5.1 Coulomb Scattering.

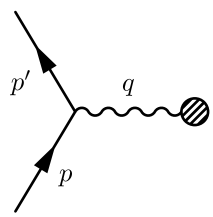
Repeat the computation of problem 4.4, part (c), this time using the full relativistic expression for the matrix element. Your should find, for the spin averaged cross section,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2|\mathbf{p}|^2\beta^2\sin^4(\theta/2)}\left(1-\beta^2\sin^2\frac{\theta}{2}\right) \tag{1}$$

where $\mathbf{p}$ is the electron's 3-momentum and β is velocity. This is the *Mott formula* for Coulomb scattering of relativistic electrons. Now derive it in a second way, by working out the cross section for electron-muon scattering, in the muon rest frame, retaining the electron mass but sending $m_\mu \to \infty$.

To answer this question we must study the single-particle cross-section and re-interpret the invariant scattering amplitude, $\mathcal{M}$ (see Appendix).

The interaction hamiltonian for a classical potential is given by $H_{int} = \int d^3x e\bar{\psi}\gamma^\mu\psi\tilde{A}_\mu$ which yields the vertex:



$$= -ie\gamma^\mu\tilde{A}_\mu(\mathbf{q}). \tag{2}$$

In the special case of Coulomb scattering we set $A^\mu = (\frac{Ze}{4\pi r}, \mathbf{0})$. The matrix element for scattering is

$$i\mathcal{M} = -ie\left(\bar{u}(p')\gamma^\mu u(p)\right)\tilde{A}_\mu(\mathbf{p'}-\mathbf{p}), \tag{3}$$

To obtain the unpolarized cross-section we calculate the spin-averaged matrix element squared

$$\begin{aligned}
\frac{1}{2}\sum_{\text{spin}}|\mathcal{M}|^2 &= \frac{1}{2}\sum_{\text{spin}}\bar{u}(p')(-ie\gamma^\mu\tilde{A}_\mu)u(p)u^\dagger(p)(ie\gamma^{\nu\dagger}\tilde{A}_\nu^\dagger)\bar{u}^\dagger(p') \\
&= \frac{e^2}{2}\sum_{\text{spin}}\mathrm{Tr}\left[u(p')\bar{u}(p')\gamma^\mu u(p)\bar{u}(p)\gamma^\nu\right]\tilde{A}_\mu\tilde{A}_\nu \\
&= \frac{e^2}{2}\mathrm{Tr}\left[(\not{p}'+m_e)\gamma^\mu(\not{p}+m_e)\gamma^\nu\right]\tilde{A}_\mu\tilde{A}_\nu \\
&= \frac{e^2}{2}\mathrm{Tr}\left[\not{p}'\gamma^\mu\not{p}\gamma^\nu + m_e^2\gamma^\mu\gamma^\nu\right]\tilde{A}_\mu\tilde{A}_\nu \\
&= 2e^2\left[-(p\cdot p')(\tilde{A}\cdot\tilde{A}) + 2(p\cdot\tilde{A})(p'\cdot\tilde{A}) + (\tilde{A}\cdot\tilde{A})m_e^2\right] \\
&= 2e^2\left[2EE' - p'\cdot p + m_e^2\right]\left(\frac{Ze}{|\mathbf{q}|^2}\right)^2 \\
&= 2e^2\left[EE' - \mathbf{p}'\cdot\mathbf{p} + m_e^2\right]\left(\frac{Ze}{|\mathbf{q}|^2}\right)^2,
\end{aligned} \tag{4}$$

where $\tilde{A}^\mu(\mathbf{q}) = \left(Ze/|\mathbf{q}|^2, \mathbf{0}\right)$, $p = (E, \mathbf{p})$, $p' = (E', \mathbf{p'})$ and the momentum transfer $q = p' - p$. Taking the $E = E'$ limit or equivalently the $|\mathbf{p}| = |\mathbf{p'}| = m\beta$ limit, the spin averaged matrix element squared becomes

$$\frac{1}{2}\sum_{\text{spin}}|\mathcal{M}|^2\Big|_{E'=E} = 2e^2\left[E^2 - |\mathbf{p}|^2\cos\theta + m_e^2\right]\left(\frac{Ze}{2|\mathbf{p}|^2(1-\cos\theta)}\right)^2 \tag{5}$$

$$= \frac{Z^2e^4}{2|\mathbf{p}|^4}\frac{E^2 - |\mathbf{p}|^2\cos\theta + m_e^2}{(1-\cos\theta)^2} \tag{6}$$

$$= \frac{4\pi^2Z^2\alpha^2m_e^2}{|\mathbf{p}|^4\sin^4(\theta/2)}\left(1+\beta^2\sin^2(\theta/2)\right). \tag{7}$$

