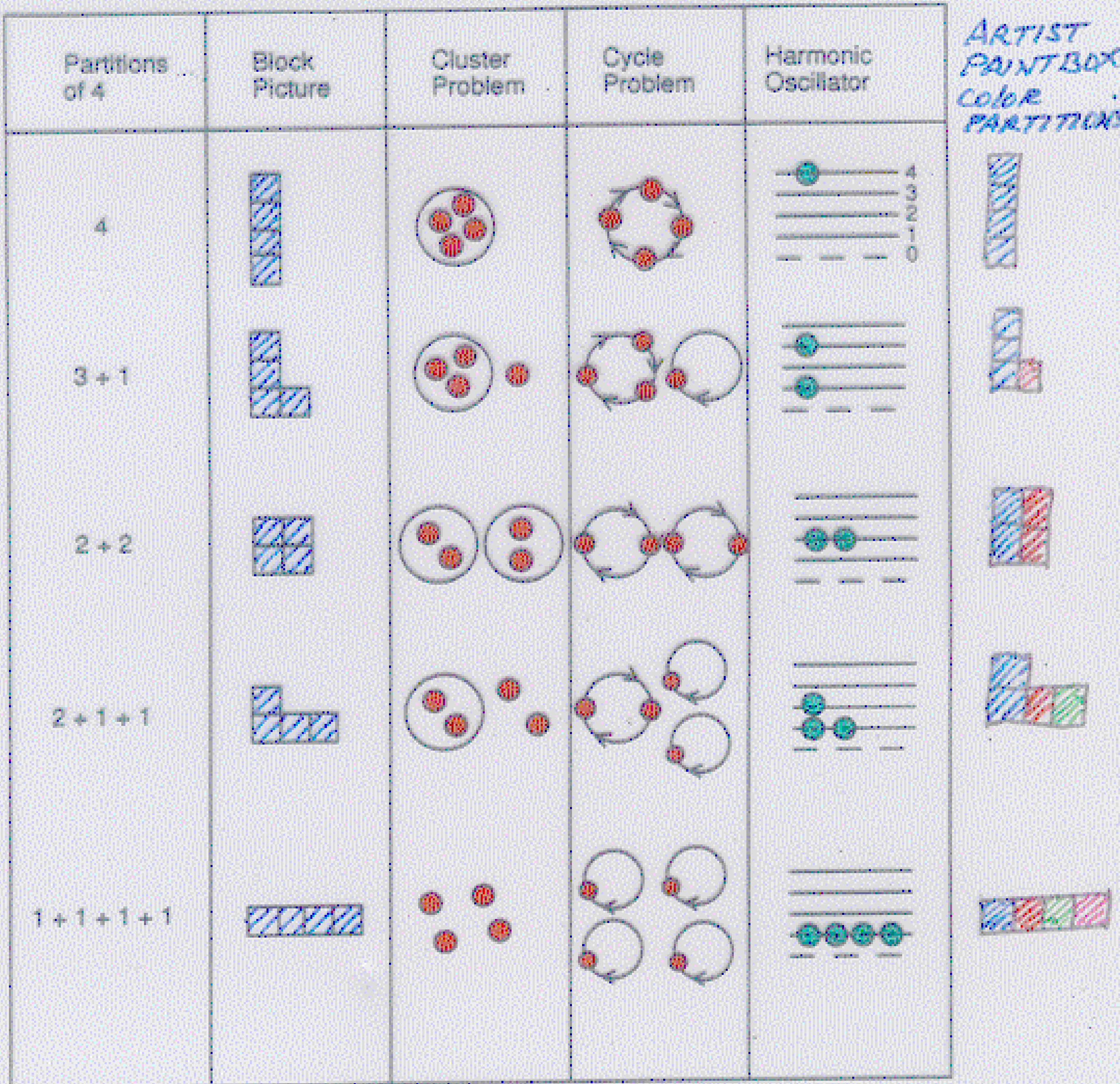




T FIXED.

