Problem 7.1

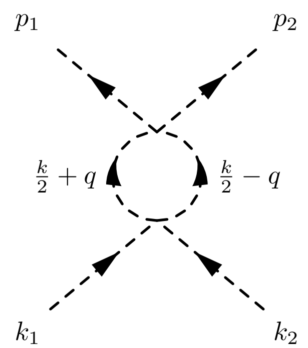
In Section 7.3 we used an indirect method to analyze the one-loop s-channel diagram for boson-boson scattering in ϕ^4 theory. To verify our indirect analysis, evaluate all three one-loop diagrams, using the standard method of Feynman parameters. Check the validity of the optical theorem.

The total scattering amplitude is given by

$$i\mathcal{M} = i\mathcal{M}_0 + i\mathcal{M}_1(s) + i\mathcal{M}_1(u) + i\mathcal{M}_1(t) + i\mathcal{M}_2 + \dots \tag{1}$$

where

$i\mathcal{M}_1(s) =$



$k = k_1 + k_2$ (2)

and the t- and u-channel diagrams can be obtained by suitable replacements of $s = (k_1 + k_2)^2$.

$$\begin{aligned} i\mathcal{M}_1(s) &= \int \frac{d^d q}{(2\pi)^d} (-i\lambda\mu^{\frac{4-d}{2}})^2 \frac{i}{(k/2-q)^2 - m_\phi^2 + i\epsilon} \frac{i}{(k/2+q)^2 - m_\phi^2 + i\epsilon} \\ &= \lambda^2 \mu^{4-d} \int \frac{d^d q}{(2\pi)^d} \int_0^1 dx \, \frac{1}{\left[k^2/4 + (2x-1)k\cdot q + q^2 - m_\phi^2 + i\epsilon\right]^2} \\ &= \lambda^2 \mu^{4-d} \int \frac{d^d \ell}{(2\pi)^d} \int_0^1 dx \, \frac{1}{\left[\ell^2 - \Delta + i\epsilon\right]^2} \end{aligned} \tag{3}$$

where $\ell = q + (2x-1)k/2$ and $\Delta = -x(1-x)k^2 + m_\phi^2$. Preforming the momentum integration we obtain

$$\begin{aligned} i\mathcal{M}_1(s) &= \lambda^2 \int_0^1 dx \, \frac{i}{16\pi^2} \left(\frac{4\pi\mu^2}{\Delta}\right)^\epsilon \Gamma(\epsilon) \\ &= \frac{\lambda^2 i}{16\pi^2} \int_0^1 dx \, \left(\frac{1}{\epsilon} + \log\left(\frac{\tilde{\mu}^2}{\Delta}\right)\right). \end{aligned} \tag{4}$$

Recalling that $s = k^2 = (k_1 + k_2)^2$ we have

$$i\mathcal{M}_1(s) = \frac{\lambda^2 i}{16\pi^2} \int_0^1 dx \, \left(\frac{1}{\epsilon} + \log\left(\frac{\tilde{\mu}^2}{m_\phi^2 - x(1-x)s}\right)\right). \tag{5}$$

Using Mathematica to preform the x integral and dividing by i we obtain

$$\mathcal{M}_1(s) = \frac{\lambda^2}{16\pi^2} \left(\frac{1}{\epsilon} + 2 - 2\sqrt{\frac{4m_\phi^2}{s} - 1} \, \text{ArcSin}\left(\frac{\sqrt{s}}{2m_\phi}\right) + \log\left(\frac{\tilde{\mu}^2}{m_\phi^2}\right)\right). \tag{6}$$

For $s < 4m_\phi^2$ the square roots are real and $\mathcal{M}_1(s)$ is real. However, for $s > 4m_\phi^2$ the square roots develop branch cuts $\implies$ that $\mathcal{M}_1(s)$ has a branch cut from threshold to infinity, $s \in \{4m_\phi^2, \infty\}$.

To find the imaginary part it is easiest to use the identity $\mathrm{Im}\log(-x \pm i\epsilon) = \pm\pi$ (for $x > 0$) in equation (5) and preform then x integration. $\mathcal{M}_1$ acquires an imaginary part when the argument of the logarithm becomes negative (branch cut from 0 to infinity). For any s this happens for $\frac{1}{2} - \frac{1}{2}\beta < x < \frac{1}{2} + \frac{1}{2}\beta$ where $\beta = \sqrt{1 - 4m_\phi^2/s}$. With these remarks we have

$$\begin{aligned}
\mathrm{Im}\mathcal{M}_1(s \pm i\epsilon) &= \frac{-\lambda^2}{16\pi^2}\int_0^1 dx\ \log\left(m_\phi^2 - x(1-x)s \pm i\epsilon\right) \\
&= \frac{-\lambda^2}{16\pi^2}\int_{\frac{1}{2}-\frac{1}{2}\beta}^{\frac{1}{2}+\frac{1}{2}\beta} dx\ (\pm\pi) \\
&= \frac{\mp\lambda^2}{16\pi}\int_{-\frac{1}{2}\beta}^{\frac{1}{2}\beta} dy \\
&= \frac{\mp\lambda^2}{16\pi}\sqrt{1 - \frac{4m_\phi^2}{s}}.
\end{aligned} \tag{7}$$

Working in the centre of mass frame $s > 0, t = 0$ and $u < 0$. This implies that only the s-channel diagrams contribute to the imaginary part of $\mathcal{M}_1$ (i.e., $\mathrm{Im}\mathcal{M}_1 = \mathrm{Im}\mathcal{M}_1(s)$).

$$\begin{aligned}
\mathcal{M}_1(t) &= \frac{\lambda^2}{16\pi^2}\left(\frac{1}{\epsilon} + 2 + \log\left(\frac{\tilde{\mu}^2}{m_\phi^2}\right)\right), \\
\mathcal{M}_1(u) &= \frac{\lambda^2}{16\pi^2}\left(\frac{1}{\epsilon} + 2 + 2\sqrt{\frac{4m_\phi^2}{|u|} + 1}\ \mathrm{ArcSinh}\left(\frac{\sqrt{|u|}}{2m_\phi}\right) + \log\left(\frac{\tilde{\mu}^2}{m_\phi^2}\right)\right).
\end{aligned} \tag{8}$$

To validate the optical theorem we need to relate the imaginary part to the amplitude of the tree-level scattering squared. The tree-level scattering amplitude squared, $|\mathcal{M}_0|^2$, is just λ^2.

Problem 7.2: Alternative regulators in QED

In Section 7.5, we saw that the Ward identity can be violated by an improperly chosen regulator. Let us check the validity of the identity $Z_1 = Z_2$, to order α, for several choices of the regulator. We have already verified that the relation holds for Pauli-Villars regularization.

(a) Recompute δZ_1 and δZ_2, defining the integrals (6.49) and (6.50) by simply placing an upper limit Λ on the integration over ℓ_E. Show that, with this definition, $\delta Z_1 \neq \delta Z_2$.

