Summary of 1709.04891 §2 & 3

- particles are irreducible unitary rep of Poincare group
- a particle is labled by its momentum and transforms under some rep of the little group.

- awkwardness in describing particles by fields: fields transform as Lorentz tensors or spinors, while particles transform under the little group.

- Feynman diagrams have the wrong transformation properties to be called scattering amplitudes - need to introduce polarization vectors/spinors that satisfy some constraints to get a proper scattering amplitude

  Ex spin-1 massless polarization vectors do not transform genuinely as Lorentz vectors : $(\Lambda p) = p$,
  $(\Lambda \varepsilon_\pm)^\mu = e^{\pm i\theta}\varepsilon_\pm^\mu + x(\Lambda, p) p^\mu$ - that is only the gauge equivalency class $\{\varepsilon_\pm^\mu \mid \varepsilon_\pm^\mu + \alpha p^\mu\}$ is invariant under Lorentz transformations

- spinor helicity variables transform nicely under both little group & Lorentz group transformations as well as represent the constrained on-shell momenta $P_{\alpha\dot\alpha}$ by unconstrained $\lambda_\alpha \tilde\lambda_{\dot\alpha}$. - Amplitudes are functions of λ & $\tilde\lambda$

- This chapter teaches us how scattering amplitudes factorize on a pole (amp only has simple poles in s,t,u)

$$\lim_{\substack{x\to 0\\ x\in\{s,t,u\}}} x\,\mathcal{M}(s,t,u) = \mathcal{M}_L(s,t,u)\,\mathcal{M}_R(s,t,u).$$

- often residues in on channel have poles in another and the consistent factorization of amplitudes place strict constraints on it.

  Ex1 there does not exist a consistent 4pt amplitude of massless particles except for those of spin $0, \frac{1}{2}, 1, \frac{3}{2}, 2$

  Ex2 self interacting spin 1 particles need a color structure to be consistent

  Ex3 self-interacting spin 2 particles do not have a color structure

  Ex4 cannot have a consistent theory of massless charged particles with spin $\geq 3/2$

  Ex5 spin > 2 cannot consistently couple to gravity

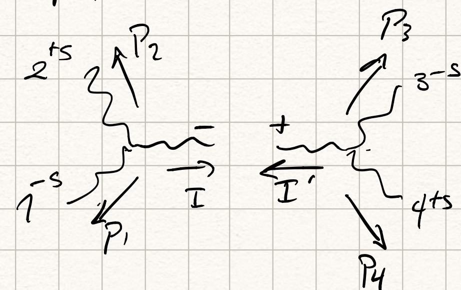
  Ex no coupling b/w 3 particles such that $h_1 + h_2 + h_3 = 0$.

3.7

Self-interacting massless particle of spin s: $A_3[1^{-s}2^{-s}3^{+s}] = \dfrac{\langle 12\rangle^{3s}}{\langle 13\rangle^s\langle 23\rangle^s}$

look at s-channel pole: $(p_1+p_2)^2 = \langle 12\rangle[12] = \langle 34\rangle[34] = (p_3+p_4)^2 \to 0$

$A_4[1^{-s}2^{+s}3^{-s}4^{+s}] = \mathcal{F}\langle 13\rangle^2[24]^2$ by little group scaling



choose $[12]\to 0$ and $\langle 34\rangle \to 0$

$[A_4] = 0 = [\mathcal{F}] + 4 \Rightarrow [\mathcal{F}] = -4$

$I = -p_1 - p_2 = p_3 + p_4$

s-channel

$$\lim_{s\to 0} sA_4 = A_3[1^{-s}2^{+s}I^{-s}]\,A_3[3^{-s}4^{+s}I'^{+s}]$$
$$= \left(\frac{\langle 1I\rangle^3}{\langle 12\rangle\langle I2\rangle}\right)^s\left(\frac{[4I]^3}{[I3][43]}\right)^s$$
$$= \left(\frac{(\langle 12\rangle[24])^3}{\langle 12\rangle\langle I2\rangle[I3][43]}\right)^s \quad ①$$
$$= \left(\frac{(-\langle 13\rangle[34])^2\cancel{\langle 12\rangle}[24]}{\cancel{\langle 12\rangle}\langle I2\rangle[I3][43]}\right)^s$$
$$= \left(\langle 13\rangle^2[24]^2\right)^s\left(\frac{[34]^2}{\langle I2\rangle[I3][43][24]}\right)^s \quad ②$$
$$= \left(\langle 13\rangle^2[24]^2\right)^s\left(\frac{\cancel{[34]^2}}{\langle 24\rangle\cancel{[43]^2}[24]}\right)^s \quad ③$$
$$= \left(\frac{\langle 13\rangle^2[24]^2}{u}\right)^s$$