The cross section is

$$d\sigma = \frac{d^3p'}{(2\pi)^3}\frac{|\mathcal{M}|^2}{2E'2E\beta}(2\pi)\delta(E-E') \tag{8}$$

(see problem the appendix for the derivation of this formula). Dividing by the spherical measure we obtain the differential cross section

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \int \frac{d|\mathbf{p}'|}{(2\pi)^3}|\mathbf{p}'|^2\frac{|\mathcal{M}|^2}{2E'2E\beta}(2\pi)\delta(E-E') \\ &= \frac{1}{64\pi^2}\int d|\mathbf{p}'|\;|\mathbf{p}'|^2\frac{|\mathcal{M}|^2}{E'E\beta}\frac{E}{|\mathbf{p}|}\delta(|\mathbf{p}|-|\mathbf{p}'|) \\ &= \frac{1}{64\pi^2}\frac{|\mathbf{p}|}{E\beta}\;|\mathcal{M}|^2\Big|_{|\mathbf{p}|=|\mathbf{p}'|=m\beta} \\ &= \frac{Z^2\alpha^2 m_e^2}{16E\beta|\mathbf{p}|^3\sin^4(\theta/2)}\left(1+\beta^2\sin^2(\theta/2)\right) \\ &\approx \frac{Z^2\alpha^2}{16|\mathbf{p}|^2\beta^2\sin^4(\theta/2)}\left(1+\beta^2\sin^2(\theta/2)\right)\end{aligned} \tag{9}$$

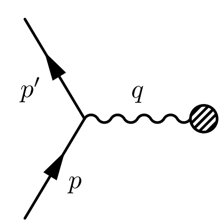
This is annoyingly off by the large numerical factor of 8... I will not look for my factors of 2, but if someone sees the mistake please let me know.

Appendix: Derivation of the cross-section for single-particle scattering off a time-independent classical potential

The interaction hamiltonian for a classical electromagnetic potential is given by $H_{int} = \int d^3xe\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)$. The T matrix element for an electron scattering off a time-independent electromagnetic potential is

$$\begin{aligned}\langle p'|iT|p\rangle &= \left({}_0\langle p'|T\left[\exp\left(-i\int dtH_{int}(t)\right)\right]|p\rangle_0\right)_{\text{connected, amputated}} \\ &\approx \left({}_0\langle p'|T\left[-i\int d^4xe\bar{\psi}\gamma^\mu\psi A_\mu\right]|p\rangle_0\right)_{\text{connected, amputated}} \\ &= -ie\int d^4xA_\mu(x){}_0\langle \contraction{}{p'}{|}{\bar{\psi}}p'|\bar{\psi}\gamma^\mu \contraction{}{\psi}{|}{p}\psi|p\rangle_0 \\ &= -ie\int d^4xA_\mu(x)\bar{u}(p')\gamma^\mu e^{i(p'-p)\cdot x}u(p) \\ &= -ie\bar{u}(p')\gamma^\mu u(p)\int dte^{i(E'-E)t}\int d^3xA_\mu(\mathbf{x})e^{-i(p'-p)\cdot\mathbf{x}} \\ &= -ie\bar{u}(p')\gamma^\mu u(p)\tilde{A}_\mu(\mathbf{p}'-\mathbf{p})(2\pi)\delta(E'-E) \\ &= i\mathcal{M}(2\pi)\delta(E'-E)\end{aligned} \tag{10}$$

It is useful to recall that *amputated connected* diagrams correspond to diagrams in which all field operators, and, *in* and *out* states are fully contracted. It is now natural to define the Feynman rule



$$= -ie\gamma^\mu\tilde{A}_\mu(\mathbf{q}). \tag{11}$$

for calculating the invariant scattering amplitude $\mathcal{M}$.