(b) Recompute δZ_1 and δZ_2, defining the integrals (6.49) and (6.50) by dimensional regularization. You may take the Dirac matrices to be 4 x 4 as usual, but note that, in d-dimensions,

$$g^{\mu\nu}\gamma_\mu\gamma_\nu = d.$$

Show that, with this definition, $\delta Z_1 = \delta Z_2$.

Preamble to parts (a) and (b)

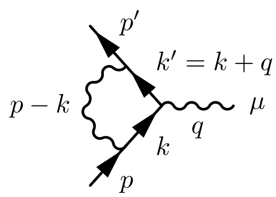
We will take the dimension to be d so that we may use our formulas for both parts (a) and (b).

We will need to know the integral,

$$\int \frac{d^d\ell}{(2\pi)^4}\frac{\ell^{2n}}{(\ell^2-\Delta+i\epsilon)^m} = \frac{(-1)^{m+n}i}{(2\pi)^4}\Omega_d\int_0^\infty d\ell_E\frac{\ell_E^{d-1+2n}}{(\ell_E^2+\Delta)^m} = \frac{(-1)^{m+n}i}{(2\pi)^4}\Omega_d I_m^n, \tag{9}$$

in order to evaluate δZ_1 and δZ_2. In the text the integral, I_m, is given by equations (6.49) and (6.50), and, regulated by Pauli-V regularization. We are asked to evaluate these integrals by (a) placing an upper bound Λ on the momentum ℓ_E and (b) using dimensional regularization.

The renormalization factor Z_1 is defined by the relation $\Gamma^\mu(p+k,p)|_{k\to 0} = Z_1^{-1}\gamma^\mu$ where Γ^μ is the electron vertex. The one-loop correction to the electron vertex is given by



$$= \bar{u}(p')\delta\Gamma^\mu(p',p)u(p). \tag{10}$$

From Peskin and Schroder we know that this correction is given by equation (6.38)

$$\begin{aligned}\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) &= \int\frac{d^dk}{(2\pi)^4}\frac{-ig_{\nu\rho}}{(k-p)^2+i\epsilon}\bar{u}(p')(-ie\gamma^\nu)\frac{i(\not{k}'+m)}{k'^2-m^2+i\epsilon}\gamma^\mu\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}(-ie\gamma^\rho)u(p)\\ &= 2ie^2\int\frac{d^dk}{(2\pi)^4}\frac{\bar{u}(p')Nu(p)}{((k-p)^2-\mu^2+i\epsilon)(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)},\end{aligned} \tag{11}$$

where $N = -\frac{1}{2}\gamma^\nu[\not{k}\gamma^\mu\not{k}+m^2\gamma^\mu+m(\not{k}'\gamma^\mu+\gamma^\mu\not{k})]\gamma_\nu$. Evaluating the above correction at $p'=p \implies k'=k$ will yield the correction to the renormalization constant Z_1

$$\bar{u}(p)\delta\Gamma^\mu(p,p)u(p) = 2ie^2\int\frac{d^dk}{(2\pi)^4}\frac{\bar{u}(p)Nu(p)}{((k-p)^2-\mu^2+i\epsilon)(k^2-m^2+i\epsilon)^2}, \tag{12}$$

where $N = (1 - \frac{4-d}{2})\not{k}\gamma^\mu\not{k} + \frac{d-2}{2}m^2\gamma^\mu - dmk^\mu$. Combining the propagators via use of Feynman parameters and simplifying the numerator using the Dirac equation, we obtain

$$\begin{aligned}
\bar{u}(p)\delta\Gamma^\mu(p,p)u(p) &= 2ie^2\int dx\int dy\; y\delta(x+y-1)\int\frac{d^dk}{(2\pi)^4}\frac{\bar{u}(p)[(1-\frac{4-d}{2})\not{k}\gamma^\mu\not{k}+\frac{d-2}{2}m^2\gamma^\mu - dmk^\mu]u(p)}{(k^2-2xk\cdot p+xp^2-x\mu^2-ym^2+i\epsilon)^3}\\
&= 2ie^2\int dx\int dy\; y\delta(x+y-1)\\
&\quad\times\int\frac{d^d\ell}{(2\pi)^4}\frac{\bar{u}(p)[(1-\frac{4-d}{2})(\not{\ell}+x\not{p})\gamma^\mu(\not{\ell}+x\not{p})+\frac{d-2}{2}m^2\gamma^\mu - dm(\ell+xp)^\mu]u(p)}{(\ell^2-\Delta+i\epsilon)^3}\\
&\to 2ie^2\int dx\int dy\; y\delta(x+y-1)\\
&\quad\times\int\frac{d^d\ell}{(2\pi)^4}\frac{(d-2)^2}{2d}\frac{\bar{u}(p)[(1-\frac{4-d}{2})(-\frac{d-2}{d}\ell^2\gamma^\mu + x^2\not{p}\gamma^\mu\not{p})+\frac{d-2}{2}m^2\gamma^\mu - dmxp^\mu]u(p)}{(\ell^2-\Delta+i\epsilon)^3}\\
&= 2ie^2\int dx\int dy\; y\delta(x+y-1)\\
&\quad\times\int\frac{d^d\ell}{(2\pi)^4}\frac{\bar{u}(p)\left[\left(-\frac{d-2}{d}(1-\frac{4-d}{2})\ell^2+((1-\frac{4-d}{2})x^2+\frac{d-2}{2})m^2\right)\gamma^\mu - dmxp^\mu\right]u(p)}{(\ell^2-\Delta+i\epsilon)^3}\\
&= 2ie^2\int dx\int dy\; y\delta(x+y-1)\\
&\quad\times\int\frac{d^d\ell}{(2\pi)^4}\frac{\bar{u}(p)\left[\left(-\frac{(d-2)^2}{2d}\ell^2+(\frac{d-2}{2}(1+x^2)-dx)m^2\right)\gamma^\mu\right]u(p)}{(\ell^2-\Delta+i\epsilon)^3}.
\end{aligned}\tag{13}$$

where $\Delta = -x(1-x)p^2 + x\mu^2 + ym^2 = -x(1-x)p^2 + x\mu^2 + (1-x)m^2$. In the last line of equation (5) we have used the Gordon identity. Now,

$$\Gamma^\mu(p,p) = \gamma^\mu + \delta\Gamma^\mu(p,p) \equiv \frac{\gamma^\mu}{1+\delta Z_1} \implies \delta Z_1 = -\delta\Gamma^\mu(p,p) \tag{14}$$

Thus, the one-loop correction to the renormalization constant Z_1 is

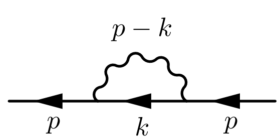
$$\delta Z_1 = -2ie^2\int dx\;(1-x)\int\frac{d^d\ell}{(2\pi)^4}\frac{-\frac{(d-2)^2}{2d}\ell^2+((\frac{d-2}{2}(1+x^2)-dx)m^2}{(\ell^2-\Delta+i\epsilon)^3} \tag{15}$$

Note that up to this part we have not imposed any regulators on the integral and worked in d-dimensions. This is the starting point for the evaluation of δZ_1 for both parts (a) and (b).