① 1) $\langle 1+2\rangle[2+3\rangle[3+4\rangle[4 = 0$

$\langle 12\rangle[24] + \langle 13\rangle[34] = 0$

$\langle 12\rangle[24] = -\langle 13\rangle[34]$

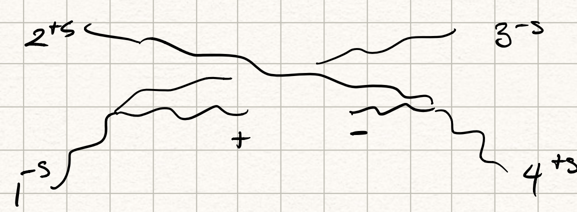
② 3) $[3+4\rangle[4+I\rangle[I = 0$

$\langle 24\rangle[43] + \langle 2I\rangle[I3] = 0$

$\langle I2\rangle[I3] = +\langle 24\rangle[43]$

③ $u = \langle 13\rangle[13] = \langle 24\rangle[24]$

u-channel



$$\lim_{u\to 0} u A_4[1^{-s}2^{+s}3^{-s}4^{+s}]$$
$$= A_3[1^{-s}3^{-s}I^{+s}]\,A_3[2^{+s}4^{+s}I^{-s}]$$
$$= \left(\frac{\langle 13\rangle^3}{\langle 1I\rangle\langle 3I\rangle}\right)^s\left(\frac{[24]^3}{[2I][4I]}\right)^s$$
$$= \left(\langle 13\rangle^2[24]^2\right)^s\left(\frac{\langle 13\rangle[24]}{\langle 1I\rangle\langle 3I\rangle[2I][4I]}\right)^s$$
$$= \left(\langle 13\rangle^2[24]^2\right)^s\left(\frac{\cancel{\langle 13\rangle}\cancel{[24]}}{\cancel{\langle 13\rangle}[32]\langle 32\rangle\cancel{[24]}}\right)^s$$
$$= \left(\frac{\langle 13\rangle[24]^2}{t}\right)^s$$

1) $[1+3\rangle[3+I\rangle[I = 0$

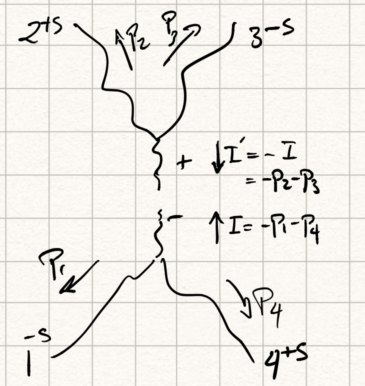
2) $[2+4\rangle[4+I\rangle[I = 0$

$\langle 1I\rangle[2I] = \langle 13\rangle[32]$

$\langle 3I\rangle[4I] = \langle 32\rangle[24]$

$t = \langle 14\rangle[14] = \langle 23\rangle[23]$

the t channel pole: $t=(p_1+p_4)^2 = \langle 14\rangle[14] = \langle 23\rangle[23] = (p_2+p_3)^2 \to 0$



$$\lim_{t\to 0} t\, A_4[1^{-s} 2^{+s} 3^{-s} 4^{+s}]$$

$$= A_3[1^{-s} I^{-s} 4^{+s}]\; A_3[2^{+s} I^{+s} 3^{-s}]$$

$$= \left(\frac{\langle 1I\rangle^3}{\langle 14\rangle\langle I4\rangle} \frac{[2I]^3}{[23][I3]} \right)^s$$

$$= \left(\frac{-\langle 13\rangle^2 [32]^2 \cancel{\langle 14\rangle}[42]}{\cancel{\langle 14\rangle}\cancel{[23]} \langle I4\rangle [I3]} \right)^s$$

