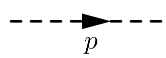
Problem 9.1: Scalar QED

This problem concerns the theory of a complex scalar field ϕ interacting with the electromagnetic field A^μ. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + (D_\mu\phi)^*(D_\mu\phi) - m_\phi^2\phi^*\phi, \tag{1}$$

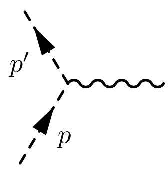
where $D_\mu = \partial_\mu + ieA_\mu$ is the usual gauge-covariant derivative.

(a) Use the functional method of Section 9.2 to show that the propagator of the complex scale field is the same as that of a real field:

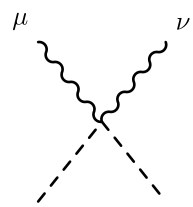


$$= \frac{i}{p^2 - m_\phi^2 + i\epsilon} \tag{2}$$

Also derive the Feynman rules for the interactions between photons and scalar particles; you should find



$$= -ie(p+p')^\mu$$



$$= 2ie^2 g^{\mu\nu} \tag{3}$$

(b) Compute, to lowest order, the differential cross section for $e^+e^- \to \phi\phi^*$. Ignore the electron mass (but not the scalar particle's mass), and average over the electron and positorn polarizations. Find the asymptotic angular dependence and total cross section. Compare your results to the corresponding formulae for $e^+e^- \to \mu^+\mu^-$.

(c) Compute the contribution of the charged scalar to the photon vacuum polarization, using dimensional regularization. Note that there are two diagrams. To put the answer into the expected form,

$$\Pi^{\mu\nu}(q^2) = (g^{\mu\nu}q^2 - q^\mu q^\nu)\Pi(q^2),$$

it is useful to add the two diagrams at the beginning, putting both terms over a common denominator before introduction a Feynman parameter. Show that, for $-q \gg m^2$, the charged boson contribution to $\Pi(q^2)$ is exactly 1/4 that of a virtual electron-positron pair.

9.1 (a)

Let us start with the action for the theory

$$\begin{aligned}
S &= \int d^4x \left(-\frac{1}{4}F_{\mu\nu}^2 + (D_\mu\phi)^*(D_\mu\phi) - m_\phi^2\phi^*\phi\right) \\
&= \int d^4x \left(-\frac{1}{4}F_{\mu\nu}^2 + \partial_\mu\phi^*\partial^\mu\phi - ieA^\mu\left[\phi^*(\partial_\mu\phi) - (\partial_\mu\phi^*)\phi\right] + e^2A^\mu A_\mu\phi^*\phi - m_\phi^2\phi^*\phi\right) \\
\text{IBP} &= \int d^4x \left(-\frac{1}{4}F_{\mu\nu}^2 - \phi^*(\partial^2 + m_\phi^2)\phi - ieA^\mu\left[\phi^*(\partial_\mu\phi) - (\partial_\mu\phi^*)\phi\right] + e^2A_\mu^2|\phi|^2\right) \\
&\equiv S_0^\gamma + S_0^\phi + S_{\text{int}}.
\end{aligned} \tag{4}$$

The free theory generating functional is given by

$$\begin{aligned} Z_0\left[J_{\rm em}, J_{\rm s}, J_{\rm s}^*\right] &= \int \mathcal{D}A \int \mathcal{D}\phi \int \mathcal{D}\phi^* e^{iS_0^\gamma + i\int \mathrm{d}^4x\ A_\mu J^\mu_{\rm em} + iS_0^\phi + i\int \mathrm{d}^4x\ \{J_{\rm s}^*\phi + \phi^* J_{\rm s}\}} \\ &= Z_0\left[0,0,0\right] e^{-i\int \mathrm{d}^4x\ \int \mathrm{d}^4y\ \left(\frac{1}{2} J^\mu_{\rm em}(x) D_{\mu\nu}(x-y) J^\nu_{\rm em}(y) + J^*_{\rm s}(x) D(x-y) J_{\rm s}(y)\right)}. \end{aligned} \tag{5}$$

Here, we assume that the Faddeev and Popov procedure has been preformed to restrict the path integral over A to physically unique field configurations. We have also ignored the overall infinite constant factor that this procedure generates (recall that we are usually interested in ratios).

Expanding the exponential with the interaction term we obtain the full generating functional as a series in e

$$\begin{aligned} Z\left[J_{\rm em}, J_{\rm s}, J_{\rm s}^*\right] = \int \mathcal{D}A \int \mathcal{D}\phi \int \mathcal{D}\phi^* Z_0\left[J_{\rm em}, J_{\rm s}, J_{\rm s}^*\right] &\left(1 + ie^2 \int \mathrm{d}^4x\ A_\mu^2(x)|\phi(x)|^2 + \mathcal{O}(e^4)\right) \\ \times &\left(1 + e\int \mathrm{d}^4z\ A^\mu(z)\left[\phi^*(z)(\partial_\mu\phi(z)) - (\partial_\mu\phi^*(z))\phi(z)\right] + \mathcal{O}(e^2)\right). \end{aligned} \tag{6}$$

We can now read off the propagator and interaction vertices. The scalar field propagator is

$$\langle 0|\mathrm{T}\phi(x_1)\phi^*(x_2)|0\rangle = D(x_1 - x_2) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{i}{p^2 - m_\phi^2 + i\epsilon} e^{ik\cdot(x_1 - x_2)} \tag{7}$$

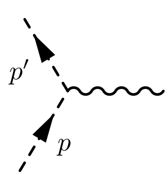
while the photon propagator is

$$\langle 0|\mathrm{T}A_\mu(x_1)A_\nu(x_2)|0\rangle = D_{\mu\nu}(x_1 - x_2) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2 - i\epsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right). \tag{8}$$

Fourier transforming the fields in the term,

$$\begin{aligned} &e\int \mathrm{d}^4z\ A^\mu(k)\left[\phi^*(z)(\partial_\mu\phi(z)) - (\partial_\mu\phi^*(z))\phi(z)\right] \\ &= (-ie)\int \mathrm{d}^4z\ \int \frac{\mathrm{d}^4k}{(2\pi)^4}\ \int \frac{\mathrm{d}^4p}{(2\pi)^4}\ \int \frac{\mathrm{d}^4p'}{(2\pi)^4}\ \tilde{A}^\mu(k)\tilde{\phi}^*(p')\tilde{\phi}(p)(p_\mu + p'_\mu)e^{-iz\cdot(p-p'-k)} \\ &= \int \frac{\mathrm{d}^4p}{(2\pi)^4}\ \int \frac{\mathrm{d}^4p'}{(2\pi)^4}\ \tilde{A}^\mu(p-p')\tilde{\phi}^*(p')\tilde{\phi}(p)(-ie)(p_\mu + p'_\mu) \end{aligned} \tag{9}$$

yields the vertex

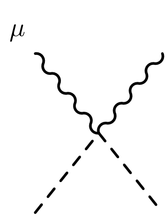


$$\mu = -ie(p+p')^\mu \tag{10}$$

while the term,

$$ie^2\int \mathrm{d}^4x\ A_\mu^2(x)|\phi(x)|^2,$$

yields the vertex

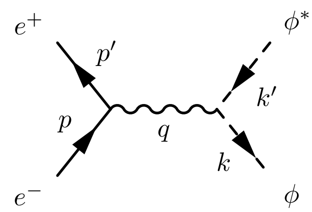


$$= 2ie^2 g^{\mu\nu}. \tag{11}$$

where the 2 comes from symmetry in the two A fields.

