An Introduction to Quantum Field Theory (Peskin and Schroeder) Solutions

Andrzej Pokraka

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Contents

4 Interacting Fields and Feynman Diagrams

4.1 Creation of Klein-Gordon particles from a classical source ✓

Recall from Chapter 2 that this process can be described by the Hamiltonian

$$H = H_0 + \int d^3x \left(-j\left(t, \mathbf{x}\right)\phi\left(t, \mathbf{x}\right)\right),$$

where H_0 is the free Klein-Gordon Hamiltonian, $\phi(x)$ is the Klein-Gordon field, and $j(x)$ is a complex scalar function. We found that, the system is in the vacuum state before the source is turned on the source will create a mean number of particles

$$\langle N \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} |\tilde{j}(p)|^2.$$

In this problem we will verify this statement, and extract more detailed information, by using a perturbation expansion in the strength of the source.

a) Show that the probability that the source creates no particles is given by

$$P(0) = \left| \langle 0| T\left\{ \exp\left[i \int d^4x j(x) \phi_I(x) \right] \right\} |0\rangle \right|^2.$$

Proof: The time evolution operator is given by

$$U(-T, T) = T\left\{ \exp[-i \int d\tau H_I(\tau) \right\}$$

and it tells us how the states of the system evolve in time. The ground state, $|0\rangle$, satisfies $H_0|0\rangle = 0$. Thus, the ground state at time, T, is given by evolving the ground state at time $-T$ to time T,

$$U(-T, T)|0\rangle.$$

To get the probability that the source produces no particles we project $U(-T, T)|0\rangle$ onto the ground state and square. Taking the $T \to \infty$ limit the probability is

$$P(0) = \left| \langle 0| T\left\{ \exp\left[i \int d^4x j(x) \phi_I(x) \right] \right\} |0\rangle \right|^2.$$

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b) Evaluate the term in $P(0)$ of order j^2, and show that $P(0) = 1 - \lambda + \mathcal{O}(j^4)$, where λ equals the expression above for $\langle N \rangle$.

Proof: We start by expanding the time ordered exponential in $P(0)$

$$\begin{aligned} P(0) &= \left| 1 + i\int d^4x \,\langle 0|T\{j(x)\phi_I(x)\}|0\rangle + \frac{i^2}{2}\int d^4x\int d^4y\,\langle 0|T\{j(x)j(y)\phi_I(x)\phi_I(y)\}|0\rangle \right. \\ &\left. + \frac{i^3}{6}\int d^4x\int d^4y\int d^4z\,\langle 0|T\{j(x)j(y)j(z)\phi_I(x)\phi_I(y)\phi_I(z)\}|0\rangle + \mathcal{O}(j^4)\right|^2. \end{aligned}$$

To simplify this further we need to evaluate the time order products of the source with the field. The terms that are odd in j have an odd number of field operators. These are proportional to the expectation value of the time ordered product of and odd number of field operators. Since the time ordered product is equal to the normal ordered product plus the normal order of all contractions, the only terms which survive the expectation value are ones where the field operators are all contracted. Thus, all odd terms vanish and

$$\begin{aligned} P(0) &= \left|1 + \frac{i^2}{2}\int d^4x\int d^4y\, j(x)j(y)\langle 0|T\{(x)\phi_I(y)\}|0\rangle + \mathcal{O}(j^4)\right|^2 \\ &= \left|1 + \frac{i^2}{2}\int d^4x\int d^4y\, j(x)j(y)\langle 0|N\left\{\phi_I(x)\phi_I(y) + \overline{\phi_I(x)\phi_I(y)}\right\}|0\rangle + \mathcal{O}(j^4)\right|^2 \\ &= \left|1 + \frac{i^2}{2}\int d^4x\int d^4y\, j(x)D_F(x-y)j(y) + \mathcal{O}(j^4)\right|^2 \\ &= 1 - \frac{1}{2}\int d^4x\int d^4y\, j(x)D_F(x-y)j(y) - \frac{1}{2}\left(\int d^4x\int d^4y\, j(x)D_F(x-y)j(y)\right)^* + \mathcal{O}(j^4) \\ &= 1 - \mathrm{Re}\left[\int d^4x\int d^4y\, j(x)D_F(x-y)j(y)\right] + \mathcal{O}(j^4) \\ &= 1 - \left[\int d^4x\int d^4y\int \frac{d^4p}{(2\pi)^4}\, j(x)\frac{ie^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}j(y)\right] + \mathcal{O}(j^4) \\ &= 1 - \mathrm{Re}\left[\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}\int d^4x\, j(x)e^{-ip\cdot x)}\int d^4y\, j(y)e^{ip\cdot y}\right] + \mathcal{O}(j^4) \\ &= 1 - \mathrm{Re}\left[\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}\tilde{j}(p)\,\tilde{j}^*(p)\right] + \mathcal{O}(j^4) \\ &= 1 - \mathrm{Re}\left[\int \frac{d^3p}{(2\pi)^3}\int \frac{d^0p}{2\pi}\frac{i}{(p^0)^2 - E_{\mathbf{p}}^2 + i\epsilon}\left|\tilde{j}(p)\right|^2\right] + \mathcal{O}(j^4) \end{aligned}$$

Since $p^0 > 0$ there is only a simple pole at $p^0 = E_{\mathbf{p}} - i\epsilon$. The small imaginary shift forces us to close in the lower plane which gives an additional minus sign due to the direction of the contour

$$\begin{aligned} P(0) &= 1 + \mathrm{Re}\left[\int \frac{d^3p}{(2\pi)^3} i\mathrm{Residue}\left\{\frac{i}{(p^0)^2 - E_{\mathbf{p}}^2 + i\epsilon}\left|\tilde{j}(p)\right|^2; p^0 = E_{\mathbf{p}}\right\}\right] + \mathcal{O}(j^4) \\ &= 1 - \mathrm{Re}\left[\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}^2}\left|\tilde{j}(p)\right|^2\right] + \mathcal{O}(j^4) \\ &= 1 - \lambda + \mathcal{O}(j^4). \end{aligned}$$

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c) Represent the term computed in part (b) as a Feynman diagram. Now represent the whole perturbation series for $P(0)$ in terms of Feynman diagrams. Show that this series exponentiates, so that it can be summed exactly: $P(0) = \exp(-\lambda)$.

Proof: Due to the presence of the propagator in the the first order term of $P(0)$ we represent $\lambda \equiv \longrightarrow$. This diagram has two points and a direction in time. We can write the whole series as

$$P(0) = \left| 1 - \longrightarrow + \begin{array}{c}\longrightarrow\\ \longrightarrow\end{array} - \begin{array}{c}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{array} + \ldots \right|^2 .$$

To get the series we must know the correct symmetry factor. In each term there are n propagators with $2n$ vertices. There are n "in" (left) vertices and n "out" (right) vertices. Each in vertex must be paired with an out vertex; there are $2^{2n/2} = 2^n$ ways to do this. Additionally, the n in vertices can be interchanged in $n!$ ways. This yields the symmetry factor

$$S_n = 2^n \cdot n!.$$

Dividing by the symmetry factor for each term the series becomes

$$\begin{aligned} P(0) &= \left| 1 - \longrightarrow + \begin{array}{c}\longrightarrow\\ \longrightarrow\end{array} - \begin{array}{c}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{array} + \ldots \right|^2 \\ &= \left| \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{S_n} \right|^2 \\ &= = \left| \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-\lambda}{2} \right)^n \right|^2 \\ &= \left| \exp\left(-\lambda/2\right) \right|^2 \\ &= \exp\left(-\lambda\right) \end{aligned}$$

as required.