$$= \left(\frac{+\langle 13\rangle^2 \; [32][24]}{\langle I4\rangle [I3]} \right)^s$$

$$= \left(\frac{\langle 13\rangle^2 [24]^2}{s} \right)^s$$

$2\rangle[2 + 3\rangle[3 + I\rangle[I = 0$

$1\rangle[1 + 4\rangle[4 + I\rangle[I = 0$

$$R_s = \left(\frac{\langle 13\rangle^2 [24]^2}{s} \right)^s \qquad R_u = \left(\frac{\langle 13\rangle [24]^2}{t} \right)^s \qquad R_t = \left(\frac{\langle 13\rangle [24]^2}{s} \right)$$

3.10

We know that

$$\lim_{s\to 0} s\mathcal{F}\,\langle 13\rangle^2[24]^2 = \frac{\langle 13\rangle^2[24]^2}{u} = R_s$$

$$\lim_{t\to 0} t\mathcal{F}\,\langle 13\rangle^2[24]^2 = \frac{\langle 13\rangle^2[24]^2}{s} = R_t$$

$$\lim_{u\to 0} u\mathcal{F}\,\langle 13\rangle^2[24]^2 = \frac{\langle 13\rangle^2[24]^2}{t} = R_u$$

Residues of one channel have poles in the other channels.

$$\Rightarrow \mathcal{F} = \left(\frac{A}{st} + \frac{B}{tu} + \frac{C}{us}\right)$$

as $s\to 0$ $t\to -u$ $\Rightarrow$ the residue in the s channel is:

$$\lim_{s\to 0} s\left(\frac{A}{us} + \frac{B}{tu} + \frac{C}{us}\right)\langle 13\rangle^2[24]^2$$

$$= \left(\frac{C}{u} - \frac{A}{u}\right)\langle 13\rangle^2[24]^2$$

$$= \frac{\langle 13\rangle^2[24]^2}{u} \Rightarrow C-A=1$$

as $t\to 0$, $u\to -s$ and:

$$\lim_{t\to 0} A_4 = \left(\frac{A}{s} - \frac{B}{s}\right)\langle 13\rangle^2[24]^2$$

$$= \frac{\langle 13\rangle^2[24]^2}{s} \Rightarrow A-B=1$$

No solution

$\Downarrow$

need more than one flavour.

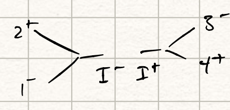
as $u\to 0$, $s\to -t$, and:

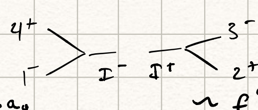
$$\lim_{u\to 0} A_4 = \left(\frac{B}{t} - \frac{C}{t}\right)\langle 13\rangle^2[24]^2$$

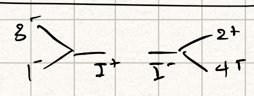
$$= \frac{\langle 13\rangle^2[24]^2}{t} \Rightarrow B-C=1$$

Self interaction: $A_3[1^{-s}2^{+s}3^{-s}] \to f^{a_1a_2a_3}\left(\frac{\langle 13\rangle^3}{\langle 12\rangle\langle 23\rangle}\right)^s$.

we know that f must be totally asym b/c the brackets are totally asym and this is an amp for bosons.

$\Rightarrow R_s \to f^{a_1a_2b}f^{a_3a_4b}R_s$ $\longrightarrow$  $\sim f^{a_1a_2b}f^{ba_3a_4}$

$R_t \to$ R_t $\longrightarrow$  $\sim f^{a_1a_4b}f^{ba_3a_2}$

$R_u \to$ R_u $\longrightarrow$  $\sim f^{a_1ba_3}f^{a_2ba_4}$

Ansatz: $\mathcal{F} = \left(\frac{A^{a_1a_2a_3a_4}}{st} + \frac{B^{a_1a_2a_3a_4}}{tu} + \frac{C^{a_1a_2a_3a_4}}{us}\right)$

from $s\to 0$: $C^{a_1a_2a_3a_4} - A^{a_1a_2a_3a_4} = f^{a_1a_2b}f^{ba_3a_4}$

" $t\to 0$: $A^{a_1a_2a_3a_4} - B^{a_1a_2a_3a_4} = f^{a_1a_4b}f^{ba_3a_2} = f^{a_2a_3b}f^{ba_4a_1}$