Now, let us set up the calculation of the correction to Z_2. This renormalization constant is defined as

$$\frac{1}{Z_2} \equiv 1 - \frac{d\Sigma}{d\not{p}}|_{\not{p}\to 0}. \tag{16}$$

The first order contribution to the electron self-energy is given by the diagram



$$= \frac{i(\not{p}-m_0)}{p^2-m_0^2}[-i\Sigma_2(p)]\frac{i(\not{p}-m_0)}{p^2-m_0^2}. \tag{17}$$

where

$$-i\Sigma_2(p) = (-ie)^2 \int \frac{d^dk}{(2\pi)^d}\gamma^\mu \frac{i(\not{k}+m_0)}{k^2-m_0^2+i\epsilon}\gamma_\mu \frac{-i}{(p-k)^2-\mu^2+i\epsilon}$$
$$= -e^2\int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d}\frac{-(d-2)x\not{p}+dm_0}{(\ell^2-\Delta+i\epsilon)^2} \tag{18}$$

where $\ell = k - xp$ and $\Delta = -x(1-x)p^2 + x\mu^2 + (1-x)m_0^2$. To find δZ_2 we must evaluate Σ_2 by regulating the integral, differentiate wrt $\not{p}$ and take the limit $\not{p} \to m$

$$\delta Z_2 = (Z_2 - 1) = \frac{d\Sigma_2}{d\not{p}}|_{\not{p}\to m}. \tag{19}$$

Part (a)

First we set $d = 4$ then regulate the integral of equation (1) by placing an upper bound Λ on the momentum, ℓ_E:

$$I_m^n \to \int_0^\Lambda d\ell_E \frac{\ell_E^{3+2n}}{(\ell_E^2+\Delta)^m} = \int_0^{\Lambda^2} \frac{1}{2}d(\ell_E^2)\frac{(\ell_E^2)^{n+1}}{(\ell_E^2+\Delta)^m}. \tag{20}$$

In particular, we will need:

$$I_3^0 = \int_0^\Lambda \frac{1}{2}d(\ell_E^2)\frac{(\ell_E^2)}{(\ell_E^2+\Delta)^3}$$
$$= \frac{1}{4}\frac{\Lambda^4}{\Delta(\Delta+\Lambda^2)^2}$$
$$= \frac{1}{2\Delta}\left(\frac{1}{2} + \mathcal{O}\left(\frac{\Delta}{\Lambda^2}\right)\right), \tag{21}$$

and

$$I_3^1 = \int_0^{\Lambda^2} \frac{d(\ell_E^2)}{2}\frac{(\ell_E^2)^2}{(\ell_E^2+\Delta)^3}$$
$$= \frac{1}{2}\left(\log\left(\frac{\Delta+\Lambda^2}{\Delta}\right) + \frac{3\Delta^2+4\Delta\Lambda^2}{2(D+\Lambda^2)^2} - \frac{3}{2}\right)$$
$$= \frac{1}{2}\left(\log\left(\frac{\Lambda^2}{\Delta}\right) - \frac{3}{2} + \mathcal{O}\left(\frac{\Delta}{\Lambda^2}\right)\right) \tag{22}$$

for the evaluation of δZ_1, and,

$$I_2^0 = \int_0^{\Lambda^2} \frac{1}{2}d(\ell_E^2)\frac{\ell_E^2}{(\ell_E^2+\Delta)^2}$$
$$= \frac{1}{2}\log\left(\frac{\Delta+\Lambda^2}{\Delta}\right) - \frac{\Lambda^2}{2\Delta(\Delta+\Lambda^2)^2}$$
$$= \frac{1}{2}\left(\log\left(\frac{\Lambda^2}{\Delta}\right) - 1 + \mathcal{O}\left(\frac{\Delta}{\Lambda^2}\right)\right), \tag{23}$$

for the evaluation of δZ_2.

Substituting equation (1) (with equations (11) and (12)) into (6) with $d = 4$ we obtain δZ_1

$$\begin{aligned}
\delta Z_1 &= -2ie^2 \int dx\,(1-x) \int \frac{d^4\ell}{(2\pi)^4} \frac{-\frac{1}{2}\ell^2 + (1-4x+x^2)m^2}{(\ell^2-\Delta+i\epsilon)^3} \\
&= -2ie^2 \frac{(-1)^3 i}{(2\pi)^4}\Omega_4 \int dx \int dy\; y\delta(x+y-1)\left(\frac{1}{2}I_3^1 + (1-4x+x^2)m^2 I_3^0\right) \\
&= -\frac{\alpha}{\pi}\int dx\,(1-x)\left(\frac{1}{4}\left(\log\left(\frac{\Lambda^2}{\Delta}\right) - \frac{3}{2}\right) + (1-4x+x^2)m^2\frac{1}{4\Delta} + \mathcal{O}\left(\frac{\Delta}{\Lambda^2}\right)\right) \\
&\approx -\frac{\alpha}{4\pi}\int dx\,(1-x)\left(\log\left(\frac{\Lambda^2}{\Delta}\right) - \frac{3}{2} + (1-4x+x^2)\frac{m^2}{\Delta}\right) \\
&= -\frac{\alpha}{4\pi}\int dx\,(1-x)\left(\log\left(\frac{\Lambda^2}{\Delta}\right) - \frac{3}{2} + (1-4x+x^2)\frac{m^2}{\Delta}\right). \qquad (24)
\end{aligned}$$

Setting $p^2 = m^2$ we have

$$\begin{aligned}
\delta Z_1 &= -\frac{\alpha}{4\pi}\int dx\,(1-x)\left(\log\left(\frac{\Lambda^2}{(1-x)^2m^2+x\mu^2}\right) - \frac{3}{2} + \frac{(1-4x+x^2)m^2}{(1-x)^2m^2+x\mu^2}\right) \\
&= -\frac{\alpha}{4\pi}\left(\frac{1}{2}\left(1+\log\left(\frac{\Lambda^2}{m^2}\right)\right) - \frac{3}{4} + \frac{5}{2} - 2\log\left(\frac{m^2}{\mu^2}\right)\right) \\
&= -\frac{\alpha}{4\pi}\left(\frac{5}{4} + \log\left(\frac{\Lambda^2}{m^2}\right) - 2\log\left(\frac{m^2}{\mu^2}\right)\right) \qquad (25)
\end{aligned}$$

where

$$\int_0^1 dx\,(1-x)\log\left(\frac{\Lambda^2}{(1-x)^2m^2+x\mu^2}\right) = \frac{1}{2}\left(1+\log\left(\frac{\Lambda^2}{m^2}\right)\right) \qquad (26)$$

$$\int_0^1 dx\,\frac{(1-x)(1-4x+x^2)m^2}{(1-x)^2m^2+x\mu^2} = \frac{5}{2} - 2\log\left(\frac{m^2}{\mu^2}\right). \qquad (27)$$