9.1 (b)

To lowest order $e^+e^- \to \phi\phi^*$ is given by the diagram:



$$= i\mathcal{M}(e^+e^- \to \phi\phi^*). \tag{12}$$

Applying the Feynman Rules we obtain

$$\begin{aligned} i\mathcal{M}(e^+e^- \to \phi\phi^*) &= \bar{v}(p')(-ie\gamma^\mu)u(p)\frac{ig_{\mu\nu}}{(p+p')^2}(ie)(-k+k')_\nu \\ &= ie^2\frac{\bar{v}(p')(-\not{k}+\not{k}')u(p)}{(p+p')^2}. \end{aligned} \tag{13}$$

The spin averaged matrix element is

$$\begin{aligned} \frac{1}{4}\sum_{\text{spin}}|\mathcal{M}(e^+e^- \to \phi\phi^*)|^2 &= \frac{1}{4}\left(\frac{e^2}{(p+p')^2}\right)^2 \mathrm{Tr}\left[(\not{p}' - m_\mathrm{e})(m_\mathrm{e} - \not{k})(\not{p} + m_\mathrm{e})(m_\mathrm{e} - \not{k})\right] \\ &= \frac{e^4}{(p+p')^4}\Big(2(p\cdot k)\left[p'\cdot k - p'\cdot k'\right] \\ &\qquad + 2(p\cdot k')\left[p'\cdot k' - p'\cdot k\right] \\ &\qquad + (p\cdot p')\left[2k\cdot k' - k\cdot k - k'\cdot k'\right] \\ &\qquad + m_\mathrm{e}^2\left[k\cdot k + k'\cdot k' - 2k\cdot k'\right]\Big). \end{aligned} \tag{14}$$

In the centre of mass frame $\mathbf{p} = -\mathbf{p}' \implies p + p' = (2E_\mathbf{p}, \mathbf{0})$ and $\mathbf{k} = -\mathbf{k}' \implies k + k' = (2E_\mathbf{k}, \mathbf{0})$. This means that $E_\mathbf{p} = E_\mathbf{k} \equiv E$. We also assume that $|\mathbf{p}| \gg m_\mathrm{e}$ so that $E \approx |\mathbf{p}|$. We need the dot products

$$p\cdot k = p'\cdot k' = E_\mathbf{p}E_\mathbf{k} - |\mathbf{p}||\mathbf{k}|\cos\theta \approx E\,(E - |\mathbf{k}|\cos\theta) \tag{15}$$

$$p'\cdot k = p\cdot k' = E_\mathbf{p}E_\mathbf{k} + |\mathbf{p}||\mathbf{k}|\cos\theta \approx E\,(E + |\mathbf{k}|\cos\theta) \tag{16}$$

$$p\cdot p' = E_\mathbf{p}^2 + |\mathbf{p}|^2 \approx 2E^2 \tag{17}$$

$$k\cdot k' = E_\mathbf{k}^2 + |\mathbf{k}|^2 = E^2 + |\mathbf{k}|^2 \tag{18}$$

With these relations (and taking $m_\mathrm{e} = 0$), the spin averaged matrix element squared becomes

$$\begin{aligned} \frac{1}{4}\sum_{\text{spin}}|\mathcal{M}(e^+e^- \to \phi\phi^*)|^2 &\approx \frac{e^4}{4}\left(\frac{|\mathbf{k}|^2}{E^2} - 2\frac{|\mathbf{k}|^2}{E^2}\cos^2\theta - \frac{m_\phi^2}{E^2}\right) \\ &= 8\pi^2\alpha^2\beta^2\sin^2\theta, \end{aligned} \tag{19}$$

where $\beta = \sqrt{1 - m_\phi^2/E^2}$ is the velocity of the particles and $\alpha = 4\pi e^2$ is the fine structure constant.

Since this is a two particle reaction we can use the simplified cross section formula (4.84)

$$\begin{aligned} \frac{\mathrm{d}\sigma(e^+e^- \to \phi\phi^*)}{\mathrm{d}\Omega} &= \frac{1}{2E^2}\frac{|\mathbf{k}|}{16\pi^2 E}\frac{1}{4}\sum_{\text{spin}}|\mathcal{M}(e^+e^- \to \phi\phi^*)|^2 \\ &\approx \frac{\alpha^2\beta^3}{4E^2}\sin^2\theta. \end{aligned} \tag{20}$$

Integrating over the polar coordinate yields the total cross section

$$\sigma(e^+e^- \to \phi\phi^*) = \frac{2\pi\alpha^2\beta^3}{3E^2}. \tag{21}$$

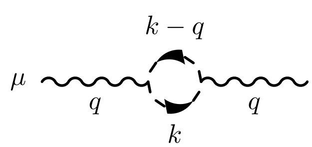
Comparing our results to those of the $e^+e^- \to \mu^-\mu^+$

$$\begin{aligned}\frac{\mathrm{d}\sigma(e^+e^- \to \mu^-\mu^+)}{\mathrm{d}\Omega} &= \frac{\alpha^2\beta}{4E}\left[\left(1+\frac{m_\mu^2}{E^2}\right)+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right],\\ \sigma(e^+e^- \to \mu^-\mu^+) &= \frac{4\pi\alpha^2\beta}{3E^2}\left(1+\frac{m_\mu^2}{2E^2}\right),\end{aligned} \tag{22}$$

we see that the angular dependence of the differential cross sections are very different. The scalar particles have a are more likely to move perpendicular to the electron beam axis while the muons are more likely to move along the electron beam axis.

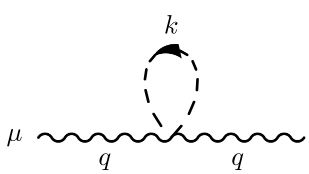
9.1 (c)

The scalar contribution to the photon vacuum polarization is given by the following diagrams



$$= i\mathcal{M}_1^{\mu\nu} \tag{23}$$

and



$$= i\mathcal{M}_2^{\mu\nu}. \tag{24}$$

Now applying the Feynman rules (in d-dimensions) we obtain

$$\begin{aligned} i\mathcal{M}_1^{\mu\nu} &= \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{(-ie)(2k-q)^\mu(i)(-ie)(2k-q)^\nu(i)}{(k^2-m_\phi^2-i\epsilon)((k-q)^2-m_\phi^2+i\epsilon)},\\ &= e^2\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{(2k-q)^\mu(2k-q)^\nu}{(k^2-m_\phi^2-i\epsilon)((k-q)^2-m_\phi^2+i\epsilon)}, \end{aligned} \tag{25}$$

$$\begin{aligned} i\mathcal{M}_2^{\mu\nu} &= \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{(2ie^2g^{\mu\nu})(i)}{k^2-m_\phi^2-i\epsilon},\\ &= e^2\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{-2g^{\mu\nu}}{k^2-m_\phi^2-i\epsilon}. \end{aligned} \tag{26}$$