" $u\to 0$: $B^{a_1a_2a_3a_4} - C^{a_1a_2a_3a_4} = f^{a_1ba_3}f^{a_2ba_4} = f^{a_1a_3b}f^{ba_2a_4}$

3.12

$S=2$: would think that there is a $1/u^2$ pole in the s-channel residue. but there is only as $s \to \infty$. We could just as well write $1/u^2 = -1/tu$

$\Rightarrow$ unique 4pt amplitude : $-\dfrac{\langle 13\rangle^4 [24]^4}{stu.}$

$$A_3[1^{-2}\,2^{-2}\,3^{+2}] = \left(\frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle}\right)^2$$

We consider the 4pt. amp : $A_4[1^{-2}\,2^{+2}\,3^{-2}\,4^{+2}] = \mathcal{F}\,(\langle 13\rangle\,[24])^4$

$$R_s = \left(\frac{\langle 13\rangle^2[24]^2}{u}\right)^2, \quad R_t = \left(\frac{\langle 13\rangle^2[24]^2}{s}\right)^2, \quad R_u = \left(\frac{\langle 13\rangle[24]^2}{t}\right)^2$$

$$\Rightarrow \mathcal{F} = \left(\frac{A}{su^2} + \frac{B}{ts^2} + \frac{C}{ut^2}\right)$$

but using $s+t+u=0$, in the $s\to 0$, $t\to 0$, and $u\to 0$ limits we can write

$$\mathcal{F} = -\frac{(A+B+C)}{stu}$$

$$= -\frac{1}{stu}$$

and

$$A_4 = -\frac{\langle 34\rangle^4[24]^2}{stu}$$

3.19 3 pt. amp for gravity and 2 spin-S particles

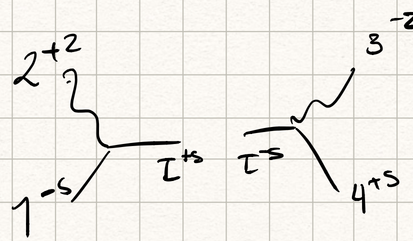
$$A_3[1^{-s}\,2^{-2}\,3^{+s}] = g\langle 12\rangle^{2s+2}\langle 23\rangle^{2-2s}\langle 31\rangle^{-2}$$

$$A_3[1^{+s}\,2^{+2}\,3^{-s}] = g[12]^{2s+2}[23]^{2-2s}[31]^{-2}$$

note that $P_I = -p_1 - p_2 = p_3 + p_4 \Rightarrow [2|P_I|3\rangle = \begin{cases} -[2|1|3\rangle = +\langle 31\rangle[12] \\ [2|4|3\rangle = -\langle 34\rangle[42] \end{cases}$

$$\Rightarrow [2|P_I|3\rangle = \tfrac{1}{2}[2|(P_4 - P_1)|3\rangle$$

S-Channel Residue



1) $[1 + 2)[2 + I)[I = 0$
3) $[3 + 4)[4 + I)[I = 0$
1) $[1 + 2)[2 + 3)[3 + 4)[4 = 0$

$$\frac{R_s}{g^2} = \frac{A_3[I^{+s}\,2^{+2}\,1^{-s}]\;A_3[I^{-s}\,3^{-2}\,4^{+s}]}{g^2}$$

$$= \frac{[I2]^{2s+2}[21]^{2-2s}}{[1I]^2}\;\frac{\langle I3\rangle^{2s+2}\langle 34\rangle^{2-2s}}{\langle I4\rangle^2}$$

$$= \frac{(\langle 3I\rangle[I2])^{2s+2}(\langle 34\rangle[12])^{2-2s}}{(\langle 4I\rangle[I1])^2}$$

$$= \frac{(\langle 13\rangle[12])^{2s+2}(\langle 34\rangle[12])^{2-2s}}{(\langle 24\rangle[12])^{-2}}$$