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d) Compute the probability that the source creates on e particle of momentum k. Preform this computation first to $\mathcal{O}(j)$ and then to all orders, using the trick of part (c) to sum the series.

Proof: The amplitude for the production of one particle of momentum $\mathbf{k}$ is

$$\langle \mathbf{k} | T \left\{ \exp\left[i \int d^4x j(x) \phi_I(x) \right] \right\} |0\rangle.$$

First we calculate this to $\mathcal{O}(j)$:

$$\begin{aligned} P(1_{\mathbf{k}}) &= \langle 0 | a_{\mathbf{k}} \left(1 + i \int d^4x\, j(x) \phi_I(x) \right) |0\rangle \\ &= \langle 0 | a_{\mathbf{k}} i \int d^4x\, j(x) \left(\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ip\cdot x} + a^\dagger_{\mathbf{p}} e^{ip\cdot x} \right) \right) |0\rangle \\ &= i \int d^4x\, j(x) \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \langle 0 | a_{\mathbf{k}} \left(a_{\mathbf{p}} e^{-ip\cdot x} + a^\dagger_{\mathbf{p}} e^{ip\cdot x} \right) |0\rangle \\ &= i \int d^4x\, j(x) \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \langle 0 | a_{\mathbf{k}} a^\dagger_{\mathbf{p}} e^{ip\cdot x} |0\rangle \\ &= i \int d^4x\, j(x) \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \langle 0 | a^\dagger_{\mathbf{p}} a_{\mathbf{k}} e^{ip\cdot x} + (2\pi)^3 \delta(\mathbf{p} - \mathbf{k}) e^{ip\cdot x} |0\rangle \\ &= i \frac{1}{\sqrt{2E_{\mathbf{k}}}} \int d^4x\, j(x) e^{ik\cdot x} \\ &= i \frac{\tilde{j}(k)}{\sqrt{2E_{\mathbf{k}}}}. \end{aligned}$$

The probability is the square of the amplitude

$$P(1_{\mathbf{k}}) = \left| i \frac{\tilde{j}(k)}{\sqrt{2E_{\mathbf{k}}}} \right|^2 = \frac{\left|\tilde{j}(k)\right|^2}{2E_{\mathbf{k}}}.$$

The probability that one particle of any momentum is created is obtained by integrating over all possible momentum

$$\begin{aligned} P(1) &= \int \frac{d^3k}{(2\pi)^3} \frac{\left|\tilde{j}(k)\right|^2}{2E_{\mathbf{k}}} \\ &= \lambda. \end{aligned}$$

The amplitude to produce one particle is $i\sqrt{\lambda}$ (I am ignoring the phase of $\tilde{j}$). We denote the particle production by $\dashrightarrow$. The full series then becomes

$$\begin{aligned} P(1) &= \left| \dashrightarrow \times \left(1 - \longrightarrow + \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} - \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} + \ldots \right) \right|^2 \\ &= \left| i\sqrt{\lambda} \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{S_n} \right|^2 \\ &= \lambda \exp(-\lambda). \end{aligned}$$

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e) Show that the probability of production of n particles is given by

$$P(n) = \frac{1}{n!} \lambda^n \exp(-\lambda).$$

This is a Poisson distribution.

Proof: The amplitude for the production of $n$particles with momentum $\mathbf{k}$ is

$$\langle 0 | a_{\mathbf{k}}^n T \left\{ \exp \left[i \int d^4x j(x) \phi_I(x) \right] \right\} | 0 \rangle.$$

The first non-zero term is $n + 1^{th}$ term of the time ordered exponential with n field operators. Only the fully contracted term of the creation and annihilation operators survives to give

$$\left(i\sqrt{\lambda} \right)^n.$$

In terms of Feynman diagrams the perturbative series reads

$$\begin{aligned} P(n) &= \frac{1}{n!} \left| (\dashrightarrow)^n \times \left(1 - \longrightarrow + \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} - \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} + \ldots \right) \right|^2 \\ &= \frac{1}{n!} \left| \left(i\sqrt{\lambda} \right)^n \sum_{l=0}^{\infty} \frac{(-1)^l \lambda^l}{S_l} \right|^2 \\ &= \frac{1}{n!} \lambda^n \exp(-\lambda). \end{aligned}$$

Note that we have included a factor of $1/n!$ to account for the n identical particles in the final state.

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f) Prove the following facts about the Poisson distribution:

$$\sum_{n=0}^{\infty} P(n) = 1; \text{ and } \langle N \rangle = \sum_{n=0}^{\infty} nP(n) = \lambda.$$

The first identity says that the $P(n)$'s are properly normalized probabilities, while the second confirms our proposal for $\langle N \rangle$. Compute the mean square fluctuations $\langle (N - \langle N \rangle)^2 \rangle$.

Proof:

$$\sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n e^{-\lambda} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n = e^{-\lambda} e^{\lambda} = 1$$

$$\langle N \rangle = \sum_{n=0}^{\infty} nP(n) = e^{-\lambda} \sum_{n=0}^{\infty} \frac{n}{n!} \lambda^n = e^{-\lambda} \sum_{n=0}^{\infty} \frac{1}{(n-1)!} \lambda^n = e^{-\lambda} \sum_{m=0}^{\infty} \frac{1}{m!} \lambda^{m+1} = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

$$\begin{aligned}
\langle (N - \langle N \rangle)^2 \rangle &= \langle N^2 \rangle - \langle N \rangle \\
&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{n}{(n-1)!} \lambda^n - \lambda \\
&= e^{-\lambda} \sum_{m=0}^{\infty} \frac{m+1}{m!} \lambda^{m+1} - \lambda \\
&= \lambda e^{-\lambda} \sum_{m=0}^{\infty} \frac{m}{m!} \lambda^m + \lambda e^{-\lambda} \sum_{m=0}^{\infty} \frac{1}{m!} \lambda^m - \lambda \\
&= \lambda^2 + \lambda - \lambda \\
&= \lambda^2
\end{aligned}$$

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4.2 Decay of a scalar particle ✓

Consider the following Lagrangian, involving two real scalar fields Φ and ϕ:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \Phi)^2 - \frac{1}{2} M^2 \Phi^2 + \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} M^2 \phi^2 - \mu \Phi \phi \phi.$$

The last term is an interaction that allows a Φ to decay into two ϕ's, provided that $M > 2m$. Assuming that this condition is met, calculate the lifetime of the Φ to lowest order in μ.