$$\begin{aligned}
\Sigma_2(p) &= -ie^2\int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\frac{-2x\not{p}+4m_0}{(\ell^2-\Delta+i\epsilon)^2} \\
&= -ie^2\int_0^1 dx\,(-2x\not{p}+4m_0)\,\frac{(-1)^2 i}{(2\pi)^4}\Omega_4 I_2^0 \\
&= \frac{-e^2 i}{(2\pi)^4}\Omega_4\int_0^1 dx\,(-2x\not{p}+4m_0)\left(\frac{1}{2}\log\left(\frac{\Delta+\Lambda^2}{\Delta}\right) - \frac{\Lambda^2}{2\Delta(\Delta+\Lambda^2)^2}\right) \\
&= \frac{e^2}{2(2\pi)^4}\Omega_4\int_0^1 dx\,(-2x\not{p}+4m_0)\left(\log\left(\frac{\Lambda^2}{\Delta}\right) - 1 + \mathcal{O}\left(\frac{\Delta}{\Lambda^2}\right)\right) \\
&\approx \frac{\alpha}{4\pi}\int_0^1 dx\,(-2x\not{p}+4m_0)\left(\log\left(\frac{\Lambda^2}{\Delta}\right) - 1\right) \\
&= \frac{\alpha}{4\pi}\int_0^1 dx\,(-2x\not{p}+4m_0)\left(\log\left(\frac{\Lambda^2}{-x(1-x)p^2+x\mu^2+(1-x)m_0^2}\right) - 1\right). \qquad (28)
\end{aligned}$$

Now,

$$\begin{aligned}
\delta Z_2 &= \frac{d\Sigma_2}{d\not{p}}|_{\not{p}\to m} \\
&= \frac{\alpha}{4\pi}\int_0^1 dx(-2x)\left(\log\left(\frac{\Lambda^2}{(1-x)^2 m_0^2}\right)-1\right) \\
&\quad + \frac{\alpha}{4\pi}\int_0^1 dx\,(2-x)\,m_0\left(\frac{-1}{(1-x)^2 m_0^2 + x\mu^2}\right)(-2x(1-x)m_0) \\
&= \frac{\alpha}{2\pi}\int_0^1 dx\left(x - x\log\left(\frac{\Lambda^2}{(1-x)^2 m_0^2 + x\mu^2}\right) + \frac{x(2-x)(1-x)m_0^2}{(1-x)^2 m_0^2 + x\mu^2}\right) \\
&= \frac{\alpha}{2\pi}\left(\frac{1}{2} - \frac{1}{2}\left(3 + \log\left(\frac{\Lambda^2}{2}\right)\right) + \frac{1}{2}\left(-1 + 2\log\left(\frac{m^2}{\mu^2}\right)\right)\right) \\
&= \frac{-\alpha}{4\pi}\left(3 + \log\left(\frac{\Lambda^2}{2}\right) - 2\log\left(\frac{m^2}{\mu^2}\right)\right) \qquad (29)
\end{aligned}$$

where

$$\int_0^1 dx\; x\log\left(\frac{\Lambda^2}{(1-x)^2 m_0^2 + x\mu^2}\right)|_{\mu\to 0} = \frac{1}{2}\left(3 + \log\left(\frac{\Lambda^2}{2}\right)\right). \qquad (30)$$

Clearly, $\delta Z_1 \neq \delta Z_2$ as $\delta Z_1 - \delta Z_2 = 7\alpha/16$. Therefore, the Ward identity is violated.

Part (b)

We are asked to repeat part (a) but regulate the integrals using dimensional regularization:

$$\begin{aligned}
\int \frac{d^d\ell}{(2\pi)^4}\frac{\ell^{2n}}{(\ell^2-\Delta+i\epsilon)^m} &= \frac{(-1)^{m+n}i}{(2\pi)^d}\Omega_d\int_0^\infty d\ell_E\frac{\ell_E^{d-1+2n}}{(\ell_E^2+\Delta)^m} \\
&= \frac{(-1)^{n+m}i}{(4\pi)^{d/2}}\left(\frac{1}{\Delta}\right)^{m-n-d/2}\frac{\Gamma(m-d/2-n)\Gamma(d/2+n)}{\Gamma(d/2)\Gamma(m)} \qquad (31)
\end{aligned}$$

where

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \qquad (32)$$

$$\begin{aligned}
I_m^n &= \int_0^\infty d(\ell_E)\frac{(\ell_E)^{d-1+2n}}{(\ell_E^2+\Delta)^m} \\
&= \frac{1}{2}\left(\frac{1}{\Delta}\right)^{m-n-d/2}\frac{\Gamma(m-d/2-n)\Gamma(d/2+n)}{\Gamma(m)}. \qquad (33)
\end{aligned}$$

With equations (20-22), δZ_1 becomes

$$\begin{aligned}
\delta Z_1 &= -2ie^2 \int dx\,(1-x) \int \frac{d^d\ell}{(2\pi)^4} \frac{-\frac{(d-2)^2}{2d}\ell^2 + (\frac{d-2}{2}(1+x^2) - dx)m^2}{(\ell^2-\Delta+i\epsilon)^3} \\
&= \frac{-2e^2}{(4\pi)^{d/2}\Gamma(d/2)} \int dx\,(1-x) \left(\frac{(d-2)^2}{2d} \left(\frac{1}{\Delta}\right)^{2-d/2} \frac{\Gamma(2-d/2)\Gamma(d/2+1)}{\Gamma(3)} \right. \\
&\qquad \left. + \left(\frac{d-2}{2}(1+x^2) - dx\right) m^2 \left(\frac{1}{\Delta}\right)^{3-d/2} \frac{\Gamma(3-d/2)\Gamma(d/2)}{\Gamma(3)} \right) \\
&= \frac{-e^2}{16\pi^2} \int dx\,(1-x) \left(\frac{(1-\epsilon)^2}{2-\epsilon} \left(\frac{4\pi}{\Delta}\right)^{\epsilon} \frac{\Gamma(\epsilon)\Gamma(3-\epsilon)}{\Gamma(2-\epsilon)} \right. \\
&\qquad \left. + \left((1-\epsilon)(1+x^2) - 2(2-\epsilon)x\right) \left(\frac{m^2}{\Delta}\right) \left(\frac{4\pi}{\Delta}\right)^{\epsilon} \Gamma(1+\epsilon) \right) \\
&= \frac{-\alpha}{4\pi} \int dx\,(1-x) \frac{(1-\epsilon)^2}{2-\epsilon} \left(\frac{4\pi}{\Delta}\right)^{\epsilon} \frac{\Gamma(\epsilon)\Gamma(3-\epsilon)}{\Gamma(2-\epsilon)} \\
&\quad + \frac{-\alpha}{4\pi} \int dx\,(1-x) \left((1-\epsilon)(1+x^2) - 2(2-\epsilon)x\right) \left(\frac{m^2}{\Delta}\right) \left(\frac{4\pi}{\Delta}\right)^{\epsilon} \Gamma(1+\epsilon) \\
&= \frac{-\alpha}{4\pi} \int dx\,(1-x)(1-\epsilon)^2 \left(\frac{4\pi}{\Delta}\right)^{\epsilon} \Gamma(\epsilon) + \frac{-\alpha}{4\pi} \int dx\,(1-x)\left(1-4x+x^2\right)\left(\frac{m^2}{\Delta}\right) \\
&= \frac{-\alpha}{4\pi} \int dx\,(1-x) \left(\frac{1}{\epsilon} - \gamma_E + \log\left(\frac{4\pi}{\Delta}\right) - 2\right) + \frac{-\alpha}{4\pi} \int dx\,(1-x)\left(1-4x+x^2\right)\left(\frac{m^2}{\Delta}\right) \\
&= \frac{-\alpha}{4\pi} \int dx\,(1-x) \left(\frac{1}{\epsilon} - \log\Delta + \log\left(4\pi e^{-\gamma_E}\right) - 2\right) + \frac{-\alpha}{4\pi} \int dx\, \frac{(1-x)(1-4x+x^2)m^2}{(1-x)^2m^2 + x\mu^2}.
\end{aligned} \tag{34}$$