$$dG = Vdp - SdT + \mu dN$$

$$N d\mu = Vdp - SdT$$



EARLY MODELS OF FRAGMENT COUNTING

SELF-ORGANIZED CRITICALITY

RULES FOR MOVING BLOCKS & CAUSING BLOCK AVALANCHES

CLUSTERS

PERCOLATION

RANDOMLY BROKEN OBJECTS

GROUP STRUCTURE

BOSE-EINSTEIN & FERMI-DIRAC STATISTICS

SPECKLE PATTERNS

FEYNMAN THEORY OF λ TRANSITION

MICROCANONICAL P.F. FOR HARMONIC OSCILLATOR

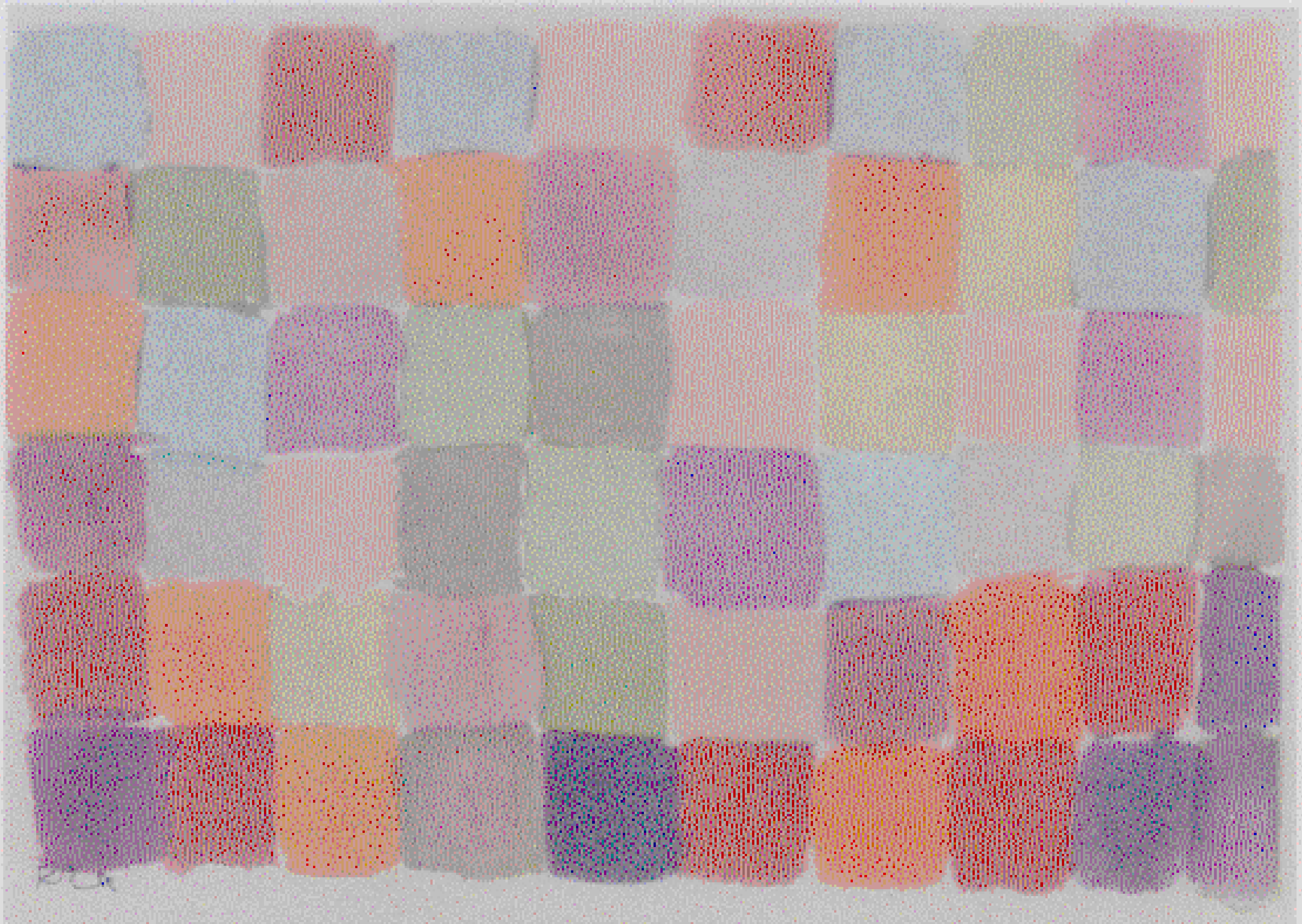
VENEZIANO HAGEDORN MASS SPECTRUM

MAP

k : CYCLE LENGTH ↔ CLUSTER SIZE

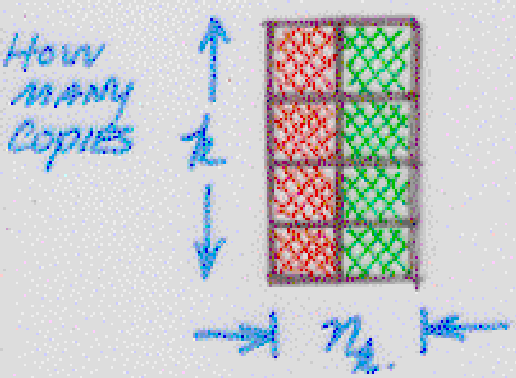
ARTISTS PAINTBOX & COLOR PARTITIONS

PAUL KLEE:



RESONANCE OF THE SOUTHERN FLORA. 1927.

n_k = # OF **DIFFERENT** COLORS EACH OCCURING k TIMES



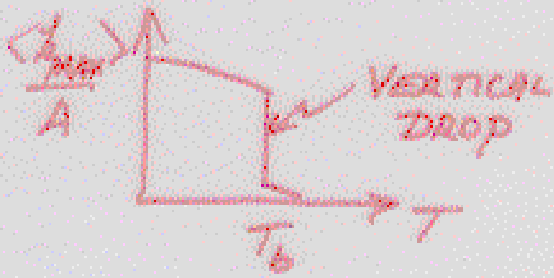
EACH DIFFERENT COLOR IS A DIFFERENT SPECIE IN BIOLOGY