Following the hint we add these diagrams together before introducing Feynman parameters

$$
\begin{aligned}
i\Pi^{\mu\nu} &= i\mathcal{M}_1^{\mu\nu} + i\mathcal{M}_2^{\mu\nu} \\
&= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left(\frac{(-ie)(2k-q)^\mu (i)(-ie)(2k-q)^\nu (i)}{(k^2 - m_\phi^2 - i\epsilon)((k-q)^2 - m_\phi^2 + i\epsilon)} + \frac{(2ie^2 g^{\mu\nu})(i)}{k^2 - m_\phi^2 - i\epsilon} \right) \\
&= e^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left(\frac{(2k-q)^\mu (2k-q)^\nu - 2g^{\mu\nu}((k-q)^2 - m_\phi^2)}{(k^2 - m_\phi^2 - i\epsilon)((k-q)^2 - m_\phi^2 + i\epsilon)} \right) \\
&= e^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^d} \left(\frac{4k^\mu k^\nu - 2k^\mu q^\nu - 2k^\nu q^\nu + q^\mu q^\nu - 2g^{\mu\nu}(k^2 - 2k\cdot q + q^2 - m_\phi^2)}{(k^2 - m_\phi^2 - i\epsilon)((k-q)^2 - m_\phi^2 + i\epsilon)} \right). \qquad (27)
\end{aligned}
$$

Introducing the Feynman parameters x, y we have

$$
\begin{aligned}
i\Pi^{\mu\nu} = e^2 &\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\, \delta(x+y-1) \int \frac{\mathrm{d}^d k}{(2\pi)^d} \\
&\times \left(\frac{4k^\mu k^\nu - 2k^\mu q^\nu - 2k^\nu q^\nu + q^\mu q^\nu - 2g^{\mu\nu}(k^2 - 2k\cdot q + q^2 - m_\phi^2)}{\Big[(x+y)k^2 - 2yk\cdot q + yq^2 - (x+y)m_\phi^2 - i\epsilon\Big]^2} \right) \\
= e^2 &\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left(\frac{4k^\mu k^\nu - 2k^\mu q^\nu - 2k^\nu q^\nu + q^\mu q^\nu - 2g^{\mu\nu}(k^2 - 2k\cdot q + q^2 - m_\phi^2)}{\Big[k^2 - 2xk\cdot q + xq^2 - m_\phi^2 - i\epsilon\Big]^2} \right). \qquad (28)
\end{aligned}
$$

By letting $\ell = k - xq$ we complete the square in the denominator

$$
\begin{aligned}
i\Pi^{\mu\nu} &= e^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{1}{[\ell^2 - \Delta]^2} \Big(4(\ell + xq)^\mu (\ell + xq)^\nu - 2(\ell + xq)^\mu q^\nu - 2(\ell + xq)^\nu q^\mu \\
&\qquad + q^\mu q^\nu - 2g^{\mu\nu}((\ell + xq)^2 - 2(\ell + xq)\cdot q + q^2 - m_\phi^2) \Big), \\
&= e^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{1}{[\ell^2 - \Delta]^2} \Big(4(\ell^\mu \ell^\nu + x\ell^{[\mu} q^{\nu]} + x^2 q^\mu q^\nu) - 2(\ell^\mu q^\nu + xq^\mu q^\nu) - 2(\ell^\nu q^\mu + xq^\nu q^\mu) \\
&\qquad + q^\mu q^\nu - 2g^{\mu\nu}(\ell^2 + 2xq\cdot\ell + x^2 q^2 - 2(\ell\cdot q + xq^2) + q^2 - m_\phi^2) \Big), \\
&= e^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{1}{[\ell^2 - \Delta]^2} \Big(4(\ell^\mu \ell^\nu + x^2 q^\mu q^\nu) - 2(xq^\mu q^\nu) - 2(xq^\nu q^\mu) + q^\mu q^\nu \\
&\qquad - 2g^{\mu\nu}(\ell^2 + (x-1)^2 q^2 - m_\phi^2) \Big), \\
&= e^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{4\ell^\mu \ell^\nu - 2g^{\mu\nu}\ell^2 + (2x-1)^2 q^\mu q^\nu - 2g^{\mu\nu}(x-1)^2 q^2 + 2g^{\mu\nu} m_\phi^2}{[\ell^2 - \Delta]^2}, \\
&= e^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{-(2 - \frac{4}{d})g^{\mu\nu}\ell^2 + (2x-1)^2 q^\mu q^\nu - 2g^{\mu\nu}(x-1)^2 q^2 + 2g^{\mu\nu} m_\phi^2}{[\ell^2 - \Delta]^2}, \qquad (29)
\end{aligned}
$$

where we have ignored terms linear in ℓ and $\Delta = -x(1-x)q^2 + m_\phi^2$. Integrating over ℓ yields

$$\begin{aligned} i\Pi^{\mu\nu} &= e^2 \int_0^1 \mathrm{d}x \left[-\left(2 - \frac{4}{d}\right) g^{\mu\nu} I_1(\Delta) + \left((2x-1)^2 q^\mu q^\nu - 2g^{\mu\nu}(x-1)^2 q^2 + 2g^{\mu\nu} m_\phi^2\right) I_0(\Delta) \right], \\ &= e^2 \int_0^1 \mathrm{d}x \left[-\left(2 - \frac{4}{d}\right) g^{\mu\nu} I_1(\Delta) + 2g^{\mu\nu} m_\phi^2 I_0(\Delta) + \left((2x-1)^2 q^\mu q^\nu - 2(x-1)^2 g^{\mu\nu} q^2\right) I_0(\Delta) \right], \end{aligned} \tag{30}$$

where

$$I_0(\Delta) = \int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^2} = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}\Gamma(2)}, \tag{31}$$

$$I_1(\Delta) = \int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2 - \Delta)^2} = \frac{-i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(1-d/2)}{\Delta^{1-d/2}\Gamma(2)} = -\frac{d}{2-d} \Delta I_0(\Delta). \tag{32}$$

Thus,

$$\begin{aligned} i\Pi^{\mu\nu} &= e^2 \int_0^1 \mathrm{d}x \left[\left(\frac{2d-4}{2-d} \Delta + 2m_\phi^2 \right) g^{\mu\nu} + \left((2x-1)^2 q^\mu q^\nu - 2(x-1)^2 g^{\mu\nu} q^2\right) \right] I_0(\Delta), \\ &= e^2 \int_0^1 \mathrm{d}x \left[2x(1-x) q^2 g^{\mu\nu} + (2x-1)^2 q^\mu q^\nu - 2(x-1)^2 g^{\mu\nu} q^2 \right] \frac{i}{(4\pi)^{2-\epsilon}} \frac{\Gamma(\epsilon)}{\Delta^\epsilon \Gamma(2)}, \\ &= e^2 \int_0^1 \mathrm{d}x \left[-(2x-1)^2 \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) + (2x-1) g^{\mu\nu} q^2 \right] \frac{i}{16\pi^2} \left(\frac{4\pi}{\Delta}\right)^\epsilon \Gamma(\epsilon), \\ &= \frac{i\alpha}{4\pi} \int_0^1 \mathrm{d}x \left[-(2x-1)^2 \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) + (2x-1) g^{\mu\nu} q^2 \right] \left[\frac{1}{\epsilon} + \log\left(\frac{4\pi e^{-\gamma_E}}{\Delta} \right) \right]. \end{aligned} \tag{33}$$