Next we must find the cross section for *single particle* scattering off a classical potential. First we must set up wave packets representing the initial-state particles, evolve this state for a very long time using the time-evolution operator $\exp(-iHt)$ of the full interacting field theory and overlap the resulting state with the desired final state wave packets. This procedure yields the probability amplitude for producing the desired final state from the initial state and is simply related to the cross-section. To get results that do not depend on the shapes of the initial and final wave packets we assume that these wave packets are peaked sharply about a single momentum.

A general wave packet may be represented as

$$|\psi\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \psi(\mathbf{k})|\mathbf{k}\rangle, \tag{12}$$

where $\psi(\mathbf{k})$ is the Fourier transform of $\psi(\mathbf{x})$ and $|\mathbf{k}\rangle$ is a one-particle state of momentum $\mathbf{k}$ in the interacting theory. In the free theory $|\mathbf{k}\rangle = \sqrt{2E_{\mathbf{k}}}a^\dagger_{\mathbf{k}}|0\rangle$. The factor of $\sqrt{2E_{\mathbf{k}}}$ converts the relativistic normalization of $|\mathbf{k}\rangle$ to the conventional normalization in which the sum of all probabilities adds up to 1:

$$\langle\psi|\psi\rangle = 1 \quad \text{if} \quad \int \frac{d^3k}{(2\pi)^3} |\psi(\mathbf{k})|^2 = 1. \tag{13}$$

The probability we wish to compute is

$$\mathcal{P} = |\langle\psi_f|\psi_i\rangle|^2, \tag{14}$$

where $|\psi_{f/i}\rangle$ are single particle wavepackets constructed in the far future and far past. *Note.* Wavepackets are localized in space and thus can be constructed independently of others.

We set up $|\psi_i\rangle$ in the remote past and then take the limit in which the wavepacket $\psi_i(\mathbf{k}_i)$ becomes concentrated about definite momenta $\mathbf{p}_i$; this defines the *in* state $|\mathbf{p}_i\rangle_{\text{in}}$ with definite initial momentum. We choose to view $\psi_i\rangle$ as a linear superposition of such states.

As in two particle scattering, we must account for the transverse displacement of the initial wavepacket relative to the origin. An incoming wavepacket with momentum $\mathbf{p}_i$ in the z-direction with impact parameter $\mathbf{b}$ can be represented by

$$|\psi_i\rangle_{\text{in}} = \int \frac{d^3p_i}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}_i}}} \psi(\mathbf{p}_i)e^{-i\mathbf{p}_i\cdot\mathbf{b}}|\mathbf{p}_i\rangle_{\text{in}}. \tag{15}$$

Note. We have extracted the explicit dependence on the impact parameter from the wavepacket $\psi(\mathbf{p}_i)$.

Similarly, we expand $\langle\psi_f|$ in terms of *out* states of definite momentum formed in the asymptotic future

$$_{\text{out}}\langle\psi_f| = \int \frac{d^3p_f}{(2\pi)^3} \frac{\psi(\mathbf{p}_f)}{\sqrt{E_{\mathbf{p}_f}}} \, _{\text{out}}\langle\mathbf{p}_f|. \tag{16}$$

However, following *Peskin* it is easier to use the *out* states of definite momentum as the final states in the probability amplitude and multiply by the various normalization factors after squaring the amplitude. As long as the detectors of final state particles mainly measure momentum and do not resolve positions at the level of de Broglie wavelengths, this is physically reasonable.

Now we can relate the probability of scattering in a real experiment to an idealized *set* of transition amplitudes between asymptotically defined *in* and *out* states of definite momentum $_{\text{out}}\langle\mathbf{p}_f|\mathbf{p}_i\rangle_{\text{in}}$. The conventions for defining the *in* and *out* states are related by time translation

$$_{\text{out}}\langle\mathbf{p}_f|\mathbf{p}_i\rangle_{\text{in}} = \lim_{T\to\infty}\langle\underbrace{\mathbf{p}_f}_{T}|\underbrace{\mathbf{p}_i}_{-T}\rangle = \lim_{T\to\infty}\langle\mathbf{p}_f|e^{-iH(2T)}|\mathbf{p}_i\rangle = \langle\mathbf{p}_f|S|\mathbf{p}_i\rangle. \tag{17}$$

Recall that S contains both the kinematic and dynamical physics $S = 1 + iT$. The invariant matrix element is related to the expectation value of the T-matrix by

$$\langle p'|iT|p\rangle = i\mathcal{M}(2\pi)\delta(E' - E) \tag{18}$$

and contains the dynamical information.