Adding in the mass scale Λ^2 (and absorbing the $\log\left(4\pi e^{-\gamma_E}\right)$) we get

$$\begin{aligned}
\delta Z_1 &= \frac{-\alpha}{4\pi} \int dx\,(1-x) \left(\frac{1}{\epsilon} + \log\left(\frac{\Lambda^2}{(1-x)^2m^2 + x\mu^2}\right) - 2\right) + \frac{-\alpha}{4\pi} \int dx\, \frac{(1-x)(1-4x+x^2)m^2}{(1-x)^2m^2 + x\mu^2} \\
&= \frac{-\alpha}{4\pi} \left(\frac{1}{2\epsilon} + \frac{1}{2} + \frac{1}{2}\log\left(\frac{\Lambda^2}{m^2}\right) - 1 + \frac{5}{2} - 2\log\left(\frac{m^2}{\mu^2}\right)\right) \\
&= \frac{-\alpha}{4\pi} \left(\frac{1}{2\epsilon} + 2 + \frac{1}{2}\log\left(\frac{\Lambda^2}{m^2}\right) - 2\log\left(\frac{m^2}{\mu^2}\right)\right) \\
&= \frac{-\alpha}{4\pi}\frac{1}{2} \left(\frac{1}{\epsilon} + 4 + \log\left(\frac{\Lambda^2}{m^2}\right) - 4\log\left(\frac{m^2}{\mu^2}\right)\right),
\end{aligned} \tag{35}$$

where

$$\int_0^1 dx\, \frac{(1-x)\{1,x,x^2\}m^2}{(1-x)^2m^2 + x\mu^2}\Big|_{\mu\to 0} = \left\{\log\left(\frac{m^2}{\mu^2}\right), -1 + \log\left(\frac{m^2}{\mu^2}\right), -\frac{3}{2} + \log\left(\frac{m^2}{\mu^2}\right)\right\} \tag{36}$$

$$\int_0^1 dx\, \frac{(1-x)(1-4x+x^2)m^2}{(1-x)^2m^2 + x\mu^2}\Big|_{\mu\to 0} = \frac{5}{2} - 2\log\left(\frac{m^2}{\mu^2}\right). \tag{37}$$

We now calculate the self-energy to 1-loop

$$\begin{aligned}
\Sigma_2(p) &= -ie^2\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{-(d-2)x\not{p}+dm_0}{(\ell^2-\Delta+i\epsilon)^2}\\
&= -ie^2\int_0^1 dx\,((2-d)x\not{p}+dm)\,\frac{i}{(4\pi)^{d/2}}\left(\frac{1}{\Delta}\right)^{2-d/2}\frac{\Gamma(2-d/2)}{\Gamma(2)}\\
&= \frac{\alpha}{4\pi}\int_0^1 dx\,(-2(1-\epsilon)x\not{p}+2(2-\epsilon)m)\left(\frac{4\pi}{\Delta}\right)^{\epsilon}\Gamma(\epsilon)\\
&= \frac{\alpha}{2\pi}\int_0^1 dx\left[(2m-x\not{p})\left(\frac{1}{\epsilon}-\log\Delta+\log(4\pi e^{-\gamma_E})\right)-(m-x\not{p})+\mathcal{O}(\epsilon)\right]\\
&= \frac{\alpha}{2\pi}\int_0^1 dx\left[(2m-x\not{p})\left(\frac{1}{\epsilon}-\log\Delta+\log(4\pi e^{-\gamma_E})\right)-(m-x\not{p})+\mathcal{O}(\epsilon)\right]
\end{aligned}\tag{38}$$

Adding mass scale in logarithm to get a dimensionless quantity yields

$$\begin{aligned}
\Sigma_2(p) &= \frac{\alpha}{2\pi}\int_0^1 dx\left[(2m-x\not{p})\left(\frac{1}{\epsilon}+\log\left(\frac{\Lambda^2}{(1-x)^2m^2+x\mu^2}\right)+\log(4\pi e^{-\gamma_E})\right)-(m-x\not{p})+\mathcal{O}(\epsilon)\right]\\
&\to \frac{\alpha}{2\pi}\int_0^1 dx\left[(2m-x\not{p})\left(\frac{1}{\epsilon}+\log\left(\frac{\Lambda^2}{(1-x)^2m^2+x\mu^2}\right)\right)-(m-x\not{p})+\mathcal{O}(\epsilon)\right].
\end{aligned}\tag{39}$$

Differentiating yields

$$\begin{aligned}
\delta Z_2 &= \frac{\alpha}{2\pi}\int_0^1 dx\left[-x\left(\frac{1}{\epsilon}+\log\left(\frac{\Lambda^2}{(1-x)^2m^2+x\mu^2}\right)\right)+\frac{2m^2x(1-x)(2-x)}{(1-x)^2m^2+x\mu^2}+x\right]\\
&= \frac{\alpha}{2\pi}\left[-\frac{1}{2}\left(\frac{1}{\epsilon}+3+\log\left(\frac{\Lambda^2}{m^2}\right)\right)-1+2\log\left(\frac{m^2}{\mu^2}\right)+\frac{1}{2}\right]\\
&= \frac{-\alpha}{4\pi}\left[\frac{1}{\epsilon}+4+\log\left(\frac{\Lambda^2}{m^2}\right)-4\log\left(\frac{m^2}{\mu^2}\right)\right]
\end{aligned}\tag{40}$$

where

$$\int_0^1 dx\,\frac{(1-x)\{1,x,x^2\}m^2}{(1-x)^2m^2+x\mu^2}|_{\mu\to 0} = \left\{\log\left(\frac{m^2}{\mu^2}\right),-1+\log\left(\frac{m^2}{\mu^2}\right),-\frac{3}{2}+\log\left(\frac{m^2}{\mu^2}\right)\right\}\tag{41}$$

$$\int_0^1 dx\frac{2x(1-x)(2-x)m^2}{(1-x)^2m^2+x\mu^2}|_{\mu\to 0} = -1+2\log\left(\frac{m^2}{\mu^2}\right).\tag{42}$$

There is a mistake δZ_1 is 1/2 of its true value. $\delta Z_1=\delta Z_2$ when regulated via dimensional regularization.