For the last term in the first brackets we change variables $y = x - \frac{1}{2}$. Then $(1-2x) \to 2y$ while $\Delta \to -(y^2 - \frac{1}{4})q^2 - m_\phi^2$. Since Δ is even in y while the last term in the first brackets is linear the integral is odd and vanishes. Thus,

$$\begin{aligned} i\Pi^{\mu\nu} &= \frac{i\alpha}{4\pi} \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \int_0^1 \mathrm{d}x \, (1-2x)^2 \left[\frac{1}{\epsilon} + \log\left(\frac{4\pi e^{-\gamma_E}}{\Delta} \right) \right], \\ &= \frac{i\alpha}{4\pi} \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \int_0^1 \mathrm{d}x \, (1-2x)^2 \left[\frac{1}{\epsilon} + \log\left(\frac{4\pi e^{-\gamma_E}}{\Delta} \right) \right], \\ &= \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) i\Pi(q^2), \end{aligned} \tag{34}$$

where

$$i\Pi(q^2) = \frac{\alpha}{4\pi} \int_0^1 \mathrm{d}x \, (1-2x)^2 \left[\frac{1}{\epsilon} + \log\left(\frac{4\pi e^{-\gamma_E}}{m_\phi^2 - x(1-x)q^2} \right) \right]. \tag{35}$$

The physically relevant part, $i\hat{\Pi}$, is

$$\begin{aligned} i\hat{\Pi}(q^2) &= i\Pi(q^2) - i\Pi(0) \\ &= \frac{\alpha}{4\pi} \int_0^1 \mathrm{d}x \, (1-2x)^2 \log\left(\frac{m_\phi^2}{m_\phi^2 - x(1-x)q^2} \right). \end{aligned} \tag{36}$$

Problem 9.2: Quantum statistical mechanics

(a) evaluate the quantum statistical partition function

$$Z = \text{Tr}[e^{-\beta H}]$$

using the strategy of section 9.1 for evaluating the matrix elements of e^{-iHt} in terms of functional integrals. Show that once again one finds a functional integral, over functions defined on a domain that is of length β and periodically connected in the time direction. Note that the Euclidean form of the Lagrangian appears in the weight.

(b) Evaluate this integral for a simple harmonic oscillator,

$$L_E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2,$$

by introducing a Fourier decomposition of $x(t)$:

$$x(t) = \sum_n x_n \frac{1}{\sqrt{\beta}} e^{2\pi i n t/\beta}.$$

The dependence of the result on β is a bit subtle to obtain explicitly, since the measure of the integral over $x(t)$ depends on β in any discretization. However, the dependence on ω should be unambiguous. Show that, up to a (possibly divergent and β-dependent) constant the integral reproduces exactly the familiar expression for the quantum partition function of an oscillator. [You many find the identity

$$\sinh z = z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{(n\pi)^2}\right)$$

useful.]

(c) Generalize this construction to field theory. Show that the quantum statistical partition function for a free scalar field can be written in terms of a functional integral. The value of this integral is given formally by

$$\left[\det(-\partial^2 + m^2)\right]^{-1/2}$$

where the operator acts on functions of Euclidean space that are periodic in the time direction with periodicity β. As before, the β dependence of this expression is difficult to compute directly. However, the dependence on m_ϕ^2 is unambiguous. (More generally, one can usually evaluate the variation of a functional determinant with respect to any explicit parameter in the Lagrangian.) Show that the determinant indeed reproduces the partition function of relativistic scalar particles.

(d) Now let $\psi(t)$, $\bar{\psi}(t)$ be two Grassmann-valued coordinates, and define a fermionic oscillator by writing the Lagrangian

$$L_E = \bar{\psi}\dot{\psi} + \omega\bar{\psi}\psi.$$

This Lagrangian corresponds to the Hamiltonian

$$H = \omega\bar{\psi}\psi, \quad \text{with } \{\bar{\psi}, \psi\} = 1;$$

that is , to a simple two-level system. Evaluate the functional integral, assuming that the fermions obey anti-periodic boundary conditions: $\psi(t+\beta) = -\psi(t)$. (Why is this reasonable?) Show that the result reproduces the partition function of a quantum-mechanical two-level system, that is, of a quantum state with Fermi statistics.

(e) Define the partition function for the photon field as the gauge-invariant functional integral

$$Z = \int \mathcal{D}A \exp\left(- \int \mathrm{d}^4 x_E \frac{1}{4} (F_{\mu\nu})^2 \right)$$

over vector fields A_μ that are periodic in the time direction with period β. Apply the gauge-fixing procedure discussed in Section 9.4 (working, for example, in Feynman gauge). Evaluate the functional determinants using the result of part (c) and show that the functional integral does give the correct quantum statistical result (including the correct counting of polarization states).

9.2 (a)

Let the system be described by the generalized coordinates $\{q_i\}$ and momenta $\{p_i\}$ where $i = 1, 2, ..., n$. The quantum mechanical partition function is given by

$$\begin{aligned} Z &= \mathrm{Tr}\left[e^{-\beta H}\right] \\ &= \int \mathrm{d}^n \boldsymbol{q}_0 \, \langle \boldsymbol{q}_0 | e^{-\beta H} | \boldsymbol{q}_0 \rangle \end{aligned} \tag{37}$$

To evaluate (37) we split the temperature interval, β, into N equal slices of size ϵ (we will eventually take the $\epsilon \to 0$ or equivalently the $N \to \infty$ limit). The partition function becomes

$$\begin{aligned} Z &= \int \mathrm{d}^n \boldsymbol{q}_0 \, \langle \boldsymbol{q}_0 | \underbrace{e^{-\epsilon H} \dots e^{-\epsilon H}}_{\text{N times}} | \boldsymbol{q}_0 \rangle \\ &= \int \mathrm{d}^n \boldsymbol{q}_0 \int \mathrm{d}^n \boldsymbol{q}_1 \dots \int \mathrm{d}^n \boldsymbol{q}_{N-1} \langle \boldsymbol{q}_0 | e^{-\epsilon H} | \boldsymbol{q}_1 \rangle \langle \boldsymbol{q}_1 | \dots | \boldsymbol{q}_{N-1} \rangle \langle \boldsymbol{q}_{N-1} | e^{-\epsilon H} | \boldsymbol{q}_0 \rangle \\ &= \int \mathrm{d}^n \boldsymbol{q}_1 \dots \int \mathrm{d}^n \boldsymbol{q}_{N-1} \langle \boldsymbol{q}_0 | 1 - \epsilon H | \boldsymbol{q}_1 \rangle \langle \boldsymbol{q}_0 | \dots | \boldsymbol{q}_{N-1} \rangle \langle \boldsymbol{q}_{N-1} | 1 - \epsilon H | \boldsymbol{q}_0 \rangle + \mathcal{O}(\epsilon^2) \end{aligned} \tag{38}$$

We insert unity in terms of a complete set of momentum states,

$$\mathbb{1} = \int \mathrm{d}\boldsymbol{p} | \boldsymbol{p} \rangle \langle \boldsymbol{p} |, \tag{39}$$