Poof: The Lagrangian can be cast into a more illuminating form

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_0^\Phi + \mathcal{L}_0^\phi - \mathcal{L}_{\text{int}} \\
\mathcal{L}_0^\Phi &= \frac{1}{2} (\partial_\mu \Phi)^2 - \frac{1}{2} M^2 \Phi^2 \\
\mathcal{L}_{\text{int}} &= -\mu \Phi \phi \phi.
\end{aligned}$$

The interaction Hamiltonian is therefore

$$H_I = \int d^3x \; \mu \Phi \phi \phi.$$

The matrix element we wish to calculate is

$$\begin{aligned}\langle \mathbf{p}_1\mathbf{p}_2|iT|\mathbf{P}\rangle &= \lim_{T\to\infty(1-i\epsilon)}\left({}_0\langle \mathbf{p}_1\mathbf{p}_2|T\left(\exp\left[-i\int_{-T}^{T}dt\ H_I\left(t\right)\right]\right)|\mathbf{P}\rangle_0\right)_{\text{connected, amputated}}\\ &= \left(2\pi\right)^4\delta^4\left(P-p_1-p_2\right)i\mathcal{M}\left(\Phi_{\mathbf{P}}\to\phi_{\mathbf{p}_1}\phi_{\mathbf{p}_2}\right)\end{aligned}$$

To first order we have

$$\langle \mathbf{p}_1\mathbf{p}_2|iT|\mathbf{P}\rangle = -i\mu\int d^4x\ {}_0\langle \mathbf{p}_1\mathbf{p}_2|\phi\phi\Phi|\mathbf{P}\rangle_0 - i\mu\int d^4x\ {}_0\langle \mathbf{p}_1\mathbf{p}_2|\phi\phi\Phi|\mathbf{P}\rangle_0$$

where only the term where all the external states are contracted contributes the the T matrix. Using these contractions for a scalar field theory equation (4.94) of Peskin we have

$$\begin{aligned}\langle \mathbf{p}_1\mathbf{p}_2|iT|\mathbf{P}\rangle &= -i\mu\int d^4x\ e^{ip_2\cdot x}e^{ip_1\cdot x}e^{-iP\cdot x} - i\mu\int d^4x\ e^{ip_1\cdot x}e^{ip_2\cdot x}e^{-iP\cdot x}\\ &= -2i\mu\int d^4x\ e^{ip_1\cdot x}e^{ip_2\cdot x}e^{-iP\cdot x}\\ &= -2i\mu\delta^4\left(p_1+p_2-P\right).\end{aligned}$$

Thus,

$$i\mathcal{M}\left(\Phi_{\mathbf{P}}\to\phi_{\mathbf{p}_1}\phi_{\mathbf{p}_2}\right) = -2i\mu.$$

Plugging this into the decay rate formula and taking into account that there are two identical bosons in the final state (divide by 2) we have

$$\begin{aligned}\Gamma_\Phi &= \frac{1}{2}\frac{1}{2m_\Phi}\int\frac{d^3p_1}{(2\pi)^3}\frac{1}{2E_1}\frac{d^3p_2}{(2\pi)^3}\frac{1}{2E_2}\left|-2i\mu\right|^2(2\pi)^4\delta^4\left(p_1+p_2-P\right)\\ &= \frac{\mu^2}{m_\Phi}\int d\Phi_2\left(P;p_1,p_2\right)\\ &= \frac{\mu^2}{m_\Phi}\frac{\tilde{\beta}\left(m_\Phi^2,m_\phi^2,m_\phi^2\right)}{8\pi}\int\frac{d\cos\theta_1}{2}\frac{d\phi_1}{2\pi}\\ &= \frac{\mu^2}{m_\Phi}\frac{1}{8\pi}\sqrt{1-\frac{2\left(m_\phi^2+m_\phi^2\right)}{m_\Phi^2}+\frac{\left(m_\phi^2-m_\phi^2\right)^2}{m_\Phi^4}}\\ &= \frac{\mu^2}{m_\Phi}\frac{1}{8\pi}\sqrt{1-4\frac{m_\phi^2}{m_\Phi^2}}.\end{aligned}$$

Thus, the life time is

$$\tau_\Phi = \frac{8\pi m_\Phi}{\mu^2\sqrt{1-4\frac{m_\phi^2}{m_\Phi^2}}}.$$

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4.3 Linear sigma model

The interaction so pions at low energy can be described by a phenomenological model called the linear sigma model. Essentially, this model consists of N real scalar fields coupled by a ϕ^4 interaction that is symmetric under rotations of the

N fields. More specifically, let $\Phi^i(x)$ $i = 1, \ldots, N$ be a set of N fields, governed by the Hamiltonian

$$H = \int d^3x \left(\frac{1}{2} \left(\Pi^i\right)^2 + \frac{1}{2} \left(\nabla \Phi^i\right)^2 + V\left(\Phi^2\right) \right),$$

where $\left(\Phi^i\right)^2 = \mathbf{\Phi} \cdot \mathbf{\Phi}$, and

$$V(\Phi^2) = \frac{1}{2} m^2 \left(\Phi^i\right)^2 + \frac{\lambda}{4} \left(\left(\Phi^i\right)^2 \right)^2$$

is a function symmetric under rotations of $\mathbf{\Phi}$. for (classical) field configurations of $\Phi^i(x)$ that are constant in space and time, this term gives the only contribution to H; hence, V is the field potential energy.

(What does this Hamiltonian have to do with the strong interactions? There are tow types of light quarks, u and d. These quarks have identical strong interactions, but different masses. If these quarks are massless, the Hamiltonian of the strong interactions is invariant to unitary transformations of the 2-component object (u, d):

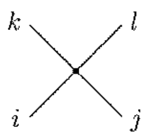
$$\begin{pmatrix} u \\ d \end{pmatrix} \to \exp\left(i \boldsymbol{\alpha} \cdot \boldsymbol{\sigma}/2\right) \begin{pmatrix} u \\ d \end{pmatrix}.$$

This transformation is called and isospin rotation. If, in addition, the strong interactions are described by a vector "gluon" field (as is true in QCD), the strong interaction Hamiltonian is invariant to the isospin rotations done separately on the left-handed and right-handed components of the quark fields. Thus, the complete symmetry of QCD with two massless quarks is $SU(2) \times SU(2)$. It happens that $SO(4)$, the group of rotations in 4 dimensions, is isomorphic to $SU(2) \times SU(2)$, so $N = 4$, the linear sigma model has the same symmetry group as the strong interactions.)

a) Analyze the linear sigma model for $m^2 > 0$ by noticing that, for $\lambda = 0$, the Hamiltonian given above is exactly N copies of the Klein-Gordon Hamiltonian. We can then calculate scattering amplitudes as perturbation series in the parameter λ. Show that the propagator is

$$\contraction{}{\Phi}{^i(x)}{\Phi}\Phi^i(x)\Phi^j(y) = \delta^{ij} D_F(x - y),$$

where D_F is the standard Klein-Gordon propagator for mass m, and that there is one type of vertex given by



$$= -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}).$$

(That is, the vertex between two $\Phi^1 s$ and two Φ^2s has the value $(-2i\lambda)$; that between four Φ^1s has the value $(-6i\lambda)$.) Compute, to leading order in λ, the differential cross sections $d\sigma/d\Omega$, in the center-of-mass frame, for the scattering processes

$$\Phi^1\Phi^2 \to \Phi^1\Phi^2, \; \Phi^1\Phi^1 \to \Phi^2\Phi^2, \text{ and } \Phi^1\Phi^1 \to \Phi^1\Phi^1$$

as functions of the center-of mass energy.