Problem 7.3

Consider a theory of elementary fermions that couple both to QED and to a Yukawa field ϕ:

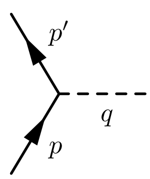
$$H_{\text{int}} = \int d^3x \left(\frac{\lambda}{\sqrt{2}} \phi \bar{\psi}\psi + e\bar{\psi}\displaystyle{\not}A\psi \right).$$

(a) Verify that the contribution to Z_1 from the vertex diagram with a virtual ϕ equals the contribution to Z_2 from the diagram with a virtual ϕ. Use dimensional regularization. Is the Ward identity generally true in this theory?

(b) Now consider the renormalization of the $\phi\bar{\psi}\psi$ vertex. Show that the rescaling of this vertex at $q^2 = 0$ is not canceled by the correction to Z_2. (It suffices to compute the ultraviolet-divergent parts of the diagrams.) In this theory, the vertex and field-strength rescaling give additional shifts of the observable coupling constant relative to its bare value.

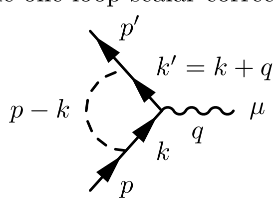
Part (a)

The scalar particle couples to the charged fermion field via the three point vertex



$$= -\frac{i\lambda}{\sqrt{2}}. \tag{43}$$

The renormalization factor Z_1 is defined by the relation $\Gamma^\mu(p+q,p)|_{q\to 0} = Z_1^{-1}\gamma^\mu$ where Γ^μ is the electron vertex. The one-loop scalar correction to the electron vertex is given by



$$= \bar{u}(p')\delta\Gamma^\mu(p',p)u(p). \tag{44}$$

Here,

$$\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) = \frac{i\lambda^2\mu^{(4-d)}}{2} \int \frac{d^dk}{(2\pi)^d} \frac{\bar{u}(p')N^\mu u(p)}{(k'^2 - m_{\text{e}}^2 + i\epsilon)(k^2 - m_{\text{e}}^2 + i\epsilon)((p-k)^2 - m_\phi^2 + i\epsilon)} \tag{45}$$

where $N^\mu = (\not{k}' + m_{\text{e}})\gamma^\mu(\not{k} + m_{\text{e}})$. Also note that the mass dimension of λ in d-dimensions is $(4-d)/2$. Thus, to keep λ dimensionless in d-dimensions we rescale $\lambda \to \mu^{(4-d)/2}\lambda$ (where μ is of mass dimension 1) for each coupling constant in the loop. We evaluate equation (37) at $q \to 0$ to obtain the vertex correction. In this limit we have

$$\begin{aligned} \bar{u}(p)\delta\Gamma^\mu(p,p)u(p) &= \frac{i\lambda^2\mu^{(4-d)}}{2} \int \frac{d^dk}{(2\pi)^d} \frac{\bar{u}(p)N^\mu u(p)}{(k^2 - m_{\text{e}}^2 + i\epsilon)^2((p-k)^2 - m_\phi^2 + i\epsilon)} \\ &= \frac{i\lambda^2\mu^{(4-d)}}{2} \int \frac{d^d\ell}{(2\pi)^d} \int_0^1 dx \int_0^1 dy\; 2y\delta(x+y-1) \frac{\bar{u}(p)N^\mu u(p)}{(\ell^2 - \Delta + i\epsilon)^3} \end{aligned} \tag{46}$$

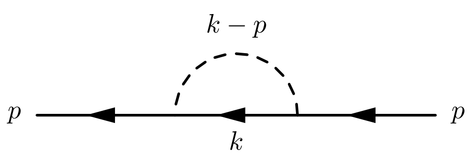
where $\ell = k - xp$ and $\Delta = -x(1-x)p^2 + xm_\phi^2 + (1-x)m_{\text{e}}^2$. Use of the Gordon identity and Dirac equation simplifies the numerator, $N^\mu = \not{\ell}\gamma^\mu\not{\ell} + xm_{\text{e}}(\not{\ell}\gamma^\mu + \gamma^\mu\not{\ell}) + (1+x)^2m_{\text{e}}^2\gamma^\mu + 2m_{\text{e}}\ell^\mu$. Averaging over the loop momentum we discard terms linear in ℓ, and write $\ell^\mu \ell^\nu = \ell^2 g^{\mu\nu}/d$. With this, the numerator becomes $N^\mu \to \frac{2-d}{d}\ell^2\gamma^\mu + (1+m_\mathrm{e})^2\gamma^\mu$.

Thus, the vertex correction is

$$
\begin{aligned}
\delta Z_1 &= -\frac{i\lambda^2\mu^{4-d}}{2}\int \frac{d^d\ell}{(2\pi)^d}\int_0^1 dx\int_0^1 dy\; 2y\delta(x+y-1)\frac{\frac{2-d}{d}\ell^2 + (1+x)^2 m_\mathrm{e}^2}{(\ell^2-\Delta+i\epsilon)^3} \\
&= -\frac{i\lambda^2\mu^{4-d}}{2}\int_0^1 dx\; 2(1-x)\int \frac{d^d\ell}{(2\pi)^d}\frac{\frac{2-d}{d}\ell^2 + (1+x)^2 m_\mathrm{e}^2}{(\ell^2-\Delta+i\epsilon)^3} \\
&= -\frac{i\lambda^2}{2}\frac{i}{16\pi^2}\int_0^1 dx\; 2(1-x)\left[-(1-\epsilon)\left(\frac{4\pi\mu^2}{\Delta}\right)^\epsilon \frac{\Gamma(\epsilon)}{2} + \frac{(1+x)^2 m_\mathrm{e}^2}{\Delta}\left(\frac{4\pi}{\Delta}\right)^\epsilon \frac{\Gamma(1+\epsilon)}{2}\right] \\
&= -\frac{\lambda^2}{64\pi^2}\int_0^1 dx\; 2(1-x)\left[\left(\frac{1}{\epsilon} + \log\left(\frac{4\pi\mu^2 e^{-\gamma_E}}{\Delta}\right) - 1\right) + \frac{(1+x)^2 m_\mathrm{e}^2}{\Delta}\right] \\
&= \frac{\lambda^2}{32\pi^2}\int_0^1 dx\; (1-x)\left[1 - \frac{1}{\epsilon} - \log\left(\frac{\tilde{\mu}^2}{(1-x)^2 m_\mathrm{e}^2 + x m_\phi^2}\right) - \frac{(1+x)^2 m_\mathrm{e}^2}{(1-x)^2 m_\mathrm{e}^2 + x m_\phi^2}\right]
\end{aligned}
\tag{47}
$$

where $\tilde{\mu}^2 = 4\pi\mu^2 e^{-\gamma_E}$.