N times to get

$$\begin{aligned} Z = &\int \mathrm{d}^n \boldsymbol{q}_0 \dots \int \mathrm{d}^n \boldsymbol{q}_{N-1} \int \mathrm{d}^n \boldsymbol{p}_0 \dots \int \mathrm{d}^n \boldsymbol{p}_{N-1} \\ &\times \langle \boldsymbol{q}_0 | \boldsymbol{p}_0 \rangle \langle \boldsymbol{p}_0 | (1 - \epsilon H) | \boldsymbol{q}_1 \rangle \langle \boldsymbol{q}_1 | \dots | \boldsymbol{q}_{N-1} \rangle \langle \boldsymbol{q}_{N-1} | \boldsymbol{p}_{N-1} \rangle \langle \boldsymbol{p}_{N-1} | (1 - \epsilon H) | \boldsymbol{q}_0 \rangle + \mathcal{O}(\epsilon^2), \\ = &\int \mathrm{d}^n \boldsymbol{q}_0 \dots \int \mathrm{d}^n \boldsymbol{q}_{N-1} \int \mathrm{d}^n \boldsymbol{p}_0 \dots \int \mathrm{d}^n \boldsymbol{p}_{N-1} \, \frac{e^{i \sum_{n=0}^{N-1} \boldsymbol{p}_n \cdot \boldsymbol{q}_n}}{(2\pi)^{(n/2)(N-1)}} \\ &\times \langle \boldsymbol{p}_0 | (1 - \epsilon H) | \boldsymbol{q}_1 \rangle \langle \boldsymbol{q}_1 | \dots | \boldsymbol{q}_{N-1} \rangle \langle \boldsymbol{p}_{N-1} | (1 - \epsilon H) | \boldsymbol{q}_0 \rangle + \mathcal{O}(\epsilon^2), \end{aligned} \tag{40}$$

At this point the derivation has been general. Let us specify to the standard form of the Hamiltonian,

$$\hat{H}(\hat{\boldsymbol{p}}, \hat{\boldsymbol{q}}) = \frac{1}{2m} \hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{p}} + V(\hat{\boldsymbol{q}}). \tag{41}$$

Insertion into (41) yields

$$\begin{aligned}
Z &= \int \mathrm{d}^n \boldsymbol{q}_0 \dots \int \mathrm{d}^n \boldsymbol{q}_{N-1} \int \mathrm{d}^n \boldsymbol{p}_0 \dots \int \mathrm{d}^n \boldsymbol{p}_{N-1} \frac{e^{i\sum_{n=0}^{N-1} \boldsymbol{p}_n \cdot \boldsymbol{q}_n}}{(2\pi)^{(n/2)(N-1)}} \\
&\quad \times \langle \boldsymbol{p}_0 | \boldsymbol{q}_1 \rangle (1 - \epsilon H(\boldsymbol{p}_0, \boldsymbol{q}_1)) \langle \boldsymbol{q}_1 | \dots | \boldsymbol{q}_{N-1} \rangle \langle \boldsymbol{p}_{N-1} | \boldsymbol{q}_0 \rangle (1 - \epsilon H(\boldsymbol{p}_{N-1}, \boldsymbol{q}_0)) + \mathcal{O}(\epsilon^2), \\
&= \int \mathrm{d}^n \boldsymbol{q}_0 \dots \int \mathrm{d}^n \boldsymbol{q}_{N-1} \int \mathrm{d}^n \boldsymbol{p}_0 \dots \int \mathrm{d}^n \boldsymbol{p}_{N-1} \\
&\quad \times \frac{e^{i\boldsymbol{p}_0 \cdot (\boldsymbol{q}_0 - \boldsymbol{q}_1)}}{(2\pi)^n} \dots \frac{e^{i\boldsymbol{p}_{N-1} \cdot (\boldsymbol{q}_0 - \boldsymbol{q}_1)}}{(2\pi)^n} e^{-\epsilon H(\boldsymbol{p}_0, \boldsymbol{q}_1)} \dots e^{-\epsilon H(\boldsymbol{p}_{N-1}, \boldsymbol{q}_0)} + \mathcal{O}(\epsilon^2), \\
&= \int \mathrm{d}^n \boldsymbol{q}_0 \dots \int \mathrm{d}^n \boldsymbol{q}_{N-1} \int \mathrm{d}^n \boldsymbol{p}_0 \dots \int \mathrm{d}^n \boldsymbol{p}_{N-1} \\
&\quad \times \frac{e^{i\boldsymbol{p}_0 \cdot (\boldsymbol{q}_0 - \boldsymbol{q}_1) - \epsilon\left(\boldsymbol{p}_0^2/2m + V(\boldsymbol{q}_1)\right)}}{(2\pi)^n} \dots \frac{e^{i\boldsymbol{p}_{N-1} \cdot (\boldsymbol{q}_0 - \boldsymbol{q}_1) - \epsilon\left(\boldsymbol{p}_{N-1}^2/2m + V(\boldsymbol{q}_0)\right)}}{(2\pi)^n} + \mathcal{O}(\epsilon^2), \\
&= \int \mathrm{d}^n \boldsymbol{q}_0 \dots \int \mathrm{d}^n \boldsymbol{q}_{N-1} \int \mathrm{d}^n \boldsymbol{p}_0 \dots \int \mathrm{d}^n \boldsymbol{p}_{N-1} \\
&\quad \times \frac{e^{i\boldsymbol{p}_0 \cdot (\boldsymbol{q}_0 - \boldsymbol{q}_1) - \epsilon \boldsymbol{p}_0^2/2m}}{(2\pi)^n} \dots \frac{e^{i\boldsymbol{p}_{N-1} \cdot (\boldsymbol{q}_0 - \boldsymbol{q}_1) - \epsilon \boldsymbol{p}_{N-1}^2/2m}}{(2\pi)^n} e^{-\epsilon \sum_{k=0}^{N-1} V(\boldsymbol{q}_k)} + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{42}$$

The momentum integrals can be preformed,

$$\int \frac{\mathrm{d}\boldsymbol{p}_k}{(2\pi)^N} e^{i\left(\boldsymbol{p}_k \cdot (\boldsymbol{q}_k - \boldsymbol{q}_{k+1}) + i\epsilon |\boldsymbol{p}_k|^2/2m\right)} = \left(\frac{m}{2\pi\epsilon}\right)^{n/2} e^{-\frac{m}{2\epsilon}|\boldsymbol{q}_k - \boldsymbol{q}_{k+1}|^2}, \tag{43}$$

to yield

$$\begin{aligned}
Z &= \left(\frac{m}{2\pi\epsilon}\right)^{n(N-1)/2} \int \mathrm{d}^n \boldsymbol{q}_0 \dots \int \mathrm{d}^n \boldsymbol{q}_{N-1} e^{\sum_{k=0}^{N-1}\left(-\frac{m}{2}\frac{|\boldsymbol{q}_k - \boldsymbol{q}_{k+1}|^2}{\epsilon} - \epsilon V(\boldsymbol{q}_k)\right)} + \mathcal{O}(\epsilon^2), \\
&= \left(\frac{m}{2\pi\epsilon}\right)^{n(N-1)/2} \int \mathrm{d}^n \boldsymbol{q}_0 \dots \int \mathrm{d}^n \boldsymbol{q}_{N-1} e^{-\epsilon \sum_{k=0}^{N-1}\left(\frac{m}{2}\frac{|\boldsymbol{q}_k - \boldsymbol{q}_{k+1}|^2}{\epsilon^2} + V(\boldsymbol{q}_k)\right)} + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{44}$$

This is of course the discretized form of the path integral with a Euclidean action,

$$U(\boldsymbol{q}_0, \boldsymbol{q}_N, i\beta) = \oint \mathcal{D}\boldsymbol{q}(i\beta) e^{-S_E[\boldsymbol{q}]}, \tag{45}$$

where

$$S_E[\boldsymbol{q}] = \int_0^\beta \mathrm{d}\tau \left(\frac{|\boldsymbol{p}|^2}{2m} + V(\boldsymbol{q})\right), \tag{46}$$

and $\oint$ represents the fact that the path integral is restricted to paths which start and end in the same place. Since the initial and final paths differ by a "time", β, we require that the paths in the trace be periodic in β (i.e., $\boldsymbol{q}(0) = \boldsymbol{q}(\beta)$).