Proof: The Feynman rules for the invariant scattering amplitude come from the quantity

$$\langle \mathbf{p}_1 \mathbf{p}_2 | iT | \mathbf{P} \rangle = \lim_{T \to \infty(1-i\epsilon)} \left({}_0\langle \mathbf{p}_1^k \mathbf{p}_2^l | T \left(\exp\left[-i \int_{-T}^{T} dt \; H_I(t) \right] \right) | \mathbf{p}_3^i \mathbf{p}_4^j \rangle_0 \right)_{\text{connected, amputated}}.$$

In the linear sigma model the interaction Hamiltonian is

$$\begin{aligned} H_I &= \int d^3x \, \frac{\lambda}{4} \left(\left(\Phi^i\right)^2 \right)^2 \\ &= \int d^3x \, \frac{\lambda}{4} \left(\sum_i \left(\Phi^i\right)^2 \right)^2 \\ &= \int d^3x \, \left(\frac{\lambda}{4} \sum_i \left(\Phi^i\right)^4 + \frac{\lambda}{2} \sum_i \sum_{j>i} \left(\Phi^i\right)^2 \left(\Phi^j\right)^2 \right). \end{aligned}$$

So we have two different types of vertices; we will combine in to one at the end. To first order in λ we have:

1. $ii \to ii$. For this case the quantity of interest is

$$\begin{aligned}
\lim_{T\to\infty(1-i\epsilon)} \left({}_0\langle \mathbf{p}_1\mathbf{p}_2|T\left(\exp\left[-i\int_{-T}^{T} dt\ H_I(t)\right]\right)|\mathbf{P}\rangle_0 \right)_{\text{connected, amputated}} &= -i4!\frac{\lambda}{4}\sum_m \int d^4x\ {}_0\langle \mathbf{p}_1^k\mathbf{p}_2^l|\Phi^m\Phi^m\Phi^m\Phi^m|\mathbf{p}_3^i\mathbf{p}_4^j\rangle_0 \\
&= -6i\lambda \int d^4x\ e^{ip_1\cdot x}e^{ip_2\cdot x}e^{-ip_3\cdot x}e^{-ip_4\cdot x} \\
&= -6i\lambda\delta^4\left(\sum p\right).
\end{aligned}$$

which yields

$$i\mathcal{M}_{ii\to ii} = -6i\lambda.$$

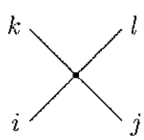
2. $ii \to jj$, $jj \to ii$ and $ij \to ij$. To vary the calculation up lets use momentum space rules instead of position space rules: $\mathcal{M}$ is the sum of all amputated and fully connected diagrams. The interaction Hamiltonian which governs such scattering processes is

$$\frac{\lambda}{2}\sum_i\sum_{j>i}\left(\Phi^i\right)^2\left(\Phi^j\right)^2.$$

The gives the vertex is given by $-i\frac{\lambda}{2}(2!)(2!) = -2i\lambda$ where the 2!'s come from the number of ways to contract two i external states with two i fields. The external legs contribute a factor of 1 and so invariant scattering amplitude is

$$\mathcal{M} = -2i\lambda.$$

Combining cases 1. and 2. we can write the amplitude as



$$= -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}).$$

Peskin has kindly provided the formula for the differential cross section for four particles with the same mass

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{|\mathcal{M}|^2}{64\pi^2E_{\text{cm}}^2}.$$

We now compute the differential cross section for three different processes:

1. $12 \to 12$. $\mathcal{M}_{12\to12} = -2i\lambda$. In the center of mass frame $\mathbf{p}_{\text{initial},1} = -\mathbf{p}_{\text{initial},2} \implies E_{\text{CM}} = 2m$.

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{|-2i\lambda|^2}{64\pi^2(2m)^2} = \frac{\lambda^2}{64\pi^2m^2}.$$

2. $11 \to 22$. $\mathcal{M}_{11\to22} = -2i\lambda$.

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{|-2i\lambda|^2}{64\pi^2(2m)^2} = \frac{\lambda^2}{64\pi^2m^2}.$$

3. $11 \to 11$. $\mathcal{M}_{11\to11} = -6i\lambda$.

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{|-6i\lambda|^2}{64\pi^2(2m)^2} = \frac{\lambda^2}{16\pi^2m^2}.$$

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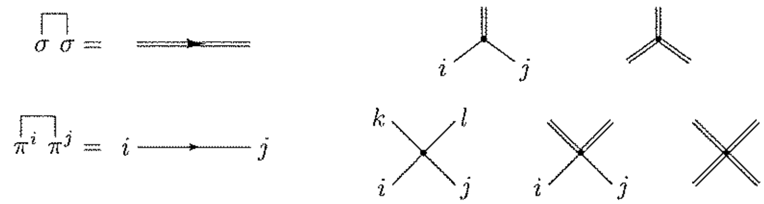
b) Now consider the case $m^2 < 0$; $m^2 = -\mu^2$. In this case, V has a local maximum, rather than a minimum, at $\Phi^i = 0$. Since V is a potential energy, this implies that the ground state of the theory is not near $\Phi^i = 0$ but rather is obtained by shifting Φ^i toward the minimum of V. By rotational invariance, we can consider this shift to be in the Nth direction. Write, then,

$$\begin{aligned} \Phi^i(x) &= \pi^i(x), \quad i = 1, \ldots, N-1, \\ \Phi^N(x) &= v + \sigma(x), \end{aligned}$$

where v is a constant chosen to minimize V. Show that, in these new coordinates (and substituting for v its expression in terms of λ and μ), we have a theory of a massive σ field and $N-1$ massless pion fields, interaction through cubic and quartic potential energy terms which all become small as $\lambda \to 0$. Construct the Feynman rules by assigning values to the propagators and vertices:



Proof:

Let us define $\mathbf{\Phi} = \left(\Phi^1, \ldots, \Phi^N\right)^T$, then $V(\Phi^2) = \frac{1}{2}m^2\mathbf{\Phi}\cdot\mathbf{\Phi} + \frac{1}{4}\lambda\left(\mathbf{\Phi}\cdot\mathbf{\Phi}\right)^2$. Replacing the mass term with a negative mass term yields the potential

$$V(\Phi^2) = -\frac{1}{2}\mu^2\mathbf{\Phi}\cdot\mathbf{\Phi} + \frac{1}{4}\lambda\left(\mathbf{\Phi}\cdot\mathbf{\Phi}\right)^2.$$

We can think about the $\Phi^1, \ldots, \Phi^N$ as N perpendicular directions in some space. We need to find the minimum of this potential. Setting the partial derivative with respect to Φ^i to zero we obtain

$$\begin{aligned} \left.\frac{\partial V}{\partial \Phi^i}\right|_{\Phi_{\min}} &= \left[-\mu^2\Phi^i + \lambda\left(\mathbf{\Phi}\cdot\mathbf{\Phi}\right)\Phi^i\right]_{\Phi_{min}} = 0 \\ \implies \left(\mathbf{\Phi}_{min}\cdot\mathbf{\Phi}_{min}\right) &= \frac{\mu^2}{\lambda} \equiv v^2 \\ \text{Or } \Phi^i_{min} &= 0. \end{aligned}$$