$$= \frac{[12]^{2s+2\,\cancel{+2-2s-2}}\langle 13\rangle^{2s+2}\langle 34\rangle^{2-2s}}{\langle 24\rangle^2}$$

$$= \left(\frac{\langle 13\rangle[24]}{[24]}\right)^{2s}\frac{[12]^2\langle 34\rangle^2\langle 13\rangle^2}{\langle 24\rangle^2\langle 34\rangle^{2s}}$$

$$= (\langle 13\rangle[24])^{2s}\frac{[12]^2\langle 34\rangle^2\langle 13\rangle^2}{([24]\langle 34\rangle)^{2s}\langle 24\rangle^2}\;\frac{([24]\langle 34\rangle)^4}{([24]\langle 34\rangle)^4}$$

$$= (\langle 13\rangle[24])^{2s}[2|4|3\rangle^{4-2s}\frac{\cancel{[12]^2}\cancel{\langle 34\rangle^2}\cancel{\langle 13\rangle^2}}{\langle 24\rangle^2[24]^2\cancel{\langle 34\rangle^2}\cancel{\langle 13\rangle^2}\cancel{[12]^2}}$$

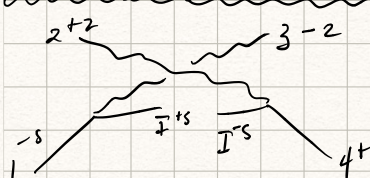
$$= \frac{(\langle 13\rangle[24])^{2s}[2|4|3\rangle^{4-2s}}{u^2} \longrightarrow -\frac{(\langle 13\rangle[24])^{2s}[2|4|3\rangle^{4-2s}}{ut}$$

$\langle 3I\rangle[I2\rangle = -\langle 31\rangle[12]$
$= -\langle 34\rangle[42\rangle$

$\langle 4I\rangle[I1] = -\langle 42\rangle[21]$
$= -\langle 43\rangle[31]$

$\langle 34\rangle[24\rangle = [2|4|3\rangle$

U-Channel Residue



1) $[1 + 3)[3 + I)[I = 0$
2) $[2 + 4)[4 + I)[I = 0$

$$= (\text{s-channel})_{2\leftrightarrow 3}$$

$$= \frac{(\langle 13\rangle[24])^{2s}[2|4|3\rangle^{4-2s}}{(u|_{2\leftrightarrow 3})^2}$$

$$= \frac{(\langle 13\rangle[24])^{2s}[2|4|3\rangle^{4-2s}}{s^2}$$

The long way:

$$\frac{R_u}{g^2} = \frac{A_3[1^{-s}\, 3^{-2}\, I^{+s}]\ A_3[4^{+s}\, 2^{+2}\, I^{-s}]}{g^2}$$

$$= \frac{\langle 13\rangle^{2s+2}\langle 3I\rangle^{2-2s}}{\langle I1\rangle^2}\ \frac{[42]^{2s+2}\,[2I]^{2-2s}}{[I4]^2}$$

$$= \frac{(\langle 3I\rangle[I2])^{2-2s}\ (\langle 13\rangle[24])^{2s+2}}{(\langle 1I\rangle[I4])^2}$$

$$= (\langle 13\rangle[24])^{2s}\ \frac{\cancel{\langle 13\rangle^2}[24]^2}{\cancel{\langle 13\rangle^2}[34]^2}\ (\langle 34\rangle[42])^{2-2s}$$

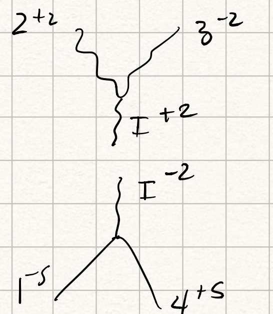
$$= \frac{(\langle 13\rangle[24])^{2s}\ [2|4|3\rangle^{4-2s}}{[34]^2\ \langle 34\rangle^2}$$

$$= \frac{(\langle 13\rangle[24])^{2s}\ [2|4|3\rangle^{4-2s}}{s^2} \to -\frac{(\langle 13\rangle[24])^{2s}\ [2|4|3\rangle^{4-2s}}{st}$$