We now must calculate how $\mathcal{M}$ relates to the cross section σ. To do this we calculate the probability for the initial state $|\mathbf{p}_i\rangle$ to scatter and become a single particle final state whose momentum lies in the small region d^3p_f in terms of the impact parameter $\mathbf{b}$. In the *Peskin* normalization this probability is

$$\mathcal{P}(i \to f; \mathbf{b}) = \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}|_{\text{out}}\langle \mathbf{p}_f|\psi_i\rangle_{\text{in}}|^2 \tag{19}$$

The cross section is simply related to the above probability

$$\sigma = \int d^2b\mathcal{P}(i \to f; \mathbf{b}). \tag{20}$$

Writing $d\sigma$ rather than σ we have

$$d\sigma = \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\int d^2b\int\frac{d^3p_i}{(2\pi)^3}\frac{\psi_i(\mathbf{p}_i)}{\sqrt{2E_{\mathbf{p}_i}}}\int\frac{d^3\bar{p}_i}{(2\pi)^3}\frac{\psi_i^*(\bar{\mathbf{p}}_i)}{\sqrt{2E_{\bar{\mathbf{p}}_i}}}e^{-i\mathbf{b}\cdot(\mathbf{p}_i-\bar{\mathbf{p}}_i)}\left({}_{\text{out}}\langle\mathbf{p}_f|\mathbf{p}_i\rangle_{\text{in}}\right)\left({}_{\text{out}}\langle\mathbf{p}_f|\bar{\mathbf{p}}_i\rangle_{\text{in}}\right)^*. \tag{21}$$

The integral over the impact parameter yields the delta function $(2\pi)^2\delta^{(2)}\left(\mathbf{p}_i^\perp - \bar{\mathbf{p}}_i^\perp\right)$. Assuming we are not interested in the trial case of forward scattering where no interaction takes place, we can drop the 1 in S and write

$$\begin{aligned}\left({}_{\text{out}}\langle\mathbf{p}_f|\mathbf{p}_i\rangle_{\text{in}}\right) &= i\mathcal{M}(i\to f)(2\pi)\delta(E_i - E_f)\\ \left({}_{\text{out}}\langle\mathbf{p}_f|\bar{\mathbf{p}}_i\rangle_{\text{in}}\right)^* &= -i\mathcal{M}^*(i\to f)(2\pi)\delta(E_i-E_f).\end{aligned} \tag{22}$$

With the above considerations, the differential cross section becomes

$$\begin{aligned}d\sigma &= \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\int\frac{d^3p_i}{(2\pi)^3}\frac{\psi_i(\mathbf{p}_i)}{\sqrt{2E_{\mathbf{p}_i}}}\int\frac{d^3\bar{p}_i}{(2\pi)^3}\frac{\psi_i^*(\bar{\mathbf{p}}_i)}{\sqrt{2E_{\bar{\mathbf{p}}_i}}}\mathcal{M}(\mathbf{p}_i\to\mathbf{p}_f)\mathcal{M}^*(\bar{\mathbf{p}}_i\to\mathbf{p}_f)\\ &\quad(2\pi)^2\delta^{(2)}\left(\mathbf{p}_i^\perp-\bar{\mathbf{p}}_i^\perp\right)(2\pi)\delta(E_i-E_f)(2\pi)\delta(\bar{E}_i-E_f).\end{aligned} \tag{23}$$