Next, we evaluate the contribution of the scalar to the electron self-energy



$$
= \frac{i(\not{p} - m_\mathrm{e})}{p^2 - m_\mathrm{e}^2}[-i\Sigma(p)]\frac{i(\not{p} - m_\mathrm{e})}{p^2 - m_\mathrm{e}^2}, \tag{48}
$$

where

$$
\begin{aligned}
\Sigma(p) &= -\frac{i\lambda^2\mu^{4-d}}{2}\int \frac{d^d k}{(2\pi)^d}\frac{i(\not{k} + m_\mathrm{e})}{k^2 - m_\mathrm{e}^2 + i\epsilon}\frac{i}{(k-p)^2 - m_\phi^2 + i\epsilon} \\
&= \frac{i\lambda^2\mu^{4-d}}{2}\int_0^1 dx\int \frac{d^d\ell}{(2\pi)^d}\frac{(\not{\ell} + y\not{p} + m_\mathrm{e})}{(\ell^2 - \Delta + i\epsilon)^2} \\
&= \frac{i\lambda^2\mu^{4-d}}{2}\int_0^1 dx\int \frac{d^d\ell}{(2\pi)^d}\frac{(x\not{p} + m_\mathrm{e})}{(\ell^2 - \Delta + i\epsilon)^2}
\end{aligned}
\tag{49}
$$

where $\ell = k - yp$ and $\Delta = -(1-x)xp^2 + (1-x)m_\mathrm{e}^2 + x m_\phi^2$. Evaluating the ℓ integral yields the self-energy,

$$
\begin{aligned}
\Sigma(p) &= \frac{i\lambda^2\mu^{4-d}}{2}\int_0^1 dx\int \frac{d^d\ell}{(2\pi)^d}\frac{(x\not{p} + m_\mathrm{e})}{(\ell^2 - \Delta + i\epsilon)^2} \\
&= \frac{i\lambda^2\mu^{4-d}}{2}\int_0^1 dx\;\; (x\not{p} + m_\mathrm{e})\frac{i}{(4\pi)^{d/2}}\left(\frac{1}{\Delta}\right)^{2-d/2}\frac{\Gamma(2-d/2)}{\Gamma(2)} \\
&= -\frac{\lambda^2}{32\pi^2}\int_0^1 dx\;\; (x\not{p} + m_\mathrm{e})\left(\frac{4\pi\mu^2}{\Delta}\right)^\epsilon \Gamma(\epsilon) \\
&= -\frac{\lambda^2}{32\pi^2}\int_0^1 dx\;\; (x\not{p} + m_\mathrm{e})\left(\frac{1}{\epsilon} + \log\left(\frac{\tilde{\mu}^2}{\Delta}\right) + \mathcal{O}(\epsilon)\right).
\end{aligned}
\tag{50}
$$

To get the electron vertex correction we differentiate with respect to p and take the limit $\not{p} \to m_{\rm e}$,

$$
\begin{aligned}
\delta Z_2 &= \frac{d\Sigma}{d\not{p}}\Big|_{\not{p}\to m_{\rm e}} \\
&= -\frac{\lambda^2}{32\pi^2}\int_0^1 dx\ \left[x\left(\frac{1}{\epsilon}+\log\left(\frac{\tilde{\mu}^2}{\Delta}\right)\right) - (x\not{p}+m_{\rm e})\,\frac{1}{\Delta}\frac{d\Delta}{dp}\right]_{\not{p}\to m_{\rm e}} \\
&= -\frac{\lambda^2}{32\pi^2}\int_0^1 dx\ \left[x\left(\frac{1}{\epsilon}+\log\left(\frac{\tilde{\mu}^2}{(1-x)^2m_{\rm e}^2+xm_\phi^2}\right)\right) + \frac{2(1-x^2)xm_{\rm e}^2}{(1-x)^2m_{\rm e}^2+xm_\phi^2}\right]. \qquad (51)
\end{aligned}
$$

To test if the Ward identity holds we take the difference between δZ_1 and δZ_2

$$
\begin{aligned}
\delta Z_2-\delta Z_1 = \frac{\lambda^2}{32\pi^2}\int_0^1 dx\ &\left[-\frac{x}{\epsilon}-\log\left(\frac{\tilde{\mu}^2}{\Delta}\right)-\frac{2x(1-x)(1+x)m_{\rm e}^2}{\Delta}\right.\\
&\left. -(1-x)+\frac{1-x}{\epsilon}+(1-x)\log\left(\frac{\tilde{\mu}^2}{\Delta}\right)+\frac{(1-x)(1+x)^2m_{\rm e}^2}{\Delta}\right]\\
&= \frac{\lambda^2}{32\pi^2}\int_0^1 dx\ \left[\frac{1-2x}{\epsilon}+(1-2x)\log\left(\frac{\tilde{\mu}^2}{\Delta}\right)-(1-x)+\frac{(1-x)(1+x)^2m_{\rm e}^2}{\Delta}\right]\\
&= \frac{\lambda^2}{32\pi^2}\int_0^1 dx\ \left[\frac{1-2x}{\epsilon}+(1-2x)\log\left(\frac{\tilde{\mu}^2}{\Delta}\right)-(1-x)+\frac{(1-x)(1+x)^2m_{\rm e}^2}{\Delta}\right]\\
&= \frac{\lambda^2}{32\pi^2}\int_0^1 dx\ \left[(1-2x)\log\left(\frac{\tilde{\mu}^2}{\Delta}\right)-(1-x)+\frac{(1-x)(1+x)^2m_{\rm e}^2}{\Delta}\right]\\
&= 0, \qquad (52)
\end{aligned}
$$

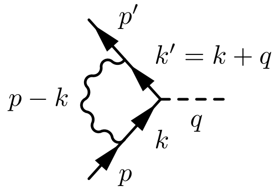
where

$$
\begin{aligned}
\int_0^1(1-2x)\log\left(\frac{\tilde{\mu}^2}{\Delta}\right) &= -\int_0^1 x(1-x)\left(\frac{-1}{\Delta}\right)\frac{d\Delta}{dx}\\
&= (1-x)-\frac{(1-x)(1+x)^2m_{\rm e}^2}{\Delta}. \qquad (53)
\end{aligned}
$$

Part (b)