SUMMARY.

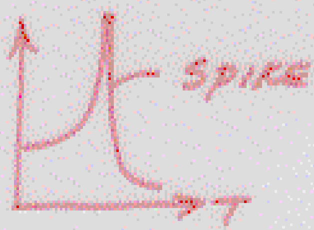
1. RECURSIVE SOLUTION $Z_A = \frac{1}{A}\sum k x_k Z_{A-k}$ EXISTS FOR
$W_A(\vec{n},\vec{x}) = \prod_k x_k^{n_k}/n_k!$
METHODS ARE FAST AND CONCISE
GIVE EXACT Z_A QUICKLY FOR LARGE SYSTEM
$A = 5000$ HAS $\sim 10^{74}$ PARTITIONS SUMMED
IN $A^2/2$ STEPS. $\sim 10^7$ STEPS

2. Application TO SSM (STANDARD STATISTICAL MODEL)
SHOWS CLEAR EVIDENCE FOR 1ST ORDER TRANSITION

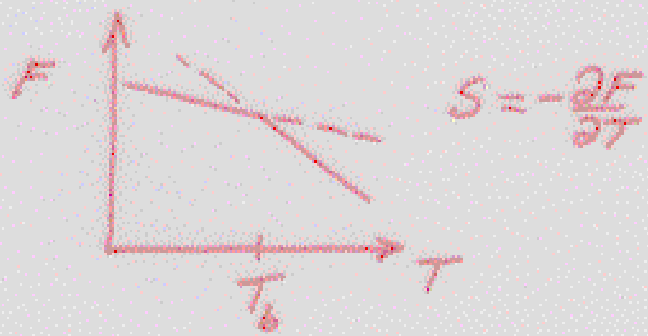
a) BOILING FEATURE



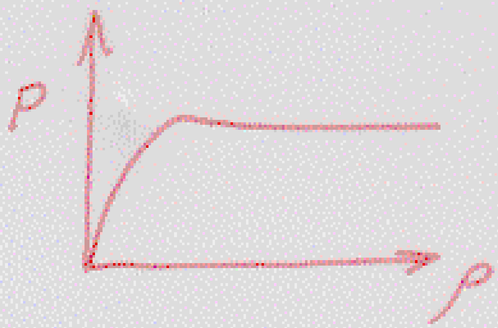
b) C_V



c) BREAK IN F



d) EOS



3. SAME FRAMEWORK CAN BE USED IN OTHER AREAS

ATOMS IN LASER TRAP
SHELL MODEL
QUANTUM OPTICS
THERMAL PIONS
PARTICLE MULTIPLICITY DISTRIBUTIONS
RANDOMIZED GAMES
SPIN GLASS THEORY
EVOLUTIONARY BIOLOGY.