9.2 (b)

We want to evaluate the integral (46) for the simple harmonic oscillator. Inserting the Fourier decomposition (know exists because from part (a) we showed that the paths in the trace are periodic) into the Euclidean Lagrangian we obtain,

$$L_E = \frac{1}{2\beta} \sum_n \sum_m x_n x_m \left(\frac{2\pi i n}{\beta}\frac{2\pi i m}{\beta} + \omega^2\right) e^{2\pi i \tau (n+m)/\beta}. \tag{47}$$

Lets first evaluate the action

$$\begin{aligned}
S_E[x(\tau)] &= \int_0^\beta \mathrm{d}\tau L_E \\
&= \int_0^\beta \mathrm{d}\tau \frac{1}{2\beta} \sum_{n,m=-\infty}^{\infty} x_n x_m \left(\frac{2\pi i n}{\beta} \frac{2\pi i m}{\beta} + \omega^2 \right) e^{2\pi i \tau (n+m)/\beta} \\
&= \frac{1}{2\beta} \sum_{n,m=-\infty}^{\infty} \left(\frac{2\pi i n}{\beta} \frac{2\pi i m}{\beta} + \omega^2 \right) (\beta \delta_{n,-m}) \\
&= \frac{1}{2} \sum_{n=-\infty}^{\infty} x_n x_{-n} \left(\frac{4\pi^2 n^2}{\beta^2} + \omega^2 \right) \\
&= \frac{1}{2} \sum_{n=-\infty}^{\infty} |x_n|^2 \left(\frac{4\pi^2 n^2}{\beta^2} + \omega^2 \right). \\
&= \frac{\omega^2}{2} x_0^2 + \sum_{n=1}^{\infty} \left(\frac{4\pi^2 n^2}{\beta^2} + \omega^2 \right) |x_n|^2.
\end{aligned} \tag{48}$$

We have used that fact that the expansion of x implies: $x_{-n} = x_n^*$ (reality condition of x). Thus, the path integral, Z, is a Gaussian

$$Z = \int \mathcal{D}x(\tau) \exp\left[-\frac{\omega^2}{2} x_0^2 - \sum_{n=1}^{\infty} \left(\frac{4\pi^2 n^2}{\beta^2} + \omega^2 \right) |x_n|^2. \right] \tag{49}$$

Here, the path integral is understood to be over the Fourier coefficients of x,

$$\begin{aligned}
Z &\propto \int \mathrm{d}x_0 \int \mathrm{dRe}x_1 \ldots \int \mathrm{dRe}x_\infty \int \mathrm{dIm}x_1 \ldots \int \mathrm{dIm}x_\infty \\
&\quad \times \exp\left[-\left(\frac{\omega^2}{2} x_0^2 + \sum_{n=1}^{\infty} \left(\frac{4\pi^2 n^2}{\beta^2} + \omega^2 \right) \left((\mathrm{Re}x_n)^2 + (\mathrm{Im}x_n)^2 \right) \right) \right], \\
&= \left(\int \mathrm{d}x_0 \exp\left[-\frac{\omega^2}{2} x_0^2 \right] \right) \left(\prod_{n=1}^{\infty} \int \mathrm{dRe}x_n \exp\left[-\left(\frac{4\pi^2 n^2}{\beta^2} + \omega^2 \right) (\mathrm{Re}x_n)^2 \right] \right) \\
&\quad \times \left(\prod_{n=1}^{\infty} \int \mathrm{dIm}x_n \exp\left[-\left(\frac{4\pi^2 n^2}{\beta^2} + \omega^2 \right) (\mathrm{Im}x_n)^2 \right] \right), \\
&= \sqrt{\frac{2\pi}{\omega^2}} \left(\prod_{n=1}^{\infty} \sqrt{\frac{\pi}{\frac{4\pi^2 n^2}{\beta^2} + \omega^2}} \right)^2, \\
&= \sqrt{\frac{2\pi}{\omega^2}} \left(\prod_{n=1}^{\infty} \frac{\beta^2}{4\pi^2 n^2} \frac{\pi}{1 + \frac{\beta^2 \omega^2}{4\pi^2 n^2}} \right), \\
&= \frac{\sqrt{2\pi}}{\omega} \left(\prod_{n=1}^{\infty} \frac{\beta^2}{4\pi n^2} \right) \frac{\beta\omega}{2} \frac{2}{\beta\omega} \left(\prod_{n=1}^{\infty} \frac{1}{1 + \frac{(\beta\omega/2)^2}{\pi^2 n^2}} \right), \\
&= \sqrt{2\pi}\beta \left(\prod_{n=1}^{\infty} \frac{\beta^2}{4\pi n^2} \right) \frac{1}{2\sinh(\beta\omega/2)}, \\
&= N(\beta) \frac{1}{2\sinh(\beta\omega/2)}.
\end{aligned} \tag{50}$$

Now the overall constant $N(\beta)$ is not well defined, however, neither was the integration measure. Dividing by β in each integral will get rid of the extra powers of β in $N(\beta)$.

Part (c)

From part (a) the field theoretic generalization follows readily:

$$Z = \langle \phi_0(x)|e^{-\beta H}|\phi_0(x)\rangle = \int \mathcal{D}_\phi e^{-S_E[\phi]} \tag{51}$$

where the ϕ are periodic, $\phi(x)|_{x^0=0} = \phi(x)_{x^0=\beta}$. The Euclidean action is obtained from the Minkowski action by Wick rotating the time component of x, $x^0 \to -ix^0$. For the free real scalar field, the Euclidean action is

$$S_E = i\int \mathrm{d}(-ix^0)\int \mathrm{d}^3x\left(-\frac{1}{2}(\partial_{E\mu}\phi)^2 - m\phi^2\right) = \int \mathrm{d}^4x\left(\frac{1}{2}(\partial_{E\mu}\phi)^2 + \frac{1}{2}m_\phi^2\phi^2\right). \tag{52}$$

To evaluate the statistical path integral we expand the field, ϕ, in its Fourier modes, as we did for the Harmonic oscillator of part (b),

$$\begin{aligned}\phi(x) &= \sum_n \frac{e^{2\pi int/\beta}}{\sqrt{\beta}}\frac{1}{\sqrt{V}}\sum_{\boldsymbol{k}} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\phi(k_n^0,\boldsymbol{k}) \\ &\equiv \sum_n \frac{e^{2\pi int/\beta}}{\sqrt{\beta}}\frac{1}{\sqrt{V}}\sum_{\boldsymbol{k}} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\phi_{n,\boldsymbol{k}}\end{aligned} \tag{53}$$

where we have taken the limit that space is a finite volume so that the $\boldsymbol{k}$ are discretized. The Fourier coefficients, $\phi_{0,\boldsymbol{k}}$ may be complex. However, since $\phi(x)$ is real, they must satisfy the reality condition $\phi_{-n,-\boldsymbol{k}} = \phi^*_{n,\boldsymbol{k}}$.