Using these delta functions we can preform all three integrals over $\bar{\mathbf{p}}_i$

$$\begin{aligned}d\sigma &= 2\pi\frac{d^3p_f}{2E_f}\int\frac{d^3p_i}{(2\pi)^3}\frac{\psi_i(\mathbf{p}_i)}{\sqrt{2E_{\mathbf{p}_i}}}\delta(E_i-E_f)\int d^3\bar{p}_i\frac{\psi_i^*(\bar{\mathbf{p}}_i)}{\sqrt{2E_{\bar{\mathbf{p}}_i}}}\mathcal{M}(\mathbf{p}_i\to\mathbf{p}_f)\mathcal{M}^*(\bar{\mathbf{p}}_i\to\mathbf{p}_f)\\ &\quad\delta^{(2)}\left(\mathbf{p}_i^\perp-\bar{\mathbf{p}}_i^\perp\right)\delta(\bar{E}_i-E_f).\end{aligned} \tag{24}$$

The perpendicular competent of $\bar{\mathbf{p}}_i$ is fixed by the delta function $\delta^{(2)}\left(\mathbf{p}_i^\perp-\bar{\mathbf{p}}_i^\perp\right)$. The only non-trivial integral is the parallel comment of $\bar{\mathbf{p}}_i$

$$\int dp_i^{\parallel}\left[\frac{\psi_i^*(\bar{\mathbf{p}}_i)}{\sqrt{2E_{\mathbf{p}_i}2E_{\bar{\mathbf{p}}_i}}}\mathcal{M}(\mathbf{p}_i\to\mathbf{p}_f)\mathcal{M}^*(\bar{\mathbf{p}}_i\to\mathbf{p}_f)\right]_{\bar{\mathbf{p}}_i^\perp=\mathbf{p}_i^\perp}\delta(\bar{E}_i-E_f) \tag{25}$$

$$= \int dp_i^{\parallel}\left[\frac{\psi_i^*(\bar{\mathbf{p}}_i)}{\sqrt{2E_{\mathbf{p}_i}2E_{\bar{\mathbf{p}}_i}}}\mathcal{M}(\mathbf{p}_i\to\mathbf{p}_f)\mathcal{M}^*(\bar{\mathbf{p}}_i\to\mathbf{p}_f)\right]_{\bar{\mathbf{p}}_i^\perp=\mathbf{p}_i^\perp}\delta(\bar{p}_i^{\parallel}-p_i^{\parallel})\frac{1}{p_i^{\parallel}/E_{\mathbf{p}_i}} \tag{26}$$

$$= \frac{\psi_i^*(\mathbf{p}_i)}{2E_{\mathbf{p}_i}v_i}|\mathcal{M}(\mathbf{p}_i\to\mathbf{p}_f)|^2 \tag{27}$$

With this the cross-section becomes

$$\begin{aligned}d\sigma &= 2\pi\frac{d^3p_f}{2E_f}\int\frac{d^3p_i}{(2\pi)^3}\delta(E_i-E_f)\frac{\psi_i(\mathbf{p}_i)\psi_i^*(\mathbf{p}_i)}{2E_{\mathbf{p}_i}v_i}|\mathcal{M}(i\to f)|^2\\ &\approx 2\pi\frac{d^3p_f}{2E_f}\int\frac{d^3p_i}{(2\pi)^3}\delta(E_i-E_f)\frac{\psi_i(\mathbf{p}_i)\psi_i^*(\mathbf{p}_i)}{2E_{\mathbf{p}_i}v_i}|\mathcal{M}(i\to f)|^2.\end{aligned} \tag{28}$$

Recall that the wavepacket is localized in momentum space about a momentum $\mathbf{p}$. This implies that we can take all smooth functions of $\mathbf{p}_i$ outside of the integral evaluated at momentum $\mathbf{p}$

$$\begin{aligned} d\sigma &= \frac{d^3p_f}{2E_{\mathbf{p}_f}2E_{\mathbf{p}}v}2\pi\delta(E_{\mathbf{p}}-E_{\mathbf{p}_f})|\mathcal{M}(\mathbf{p}\to\mathbf{p}_f)|^2\int\frac{d^3}{(2\pi)^3}\psi_i(\mathbf{p}_i)\psi_i^*(\mathbf{p}_i) \\ &= \frac{d^3p_f}{2E_{\mathbf{p}_f}2E_{\mathbf{p}}v}2\pi\delta(E_{\mathbf{p}}-E_{\mathbf{p}_f})|\mathcal{M}(\mathbf{p}\to\mathbf{p}_f)|^2 \end{aligned} \tag{29}$$

This is our final expression for the differential cross-section for the scattering off a time-independent classical potential.