For $N = 2$, the potential looks like a sombrero hat and the solution $\Phi^i = 0$ is a saddle point. The minimum is displaced from the origin by a distance v and is degenerate. This degeneracy can be traced back to the fact that the potential, V, is invariant under $SO(N)$ rotations (special orthogonal group). We can thus, pick any point of the minimum to expand our fields. The book gives us the choice

$$\begin{aligned} \Phi^i(x) &= \pi^i(x), \quad i = 1, \ldots, N-1, \\ \Phi^N(x) &= v + \sigma(x) = \frac{\mu}{\sqrt{\lambda}} + \sigma(x). \end{aligned}$$

Let us define $\mathbf{\Phi} = (\boldsymbol{\pi}, \frac{\mu}{\sqrt{\lambda}} + \sigma)^T$ where $\boldsymbol{\pi} = (\pi^1, \ldots, \pi^{N-1})$. Expanding V in the new coordinates we obtain

$$
\begin{aligned}
V(\Phi^2) &= -\frac{1}{2}\mu^2 \mathbf{\Phi}\cdot\mathbf{\Phi} + \frac{1}{4}\lambda\left(\mathbf{\Phi}\cdot\mathbf{\Phi}\right)^2 \\
&= -\frac{1}{2}\mu^2\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi} + \left(\frac{\mu}{\sqrt{\lambda}} + \sigma\right)^2\right) + \frac{1}{4}\lambda\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi} + \left(\frac{\mu}{\sqrt{\lambda}} + \sigma\right)^2\right)^2 \\
&= -\frac{1}{2}\mu^2\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right) - \frac{1}{2}\mu^2\left(\frac{\mu^2}{\lambda} + 2\frac{\mu}{\sqrt{\lambda}}\sigma + \sigma^2\right) + \frac{1}{4}\lambda\left(\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right) + \left(\frac{\mu^2}{\lambda} + 2\frac{\mu}{\sqrt{\lambda}}\sigma + \sigma^2\right)\right)^2 \\
&= -\frac{1}{2}\mu^2\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right) - \frac{1}{2}\mu^2\left(\frac{\mu^2}{\lambda} + 2\frac{\mu}{\sqrt{\lambda}}\sigma + \sigma^2\right) \\
&\quad + \frac{1}{4}\lambda\left(\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)^2 + 2\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)\left(\frac{\mu^2}{\lambda} + 2\frac{\mu}{\sqrt{\lambda}}\sigma + \sigma^2\right) + \left(\frac{\mu^2}{\lambda} + 2\frac{\mu}{\sqrt{\lambda}}\sigma + \sigma^2\right)^2\right) \\
&= -\frac{1}{2}\mu^2\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right) - \frac{1}{2}\frac{\mu^4}{\lambda} - \frac{\mu^3}{\sqrt{\lambda}}\sigma + -\frac{1}{2}\mu^2\sigma^2 \\
&\quad + \frac{1}{4}\lambda\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)^2 + \frac{1}{4}\lambda\left(2\frac{\mu^2}{\lambda}\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right) + 4\frac{\mu}{\sqrt{\lambda}}\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)\sigma + 2\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)\sigma^2\right) \\
&\quad + \frac{1}{4}\lambda\left(\frac{\mu^4}{\lambda^2} + 4\frac{\mu^2}{\lambda}\sigma^2 + \sigma^4 + 4\frac{\mu^3}{\lambda^{3/2}}\sigma + 2\frac{\mu^2}{\lambda}\sigma^2 + 4\frac{\mu}{\sqrt{\lambda}}\sigma^3\right) \\
&= -\frac{1}{2}\mu^2\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right) - \frac{1}{2}\frac{\mu^4}{\lambda} - \frac{\mu^3}{\sqrt{\lambda}}\sigma + -\frac{1}{2}\mu^2\sigma^2 \\
&\quad + \frac{1}{4}\lambda\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)^2 + \frac{1}{2}\mu^2\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right) + \sqrt{\lambda}\mu\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)\sigma + \frac{1}{2}\lambda\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)\sigma^2 \\
&\quad + \frac{1}{4}\frac{\mu^4}{\lambda} + \mu^2\sigma^2 + \frac{1}{4}\lambda\sigma^4 + \frac{\mu^3}{\sqrt{\lambda}}\sigma + \frac{1}{2}\mu^2\sigma^2 + \sqrt{\lambda}\mu\sigma^3 \\
&= \cancel{-\frac{1}{2}\mu^2\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)} - \frac{1}{2}\frac{\mu^4}{\lambda} - \cancel{\frac{\mu^3}{\sqrt{\lambda}}}\sigma + -\cancel{\frac{1}{2}\mu^2}\sigma^2 \\
&\quad + \frac{1}{4}\lambda\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)^2 + \cancel{\frac{1}{2}\mu^2\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)} + \sqrt{\lambda}\mu\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)\sigma + \frac{1}{2}\lambda\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)\sigma^2 \\
&= +\frac{1}{4}\frac{\mu^4}{\lambda} + \mu^2\sigma^2 + \frac{1}{4}\lambda\sigma^4 + \cancel{\frac{\mu^3}{\sqrt{\lambda}}}\sigma + \cancel{\frac{1}{2}\mu^2}\sigma^2 + \sqrt{\lambda}\mu\sigma^3 \\
&= -\frac{1}{4}\frac{\mu^4}{\lambda} + \sqrt{\lambda}\mu\sigma^3 + \mu^2\sigma^2 + \frac{1}{4}\lambda\sigma^4 + \frac{1}{4}\lambda\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)^2 \\
&= +\sqrt{\lambda}\mu\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)\sigma + \frac{1}{2}\lambda\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)\sigma^2.
\end{aligned}
$$

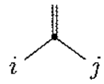
The mass term for the π^i disappears. The field σ has mass $\sqrt{2}\mu$. The massless pion fields have a quartic self-interaction. The massive σ field has both quartic and cubic self-interactions. The pion and sigma field interact through the cubic $\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)\sigma$ vertex and the quartic $\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi}\right)\sigma^2$ vertex. The strength of the every interaction depends on the coupling constant λ and becomes small in the $\lambda \to 0$ limit.

The propagators are as follows:

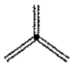
$$
\overline{\sigma(x)\sigma}(y) = D_F(x-y)|_{m=\sqrt{2}\mu} = \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{p^2 - 2\mu^2 + i\epsilon}
$$

$$
\overline{\pi^i(x)\pi^j}(y) = \delta^{ij} D_F(x-y)|_{m=0} = \delta^{ij}\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{p^2 + i\epsilon}
$$

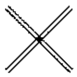
To first lowest order in λ, the vertices are as follows:



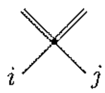
$$= -i\,(2!)\,\sqrt{\lambda}\mu\delta^{ij} = -2i\sqrt{\lambda}\mu\delta^{ij}$$



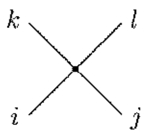
$$= -i\,(3!)\,\sqrt{\lambda}\mu = -6i\sqrt{\lambda}\mu$$



$$= -i\,(4!)\,\tfrac{1}{4}\lambda = -6i\lambda$$



$$= -i\,(2!)\,(2!)\,\tfrac{1}{2}\lambda\delta^{ij} = -2i\lambda\delta^{ij}$$



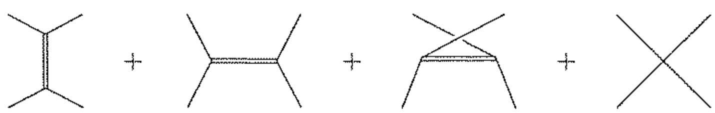
$$= -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}).$$

■

c) Compute the scattering amplitude for the process

$$\pi^i(p_1)\pi^j(p_2) \to \pi^k(p_3)\pi^l(p_4)$$

to leading order in λ. There are four Feynman diagrams that contribute:



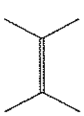
Show that, at threshold ($\mathbf{p}_i = 0$), these diagrams sum to zero. (Hint: It may be easiest to first consider the specific process $\pi^1\pi^1 \to \pi^2\pi^2$, for which only the first and fourth diagrams are nonzero, before tackling the general case.) Show that, in the special case $N = 2$ (1 species of pion), the term of $\mathcal{O}(p^2)$ also cancels.