Inserting the mode expansion of ϕ into the action, we obtain

$$\begin{aligned}S_E &= \frac{1}{2}\int \mathrm{d}^4x\left(\partial_{E\mu}\phi\partial_E^\mu\phi + m_\phi^2\phi^2\right) \\ &= \frac{1}{2}\int \mathrm{d}^4x\frac{1}{\beta V}\sum_{mn}\sum_{\boldsymbol{k}\boldsymbol{k}'}\Big(\partial_{E\mu}e^{2\pi imt/\beta}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\phi_{m,\boldsymbol{k}}\partial_E^\mu e^{2\pi int/\beta}e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}}\phi_{n,\boldsymbol{k}'} \\ &\qquad\qquad + m_\phi^2 e^{2\pi imt/\beta}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\phi_{m,\boldsymbol{k}}e^{2\pi int/\beta}e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}}\phi_{n,\boldsymbol{k}'}\Big) \\ &= \frac{1}{2}\int \mathrm{d}^4x\frac{1}{\beta V}\sum_{mn}\sum_{\boldsymbol{k}\boldsymbol{k}'}e^{2\pi imt/\beta}e^{2\pi int/\beta}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}}\phi_{m,\boldsymbol{k}}\phi_{n,\boldsymbol{k}'}\left(-\frac{4\pi^2 mn}{\beta^2} - \boldsymbol{k}\cdot\boldsymbol{k}' + m_\phi^2\right) \\ &= \frac{1}{2}\sum_m\sum_{\boldsymbol{k}}\phi_{m,\boldsymbol{k}}\phi_{-m,-\boldsymbol{k}}\left(\frac{4\pi^2m^2}{\beta^2} + \boldsymbol{k}\cdot\boldsymbol{k} + m_\phi^2\right) \\ &= \frac{1}{2}\sum_m\sum_{\boldsymbol{k}}|\phi_{m,\boldsymbol{k}}|^2\left(\left(\frac{2\pi m}{\beta}\right)^2 + E_{\boldsymbol{k}}^2\right) \\ &= \frac{1}{2}|\phi_{0,\mathbf{0}}|^2 + \sum_{m>0}\sum_{\boldsymbol{k}}|\phi_{m,\boldsymbol{k}}|^2\left(\left(\frac{2\pi m}{\beta}\right)^2 + E_{\boldsymbol{k}}^2\right).\end{aligned} \tag{54}$$

Substituting the above into the statistical path integral we get

$$\begin{aligned} Z &= \int \prod_{m>0,\boldsymbol{k}>0} \mathrm{dRe}\phi_{n,\boldsymbol{k}} \mathrm{dIm}\phi_{n,\boldsymbol{k}} \exp\left\{-\frac{1}{2}|\phi_{0,\mathbf{0}}|^2 - \sum_{m>0}\sum_{\boldsymbol{k}} |\phi_{m,\boldsymbol{k}}|^2 \left(\left(\frac{2\pi m}{\beta}\right)^2 + E_{\boldsymbol{k}}^2\right)\right\} \\ &= N(\beta)\prod_{\boldsymbol{k}} \frac{1}{\sinh(\beta E_{\boldsymbol{k}}/2)} \\ &= N(\beta)\prod_{\boldsymbol{k}} \frac{e^{-\beta E_{\boldsymbol{k}}/2}}{1-e^{\beta E_{\boldsymbol{k}}}} \end{aligned} \tag{55}$$

which is just the relativistic partition function for $E_{\boldsymbol{k}}^2 = \boldsymbol{k}\cdot\boldsymbol{k} + m_\phi^2$.

To relate the partition function to a functional determinant, recall that for some operator $\hat{\mathcal{O}}$ the functional determinant is defined by

$$\int \mathcal{D}_\phi e^{\int d^4x \phi\hat{\mathcal{O}}\phi} = \text{const} \times \sqrt{\frac{1}{\det\hat{\mathcal{O}}}}. \tag{56}$$

Integrating the Euclidean action by parts we get that the partition function is

$$Z = \int \mathcal{D}_\phi \exp\left\{\frac{1}{2}\int \mathrm{d}^4x\phi\left(-\partial_E^2\phi + m_\phi^2\phi^2\right)\right\} \equiv \text{const} \times \frac{1}{\sqrt{\det\left(-\partial_E^2\phi + m_\phi^2\phi^2\right)}}. \tag{57}$$

Part (d)

We are given the Lagrangian for a fermionic harmonic oscillator,

$$\mathcal{L}_E = \bar{\psi}\dot{\psi} + \omega\bar{\psi}\psi. \tag{58}$$

The action is given by

$$S_E = \oint_0^\beta d\tau \left(\bar{\psi}\dot{\psi} + \omega\bar{\psi}\psi\right) \tag{59}$$

where the Grassman fields, $\bar{\psi}, \psi$ are anti-periodic $\bar{\psi}(\tau+\beta) = -\bar{\psi}(\tau), \psi(\tau+\beta) = -\psi(\tau)$. We can expand the anti-periodic Grassman field as

$$\psi(\tau) = \sum_{n=-\infty}^{\infty} \psi_n e^{2\pi i(n-1/2)\tau/\beta} \tag{60}$$

and the complex conjugate field as

$$\bar{\psi}(\tau) = \sum_{n=-\infty}^{\infty} \bar{\psi}_n e^{-2\pi i(n-1/2)\tau/\beta} \tag{61}$$

where ψ_n and $\bar{\psi}$ are Grassman numbers. It is easy to see that this expansion is indeed anti-periodic

$$\psi(\tau+\beta) = \sum_{n=-\infty}^{\infty} \psi_n e^{2\pi i(n-1/2)\tau/\beta} e^{2\pi i n} e^{-i\pi} = -\psi(\tau). \tag{62}$$

After substituting the expansion for the Grassman fields the Euclidean action becomes

$$S_E = \sum_{n=-\infty}^{\infty} \bar{\psi}_n\psi_n \left[\frac{2\pi i(n-1/2)}{\beta} + \omega\right]. \tag{63}$$

The functional integral is then,

$$\begin{aligned}
Z &= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-\sum_{n=-\infty}^{\infty} \bar{\psi}_n \psi_n \left[\frac{2\pi i(n-1/2)}{\beta}+\omega\right]} \\
&\propto \prod_{n=-\infty}^{\infty} \int \mathrm{d}\bar{\psi}_n \int \mathrm{d}\psi_n \; e^{-\bar{\psi}_n \psi_n \left[\frac{2\pi i(n-1/2)}{\beta}+\omega\right]} \\
&= \prod_{n=-\infty}^{\infty} \int \mathrm{d}\bar{\psi}_n \int \mathrm{d}\psi_n \; \left(1 - \bar{\psi}_n \psi_n \left[\frac{2\pi i(n-1/2)}{\beta} + \omega\right]\right) \\
&= \prod_{n=-\infty}^{\infty} \int \mathrm{d}\bar{\psi}_n \int \mathrm{d}\psi_n \; \left(1 + \psi_n \bar{\psi}_n \left[\frac{2\pi i(n-1/2)}{\beta} + \omega\right]\right) \\
&= \prod_{n=-\infty}^{\infty} \left[\frac{2\pi i(n-1/2)}{\beta} + \omega\right] \\
&= \left[\frac{-\pi i}{\beta} + \omega\right] \prod_{n=1}^{\infty} \left[\frac{2\pi i(n-1/2)}{\beta} + \omega\right] \prod_{n=1}^{\infty} \left[\frac{2\pi i(-n+1/2)}{\beta} + \omega\right] \\
&= \left[\frac{-\pi i}{\beta} + \omega\right] \prod_{n=1}^{\infty} \left[\left(\frac{2\pi(n-1/2)}{\beta}\right)^2 + \omega^2\right]. \qquad (64)
\end{aligned}$$

This can be simplified by using the infinite product definition of cosh:

$$\cosh x = \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{\pi^2(n-1/2)^2}\right). \qquad (65)$$

With this, the partition function becomes

$$Z \propto \cosh(\beta\omega/2) = e^{\beta\omega/2} + e^{-\beta\omega/2}. \qquad (66)$$

The last equation is just the partition function for a two-level system.