At the one-loop level the $\phi\bar{\psi}\psi$ vertex renormalization constant has contributions from the diagrams

$$i\mathcal{M} =$$



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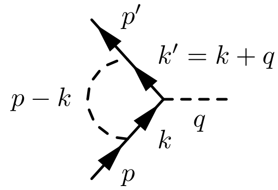


$$= -i\frac{\lambda}{\sqrt{2}}\bar{u}(p')\delta V_{\phi\bar{\psi}\psi}(p',p)u(p), \qquad (54)$$

where

$$
\begin{aligned}
&\bar{u}(p')\delta V_{\phi\bar{\psi}\psi}(p',p)u(p) = \\
&\int\frac{d^dk}{(2\pi)^d}\bar{u}(p')\left\{(-ie\mu^{(4-d)/2}\gamma^\mu)\frac{i(\not{k}'+m_{\rm e})}{k'^2-m_{\rm e}^2+i\epsilon}\frac{i(\not{k}+m_{\rm e})}{k^2-m_{\rm e}^2+i\epsilon}(-ie\mu^{(4-d)/2}\gamma_\mu)\frac{-i}{(p-k)^2-m_\gamma^2+i\epsilon}\right.\\
&\left.+\left(-i\frac{\lambda\mu^{(4-d)/2}}{\sqrt{2}}\right)\frac{i(\not{k}'+m_{\rm e})}{k'^2-m_{\rm e}^2+i\epsilon}\frac{i(\not{k}+m_{\rm e})}{k^2-m_{\rm e}^2+i\epsilon}\left(-i\frac{\lambda\mu^{(4-d)/2}}{\sqrt{2}}\right)\frac{i}{(p-k)^2-m_\phi^2+i\epsilon}\right\}u(p). \qquad (55)
\end{aligned}
$$

Setting $q = 0$ we obtain

$$\bar{u}(p)\delta V_{\phi\bar{\psi}\psi}(p,p)u(p) = \int_0^1 dx\ 2(1-x)\int \frac{d^d\ell}{(2\pi)^d}\bar{u}(p)\left\{\left(-ie^2\mu^{4-d}\right)\frac{N_1}{(\ell-\Delta_1+i\epsilon)^3}+\left(i\frac{\lambda^2\mu^{4-d}}{2}\right)\frac{N_2}{(\ell-\Delta_2+i\epsilon)^3}\right\}u(p), \tag{56}$$

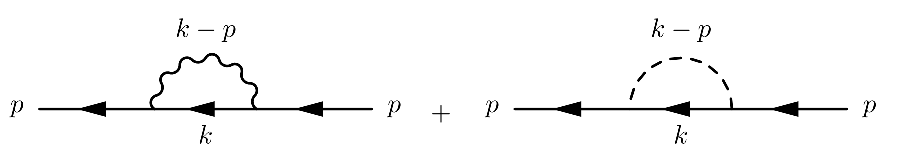
where

$$\begin{aligned}\ell &= k - yp,\\ N_1 &= d\ell^2 + \left((1+x^2)d - 2x(d-2)\right)m_e^2,\\ N_2 &= \ell^2 + (1+x)^2 m_e^2,\\ \Delta_1 &= -x(1-x)p^2 + (1-x)m_e^2 + xm_\gamma^2,\\ \Delta_2 &= -x(1-x)p^2 + (1-x)m_e^2 + xm_\phi^2.\end{aligned} \tag{57}$$

Preforming the d-dimensional momentum integral we obtain

$$\begin{aligned}\delta Z_1 = &-\int_0^1 dx\ 2(1-x)\frac{i\mu^{4-d}}{(4\pi)^{d/2}}\left(\frac{1}{\Delta_1}\right)^{2-d/2}\frac{\Gamma(2-d/2)}{\Gamma(3)}\left(-ie^2\right)\left(\frac{d^2}{2}-\left((1+x^2)d-2x(d-2)\right)m_e^2\frac{(2-d/2)}{\Delta_1}\right)\\ &-\int_0^1 dx\ 2(1-x)\frac{i\mu^{4-d}}{(4\pi)^{d/2}}\left(\frac{1}{\Delta_2}\right)^{2-d/2}\frac{\Gamma(2-d/2)}{\Gamma(3)}\left(i\frac{\lambda^2}{2}\right)\left(\frac{d}{2}-(1+x)^2m_e^2\frac{(2-d/2)}{\Delta_1}\right)\\ =&-\left(-ie^2\right)\frac{i}{16\pi^2}\int_0^1 dx\ 2(1-x)\left(\frac{4\pi\mu^2}{\Delta_1}\right)^\epsilon\frac{\Gamma(\epsilon)}{2}\left(\frac{4(2-\epsilon)^2}{2}-\epsilon\frac{\left(2(2-\epsilon)(1+x^2)-4(1-\epsilon)x\right)m_e^2}{\Delta_1}\right)\\ &-\left(i\frac{\lambda^2}{2}\right)\frac{i}{16\pi^2}\int_0^1 dx\ 2(1-x)\left(\frac{4\pi\mu^2}{\Delta_2}\right)^\epsilon\frac{\Gamma(\epsilon)}{2}\left((2-\epsilon)-\epsilon\frac{(1+x)^2m_e^2}{\Delta_1}\right)\\ =&\frac{-e^2}{16\pi^2}\int_0^1 dx\ (1-x)\left(\frac{4\pi\mu^2}{\Delta_1}\right)^\epsilon\Gamma(\epsilon)\left(2(2-\epsilon)^2-2\epsilon\frac{\left((2-\epsilon)(1+x^2)-2(1-\epsilon)x\right)m_e^2}{\Delta_1}\right)\\ &+\frac{\lambda^2}{32\pi^2}\int_0^1 dx\ (1-x)\left(\frac{4\pi\mu^2}{\Delta_2}\right)^\epsilon\Gamma(\epsilon)\left((2-\epsilon)-\epsilon\frac{(1+x)^2m_e^2}{\Delta_1}\right)\\ =&\frac{-e^2}{8\pi^2}\int_0^1 dx\ (1-x)\left(\frac{1}{\epsilon}+\epsilon\log\left(\frac{\tilde{\mu}^2}{\Delta_1}\right)\right)\left((2-\epsilon)^2-\epsilon\frac{\left((2-\epsilon)(1+x^2)-2(1-\epsilon)x\right)m_e^2}{\Delta_1}\right)\\ &+\frac{\lambda^2}{32\pi^2}\int_0^1 dx\ (1-x)\left(\frac{1}{\epsilon}+\epsilon\log\left(\frac{\tilde{\mu}^2}{\Delta_1}\right)\right)\left((2-\epsilon)-\epsilon\frac{(1+x)^2m_e^2}{\Delta_1}\right)\\ =&\frac{-\alpha}{\pi\epsilon}+\frac{\lambda^2}{32\pi^2\epsilon}+\text{finite terms.}\end{aligned} \tag{58}$$

The wave function renormalization factor Z_2 comes from the sum



(59)

We have already calculated these contributions in the text and in question 7.2:

$$\delta Z_2 = \frac{-\alpha}{4\pi\epsilon}+\frac{-\lambda^2}{64\pi^2\epsilon}+\text{finite terms.} \tag{60}$$

Taking the difference we see that

$$\begin{aligned}\delta Z_2 - \delta Z_1 &= \frac{-\alpha}{4\pi\epsilon} + \frac{-\lambda^2}{64\pi^2\epsilon} - \frac{-\alpha}{\pi\epsilon} - \frac{\lambda^2}{32\pi^2\epsilon} + \text{finite terms} \\ &= \frac{-3\alpha}{4\pi\epsilon} + \frac{-3\lambda^2}{64\pi^2\epsilon} + \text{finite terms} \\ &\neq 0.\end{aligned} \tag{61}$$