Proof:

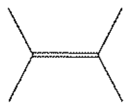
In the diagrams above let the species index be i, j, k, l starting from the top left and moving in an anti-clockwise motion:

$$\begin{matrix} i & l \\ j & k \end{matrix}.$$

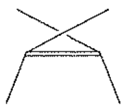
With these conventions the diagrams become:



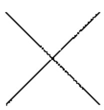
$$\begin{aligned}
\mathcal{M}_{il\to jk} &= (-2i)\,\sqrt{\lambda}\mu\delta^{il}\frac{1}{p^2 - 2\mu^2 + i\epsilon}\,(-2i)\,\sqrt{\lambda}\mu\delta^{jk} \\
&= -4\lambda\mu^2\frac{\delta^{il}\delta^{jk}}{p^2 - 2\mu^2 + i\epsilon}
\end{aligned}$$



$$\mathcal{M}_{ij\to kl} = (-2i)\sqrt{\lambda}\mu\delta^{ij}\frac{1}{p^2-2\mu^2+i\epsilon}(-2i)\sqrt{\lambda}\mu\delta^{kl}$$
$$= -4\lambda\mu^2\frac{\delta^{ij}\delta^{kl}}{p^2-2\mu^2+i\epsilon}$$



$$\mathcal{M}_{ik\to jl} = (-2i)\sqrt{\lambda}\mu\delta^{ik}\frac{1}{p^2-2\mu^2+i\epsilon}(-2i)\sqrt{\lambda}\mu\delta^{jl}$$
$$= -4\lambda\mu^2\frac{\delta^{ik}\delta^{jl}}{p^2-2\mu^2+i\epsilon}$$



$$\mathcal{M}_4 = -2i\lambda(\delta^{ij}\delta^{kl}+\delta^{il}\delta^{jk}+\delta^{ik}\delta^{jl})$$

■

d) Add to V a symmetry-breaking term.

$$\Delta V = -a\Phi^N,$$

where a is a (small) constant. (In QCD, a term of this form is produced if the u and d quarks have the same non-vanishing mass.) Find the new value of $v$that minimizes V, and work out the content of the theory about that point. Show that the pion acquires a mass such that $m_\pi^2 \sim a$, and show that the pion scattering amplitude at threshold is now non-vanishing and also proportional to a.

Proof:

■

Problem 4.4

classical potential $A_\mu(x)$

$$H_I = \int d^3x \; e\bar{\psi}\gamma^\mu\psi A_\mu$$

where ψ is the quantized fermion field

a) Show that the T-matrix element for e^- scattering off a localized potential to lowest order is

$$\langle p'|iT|p\rangle = -ie\,\bar{u}(p')\gamma^\mu u(p)\cdot\tilde{A}_\mu(p'-p)$$

where

$$\tilde{A}_\mu(p) = \mathcal{F}[A_\mu(x);\, x\to p]$$

Proof:

$$\langle p'|iT|p\rangle \equiv \left({}_0\langle p'|\,T\{\exp\{-i\int dt\, H_I(t)\}\}|p\rangle_0\right)_{\text{connected, amputated}}$$

$$\doteq \left({}_0\langle p'|T\left[-i\int d^4x\; e\bar{\psi}\gamma^\mu\psi A_\mu\right]|p\rangle_0\right)_{\text{connected, amputated}}$$

$$= -ie\int d^4x \left({}_0\langle p'|\,N[\bar{\psi}\gamma^\mu\psi A_\mu + \text{all contractions}]\,|p\rangle_0\right)_{\text{connected, amputated}}$$

$$= -ie\int d^4x\, A_\mu(x)\; {}_0\langle p'|\,\bar{\psi}(x)\gamma^\mu\psi(x)\,|p\rangle_0$$

$$= -ie\int d^4x\, A_\mu(x)\;\bar{u}(p')e^{ip'\cdot x}\gamma^\mu u(p)e^{-ip\cdot x}$$

$$= -ie\;\bar{u}(p')\gamma^\mu u(p)\int d^4x\; A_\mu(x)\,e^{ix\cdot(p'-p)}$$

$$= -ie\,\bar{u}(p')\gamma^\mu u(p)\;\tilde{A}_\mu(p'-p)$$

∎

b) if A_μ is t-independent then

$$\tilde{A}_\mu(p'-p) = 2\pi\,\delta(E'-E)\int d^3x\; A_\mu(\vec{x})\,e^{-i(\vec{p}'-\vec{p})\cdot\vec{x}}$$

$$\equiv 2\pi\,\delta(E'-E)\;\tilde{A}_\mu(\vec{p}'-\vec{p})$$

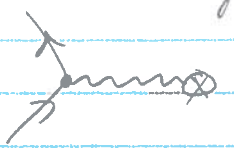
here

$\tilde{A}_\mu(\vec{q})$ is the 3D FT of A_μ

it is then natural to define

$$\langle p' | iT | p \rangle \equiv i\mathcal{M}\ \delta(E'-E)\ (2\pi)$$

and adopt the Feynman rule:



$$= -ie\gamma^\mu \tilde{A}_\mu(\vec{q})$$

Find the scattering cross-section:

Proof: $$|\psi_i\rangle_{in} = \int \frac{d^3p}{(2\pi)^3} \frac{\psi(\vec{k})\, e^{-i\vec{b}\cdot\vec{p}}}{\sqrt{2E_{\vec{p}}}} |\vec{p}\rangle_{in}$$

for the out states we use states of definite momentum and multiply by the various normalizations factors after squaring the amplitude.

$$P(b) = \frac{d^3p'}{(2\pi)^3} \frac{1}{2E'} \left|{}_{out}\langle \vec{p}' | \psi_i \rangle_{in}\right|^2$$