Part (e)

We are given the Euclidean Lagrangian for the photon field,

$$\begin{aligned}
\mathcal{L}_E &= -\frac{1}{4}(F_{\mu\nu})^2 \\
&= -\frac{1}{2}\left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu\right) \\
&= -\frac{1}{2}\partial_\mu\left(A_\nu \partial^\mu A^\nu - A_\nu \partial^\nu A^\mu\right) + \frac{1}{2}A_\nu\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)A_\mu \qquad (67)
\end{aligned}$$

where A is periodic in τ with a period of β. The Euclidean action is

$$S_E = \frac{1}{2}\int_0^\beta \mathrm{d}^4 x_E \; A_\nu\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)A_\mu \qquad (68)$$

and the partition function is

$$Z = \int \mathcal{D}_A \exp\left[-\frac{1}{2}\int_0^\beta \mathrm{d}^4 x_E \; A_\nu\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)A_\mu\right]. \qquad (69)$$

We must use the FP procedure to impose gauge invariance, so that we are only integrating over unique field configurations. Let $G(A)$ be a function of the photon field that when set to zero encodes the gauge condition. We then insert unity into the functional integral

$$Z = \int \mathcal{D}_A \int \mathcal{D}_\alpha \delta\left(G(A^\alpha)\right) \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) e^{-S_E[A]}. \tag{70}$$

where $A^\alpha_\mu = A_\mu + \frac{1}{e}\partial_\mu \alpha$. We change variables from A to A^α to get

$$Z = \int \mathcal{D}_{A^\alpha} \int \mathcal{D}_\alpha \delta\left(G(A^\alpha)\right) \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) e^{-S_E[A^\alpha]}. \tag{71}$$

Now we choose the generalized Lorenz gauge for G,

$$G(A) = \partial^\mu A_\mu(x) - \omega(x) \tag{72}$$

for any scalar function ω. Next note that the determinant is independent of A,

$$\det\left(\frac{\delta(\partial^\mu A^\alpha_\mu - \omega(x))}{\delta\alpha}\right) = \det\left(\frac{\partial^2}{e}\right). \tag{73}$$

Therefore the functional integral becomes

$$\begin{aligned} Z &= \det\left(\frac{\partial^2}{e}\right) \int \mathcal{D}_{A^\alpha} \int \mathcal{D}_\alpha \delta\left(\partial^\mu A^\alpha_\mu(x) - \omega(x)\right) e^{-S_E[A^\alpha]} \\ &= \det\left(\frac{\partial^2}{e}\right)\left(\int \mathcal{D}_\alpha\right) \int \mathcal{D}_A \delta\left(\partial^\mu A_\mu(x) - \omega(x)\right) e^{-S_E[A]}. \end{aligned} \tag{74}$$

Next we integrate over Z with a Gaussian wight centred at $\omega = 0$, to get

$$\begin{aligned} N(\xi)\int \mathcal{D}_\omega e^{-\int \mathrm{d}^4x \frac{\omega^2}{2\xi}} Z &= N(\xi)\det\left(\frac{\partial^2}{e}\right)\left(\int \mathcal{D}_\alpha\right) \int \mathcal{D}_A \int \mathcal{D}_\omega \delta\left(\partial^\mu A_\mu(x) - \omega(x)\right) \\ &\qquad \times e^{-\int \mathrm{d}^4x_E \frac{\omega^2}{2\xi}} e^{-\frac{1}{2}\int_0^\beta \mathrm{d}^4x_E\ A_\nu\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)A_\mu} \\ &= N(\xi)\det\left(\frac{\partial^2}{e}\right)\left(\int \mathcal{D}_\alpha\right) \int \mathcal{D}_A e^{-\int \mathrm{d}^4x_E \frac{(\partial^\mu A_\mu)^2}{2\xi}} e^{-\frac{1}{2}\int_0^\beta \mathrm{d}^4x_E\ A_\nu\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)A_\mu} \\ &= N(\xi)\det\left(\frac{\partial^2}{e}\right)\left(\int \mathcal{D}_\alpha\right) \int \mathcal{D}_A e^{-\frac{1}{2}\int_0^\beta \mathrm{d}^4x_E\ A_\nu\left(g^{\mu\nu}\partial^2 - (1-\frac{1}{\xi})\partial^\mu\partial^\nu\right)A_\mu} \end{aligned} \tag{75}$$

where $N(\xi)$ is a function of that normalizes the Gaussian integral and the divergent integral over the field α will cancel in the ratios that define correlation functions. Therefore we simply write

$$Z = \det\left(\frac{\partial^2}{e}\right) \int \mathcal{D}_A e^{-\frac{1}{2}\int_0^\beta \mathrm{d}^4x_E\ A_\nu\left(g^{\mu\nu}\partial^2 - (1-\frac{1}{\xi})\partial^\mu\partial^\nu\right)A_\mu}. \tag{76}$$

Choosing the Feynman gauge $\xi = 1$ the functional integral becomes

$$\begin{aligned} Z &= \det\left(\frac{\partial^2}{e}\right) \int \mathcal{D}_A e^{\frac{1}{2}\int_0^\beta \mathrm{d}^4x_E\ A_\nu \partial^2 A^\nu} \\ &= \det\left(\frac{\partial^2}{e}\right) \int \mathcal{D}_{A^0} e^{\frac{1}{2}\int_0^\beta \mathrm{d}^4x_E\ A^0\partial^2 A^0} \int \mathcal{D}_{A^1} e^{\frac{1}{2}\int_0^\beta \mathrm{d}^4x_E\ A^1\partial^2 A^1} \\ &\qquad \times \int \mathcal{D}_{A^2} e^{\frac{1}{2}\int_0^\beta \mathrm{d}^4x_E\ A^2\partial^2 A^2} \int \mathcal{D}_{A^3} e^{\frac{1}{2}\int_0^\beta \mathrm{d}^4x_E\ A^3\partial^2 A^3} \\ &\propto \det\left(\partial^2\right)\left[\frac{1}{\sqrt{\det\left(-\partial^2\right)}}\right]^4 \\ &\propto \frac{1}{\det\left(-\partial^2\right)} \end{aligned} \tag{77}$$

We can now use the results of part (c) in to evaluate the functional determinant:

$$\begin{aligned}\frac{1}{\det\left(-\partial^2\right)} &= \lim_{m\to 0}\frac{1}{\det\left(-\partial^2+m^2\right)}\\ &= \lim_{m\to 0} N(\beta)\prod_{\boldsymbol{k}>0}\frac{e^{-\beta E_{\boldsymbol{k}}/2}}{1-e^{\beta E_{\boldsymbol{k}}}}\\ &= N(\beta)\prod_{\boldsymbol{k}}\frac{e^{-\beta|\boldsymbol{k}|/2}}{1-e^{\beta|\boldsymbol{k}|}} \qquad (78)\end{aligned}$$