~~$$= \frac{d^3p'}{(2\pi)^3} \frac{1}{2E'} \left|{}_0\langle \vec{p}' | iT | \psi_i \rangle\right|^2$$~~

$$= \frac{d^3p'}{(2\pi)^3} \frac{1}{2E'} \int \frac{d^3p_1}{(2\pi)^3} \int \frac{d^3p_2}{(2\pi)^3} \frac{\psi^*(p_1)\psi(p_2)\, e^{-i\vec{b}\cdot(p_2-p_1)}}{\sqrt{2E_{p_1} 2E_{p_2}}}$$

$$\times\ {}_{out}\langle \vec{p}' | p \rangle_0^*\ {}_{out}\langle \vec{p}' | \vec{p}_2 \rangle_0$$

$${}_{out}\langle \vec{p}' | \vec{p} \rangle_0 = {}_0\langle \vec{p}' | S | \vec{p} \rangle_0$$

$$= \delta_{\vec{p}\vec{p}'} + \underbrace{{}_0\langle \vec{p}' | iT | \vec{p} \rangle_0}_{\text{only want this part}}$$

$${}_0\langle \vec{p}' | iT | \vec{p} \rangle_0 = i\mathcal{M}(\vec{p}', p)\ (2\pi)\ \delta(E'-E)$$

$$\Rightarrow P(b) = \frac{d^3p'}{(2\pi)^3} \frac{1}{2E'} \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{\psi^*(p_1)\psi(p_2)}{\sqrt{2E_{p_1} 2E_{p_2}}} e^{-i\vec{b}\cdot(\vec{p}_2-\vec{p}_1)}$$

$$\times\ \mathcal{M}^*(\vec{p}', \vec{p}_1)\, \mathcal{M}(\vec{p}', \vec{p}_2)\ (2\pi)^2$$

$$\times\ \delta(E'-E_1)\ \delta(E'-E_2).$$

to get the differential cross section we integrate the Probability over the impact parameter $\vec{b}$

$$d\sigma = \int d^2\vec{b}\ P(b)$$

$$= \frac{d^3p'}{(2\pi)^3}\frac{1}{2E'}\int\frac{d^3p_1}{(2\pi)^3}\int\frac{d^3p_2}{(2\pi)^3}\frac{\psi^*(\vec{p}_1)\psi(\vec{p}_2)}{\sqrt{2E_1\,2E_2}}\mathcal{M}^*(\vec{p}'\to\vec{p}_1)\mathcal{M}(\vec{p}'\to\vec{p}_2)$$

$$\times(2\pi)^2\,\delta(E'-E_1)\,\delta(E'-E_2)$$

$$\times\int db^2\, e^{-i\vec{b}\cdot(\vec{p}_2-\vec{p}_1)}$$

$$= \frac{d^3p'}{(2\pi)^3}\frac{1}{2E'}\int\frac{d^3p_1}{(2\pi)^3}\int\frac{d^3p_2}{(2\pi)^3}\frac{\psi^*(\vec{p}_1)\psi(\vec{p}_2)}{\sqrt{2E_1\,2E_2}}\mathcal{M}^*(p'\to p_1)\mathcal{M}(p'\to p_2)$$

$$\times(2\pi)^2\,\delta(E'-E_1)\,\delta(E'-E_2)\,(2\pi)^2\delta^{(2)}(\vec{p}_1^{\perp}-\vec{p}_2^{\perp})$$

$$(*) = \int\frac{d^3p_1}{(2\pi)^3}\frac{\psi^*(\vec{p}_1)}{\sqrt{2E_{p_1}}}\mathcal{M}^*(p_1\to p_1')\,\delta(E'-E_1)\,\delta^{(2)}(\vec{p}_1^{\perp}-\vec{p}_2^{\perp})$$

$$\int\frac{d^3p_1^z}{(2\pi)^3}\left[\frac{\psi^*(\vec{p}_1)}{\sqrt{2E_{p_1}}}\mathcal{M}^*(p_1\to p')\,\delta(E'-E_1)\right]_{p_1^x=p_2^x,\ p_1^y=p_2^y}$$

$$\delta(E'-E_1) = \delta\left(\sqrt{\vec{p}_1^{\,2}+m^2}-E'\right)$$
$$= \delta\left(\sqrt{(p_1^x)^2+(p_1^y)^2+(p_1^z)^2}-E'\right)$$

$$f(p_1^z) = \sqrt{(p_1^x)^2+(p_1^y)^2+(p_1^z)^2}-E'$$

Roots at $\quad p_1^z = \sqrt{E'^2-(p_1^x)^2-(p_1^y)^2}$

$$f'(p_1^z) = \frac{p_1^z}{\sqrt{(p_1^x)^2+(p_1^y)^2+(p_1^z)^2}}$$

$$f'(p_1^z)\big|_{Root} = \sqrt{\frac{E'^2-(p_1^x)^2-(p_1^y)^2}{(p_1^x)^2+(p_1^y)^2+E'^2-(p_1^x)^2-(p_1^y)^2}}$$

$$= \sqrt{\frac{E'^2-(p_1^x)^2-(p_1^y)^2}{E'^2}}$$

$$\circledast = \int \frac{(dp_1)^z}{(2\pi)^3} \left[\frac{\psi^*(\vec{p}_1)\, \mathcal{M}^*(p_1 \to p')}{\sqrt{2E_1}} \frac{\delta\left(p_1^z - \sqrt{E'^2 - (p_1^x)^2 - (p_1^y)^2}\right)}{\sqrt{\frac{E'^2 - (p_1^x)^2 - (p_1^y)^2}{E'^2}}} \right]_{\substack{p_1^x = p_2^x \\ p_1^y = p_2^y}}$$

$$= \frac{d^3p'}{(2\pi)^3} \frac{1}{2E'} \left(\frac{1}{2\pi}\right)^2 \int dp_1^z \int d^3p_2 \left[\frac{\psi^*(p_1)\, \psi(p_2)}{\sqrt{2E_1\, 2E_2}} \mathcal{M}^*(p_1 \to p')\, \mathcal{M}(p_2 \to p') \right]_{\substack{p_1^x = p_2^x \\ p_1^y = p_2^y}}$$

$$\times \frac{\delta\left(p_1^z - \sqrt{E'^2 - (p_2^x)^2 - (p_2^y)^2}\right)}{\sqrt{E'^2 - (p_2^x)^2 - (p_2^y)^2}\, / E'} \quad \delta(E_2 - E')$$

$$\frac{\delta\left(p_1^z - \sqrt{E_2^2 - (p_2^x)^2 - (p_2^y)^2}\right)}{\sqrt{E_2^2 - (p_2^x)^2 - (p_2^y)^2}\, / E_2} = \frac{\delta(p_1^z - p_2^z)}{p_2^z / E_2}$$

v_i

Recall that the wave packets are localized in momentum space centered about $\vec{p}$

$\Rightarrow$ we can take all smooth fn's of $\vec{p}_2$ outside of the integral and evaluate them at $\vec{p}$

$$= \frac{d^3p'}{(2\pi)^3} \frac{1}{2E'} \int \frac{d^3p_2}{(2\pi)^3} \cdot \frac{|\psi(p_2)|^2}{2E_2} \frac{|\mathcal{M}(p_2 \to p')|^2}{|v_i|} (2\pi)\, \delta(E_2 - E')$$

$$\simeq \frac{d^3p'}{(2\pi)^3} \frac{1}{2E'} \frac{1}{2E_{\vec{p}}} \frac{|\mathcal{M}(\vec{p} \to \vec{p}')|^2}{|v_i|} \int \frac{d^3p_2}{(2\pi)^3} (2\pi)\, \delta(E_2 - E')\, |\psi(\vec{p}_2)|^2$$

Real detectors cannot resolve the spread in momentum coming from the wave functions so it is okay to take the δ fn out of the integral

$$\simeq \frac{d^3p'}{(2\pi)^3} \frac{1}{2E'2E} \frac{|\mathcal{M}(\vec{p} \to \vec{p}')|^2}{|v_i|} (2\pi)\, \delta(E - E')$$

Where $\displaystyle\int \frac{d^3p_2}{(2\pi)^3} |\psi(\vec{p}_2)|^2 = 1$

The term in part (b) is

$$\rightarrow = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\bar{p}}} |j(p)|^2$$

$$= \int d^4x\, d^4y \quad j(x)\, D_F(x-y)\, j(y).$$

$$= \int d^4x\, d^4y$$ 

$$\equiv \rightarrow$$

The series may be represented as

$$P(0) = \left| \langle 0 | \, T \{ \exp [i \int d^4x \; j(x) \phi_I(x)] \} | 0 \rangle \right|^2$$

$$= \left| 1 - \rightarrow + \underset{\rightarrow}{\rightarrow} - \underset{\rightarrow}{\overset{\rightarrow}{\rightarrow}} + \dots \right|^2$$

to get the correct answer we need to get the correct symmetry factor for each diagram.

- each diagram has $2n$ vertices (n = # of $\rightarrow$)
- n must be chosen as "in" and n as "out" vertices (left and right)
- the n "in" must be paired with ~~the~~ n "out"

  $$\#\text{ of ways} = 2^{2n/2} = 2^n$$

- The "in" vertices can be interchanged in $n!$ ways
- The symmetry factor is thus

$$S = 2^n n!$$

c). Specialize to the case of e^- Coulomb scattering

$$A_0 = \frac{Ze}{4\pi r}$$

Work in the NR limit and derive the Rutherford formula

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Z^2}{4m^2 v^4 \sin^4(\theta/2)}$$

Proof:

From part (a) the matrix element is

$$\mathcal{M} = -ie\,\bar{u}(p')\gamma^\mu u(p')\,\tilde{A}_\mu(\vec{p}\,'-\vec{p})$$
$$= -ie\,\bar{u}(p')\gamma^0 \tilde{A}_0(\vec{p}\,'-\vec{p})\,u(p')$$

$\tilde{A}_0(\vec{p}) = ?$ to do the FT we need to add a convergence factor a which we will take to zero.

$$\hat{A}_0(\vec{p},a) = \int d^3x \left(\frac{Ze}{4\pi r}e^{-ar}\right) e^{-i\vec{p}\cdot\vec{r}}$$
$$= \frac{Ze}{4\pi}\int dr\, r^2\, d\cos\theta\, d\varphi\, \frac{e^{-ipr\cos\theta}}{r} e^{-ar}$$
$$= \frac{Ze}{2}\int dr\, r\, \frac{e^{-ipr\cos\theta}\Big|_{\cos\theta=-1}^{\cos\theta=1}}{-ipr} e^{-ar}$$
$$= \frac{Zei}{2p}\int_0^\infty dr \left(e^{(-ip-a)r} - e^{(ip-a)r}\right)$$
$$= \frac{Zei}{2p}\left(\frac{0-1}{-ip-a} - \frac{0-1}{ip-a}\right)$$
$$= \frac{Zei}{2p}\left(\frac{(-ip-a)-(ip-a)}{p^2+a^2}\right)$$
$$= \frac{Zei}{2p}\,\frac{-2ip}{p^2+a^2}$$
$$= \frac{Ze}{p^2+a^2}$$

$$\Rightarrow \hat{A}_0(\vec{p}) = \hat{A}_0(\vec{p},0) = \frac{Ze}{p^2}$$

in the NR limit

$$u_s(p) \simeq \begin{pmatrix} \xi_s \\ 0 \end{pmatrix} \sqrt{2m}$$

$$\bar{u}_s(p) = (\xi_s^+, 0)\, \gamma^0 \sqrt{2m}$$

$$\Rightarrow \bar{u}_{s'}(p')\, \gamma^0\, u_s(p) \to \overset{2m}{\downarrow}(\xi_{s'}^+, 0)\, \gamma^0 \gamma^0 \begin{pmatrix} \xi_s \\ 0 \end{pmatrix} = \overset{2m}{\downarrow}\xi_{s'}^+ \xi_s \overset{2m}{\downarrow}= \delta_{ss'}$$

$$\Rightarrow \mathcal{M} = -ie \frac{Ze}{(\bar{p}'-\bar{p})^2} \delta_{ss'}\, 2m$$

$$= \frac{-iZe^2}{(\bar{p}'-\bar{p})^2} \delta_{ss'} (2m) \quad \text{say } \vec{p}' \cdot \vec{p} = pp' \cos\theta$$

$$= \frac{-iZe^2}{p^2 + p'^2 - pp' \cos\theta} \delta_{ss'} (2m)$$

Let us now find $\frac{d\sigma}{d\Omega}$

$$\sigma = \int \frac{d^3p'}{(2\pi)^3} \frac{1}{2E' 2E} \frac{|\mathcal{M}|^2}{v_i} (2\pi)\, \delta(E-E')$$

$$= \int d\Omega \int \frac{dp'}{(2\pi)^3} \frac{p'^2}{2E'2E} \frac{|\mathcal{M}|^2}{v_i} (2\pi)\, \delta\left(\sqrt{p'^2+m^2} - E'\right)$$

$$f(p) = \sqrt{p^2+m^2} - E$$

$$\text{Roots } p' = \sqrt{E - m^2} = P$$

$$f'(p) = \frac{P}{\sqrt{P^2+m^2}} = \frac{P}{E} = v_f$$

$$= \int d\Omega \int \frac{dp'}{(2\pi)^3} \frac{p'^2}{2E'2E} \frac{|\mathcal{M}|^2}{v_i} (2\pi) \frac{\delta(p'-p)}{v_f}$$

$$= \int d\Omega \left(\frac{1}{2\pi}\right)^2 \frac{p^2}{4E^2} \frac{|\mathcal{M}|^2}{|v|^2}$$

$$= \int d\Omega \frac{1}{16\pi^2} |\mathcal{M}|^2$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2} |\mathcal{M}|^2_{|\vec{p}'| = |\vec{p}|}$$

$$= \frac{1}{16\pi^2} \left| \frac{-iZe^2}{(\vec{p}' - \vec{p})^2} 2m\, \delta_{ss'} \right|^2 \Bigg|_{p'=p} \qquad e^2 = 4\pi\alpha$$

$$= \frac{1}{16\pi^2} \frac{Z^2 e^4}{(\vec{p}' - \vec{p})^4} 4m^2 \delta_{ss'} \Bigg|_{p'=p}$$

$$= \frac{m^2}{4\pi^2} (4\pi\alpha)^2 \frac{Z^2}{(\vec{p}' - \vec{p})^4} \delta_{ss'} \Bigg|_{p'=p}$$

$$= 4m^2 \frac{\alpha^2 Z^2}{(2p^2 - 2p^2 \cos\theta)^2}$$

$$= 4m^2 \frac{\alpha^2 Z^2}{4p^4 (1 - \cos\theta)^2}$$

$$= \frac{m^2 \alpha^2 Z^2}{4m^4 v^4 \sin^2\theta/2} \qquad (1 - \cos\theta)^2 = 2\sin\theta/2$$

$$= \frac{\alpha^2 Z^2}{4m^2 v^4 \sin^2\theta/2}$$