$\langle 1I\rangle[I4] = -\langle 13\rangle[34]$
$= -\langle 12\rangle[24]$

$\langle 3I\rangle[I2] = -\langle 31\rangle[12]$
$= -\langle 34][42]$

the "t"-channel:



$$A_3[1^{-2}2^{+2}3^{-2}] = \kappa\langle 12\rangle^{-2}\langle 23\rangle^{-2}\langle 31\rangle^{6}$$

$$A_3[1^{-s}\,2^{-2}\,3^{+s}] = g\langle 12\rangle^{2s+2}\langle 23\rangle^{2-2s}\langle 31\rangle^{-2}$$

1) $[1 + 4\rangle[4 + I\rangle[I = 0$
2) $[2 + 3\rangle[3 + I\rangle[I = 0$

$\langle 4I\rangle[I3] = -\langle 41\rangle[13]$
$= -\langle 42\rangle[23]$

$\langle 1I\rangle[I3] = -\langle 14\rangle[43] = -\langle 12\rangle[23]$

$\langle 1I\rangle[I2] = -\langle 14\rangle[42] = -\langle 13\rangle[32]$

$\langle 4I\rangle[I2] = -\langle 41\rangle[12]$
$= -\langle 43\rangle[32]$

$$[2|P_I|3\rangle = \begin{cases} -[2|1|3\rangle = +\langle 31\rangle[12] \\ \ \ [2|4|3\rangle = -\langle 34\rangle[42] \end{cases}$$

$$\frac{R_t}{\kappa g} = \frac{A_3[1^{-s}\, I^{-2}\, 4^{+s}]}{g}\ \frac{A_3[2^{+2}\, 3^{-2}\, I^{+2}]}{\kappa}$$

$$= \frac{\langle 1I\rangle^{2s+2}\langle I4\rangle^{2-2s}}{\langle 41\rangle^2}\ \frac{[I2]^6}{[23]^2\,[3I]^2}$$

$$= \frac{\langle 1I\rangle^{2s}}{\langle 4I\rangle^{2s}}\ \frac{\langle 1I\rangle^2\langle I4\rangle^2\,[I2]^6}{[3I]^2}\ \frac{1}{\langle 41\rangle^2[23]^2}$$

$$= \left(\frac{\langle 1I\rangle[I2]}{\langle 4I\rangle[I2]}\right)^{2s}\ \frac{\langle 1I\rangle^2\langle I4\rangle^2[I2]^6}{[3I]^2}\ \frac{1}{\langle 41\rangle^2[23]^2}$$

$$= \frac{\langle 13\rangle^{2s}\cancel{[32]^{2s}}}{\langle 43\rangle^{2s}\cancel{[32]^{2s}}}\ \frac{\langle 1I\rangle^2\langle I4\rangle^2[I2]^6}{[3I]^2}\ \frac{1}{\langle 41\rangle^2[23]^2}$$

$$= \frac{\langle 13\rangle^{2s}\,[24]^{2s}}{\langle 43\rangle^{2s}\,[24]^{2s}}\ \frac{\langle 1I\rangle^2\langle I4\rangle^2[I2]^6}{[3I]^2}\ \frac{1}{\langle 41\rangle^2[23]^2}$$

$$= (\langle 13\rangle[24])^{2s}\ [2|4|3\rangle^{-2s}\ \underbrace{\frac{\langle 1I\rangle^2\langle I4\rangle^2[I2]^6}{[3I]^2}}\ \frac{1}{\langle 41\rangle^2[23]^2}$$

$$(\langle 4I\rangle^2[I2]^2)\ (\langle 1I\rangle^2[I2]^2)\ \frac{[I2]^2}{[3I]^2}$$

$$\langle 43\rangle^2[32]^2\ \ \langle 14\rangle^2[42]^2\ \frac{(\langle 1I\rangle[I2])^2}{(\langle 1I\rangle[I2])^2}$$

$$\frac{(\langle 43\rangle[42])^4 \; (\langle 14\rangle[23])^2}{(\langle 12\rangle[12])^2}$$

$$\frac{[2|4|3\rangle^4 \; (\langle 14\rangle[23])^2}{s^2}$$

$$\frac{R_t}{\kappa g} = \frac{(\langle 13\rangle[24])^{2s} \; [2|4|3\rangle^{-2s+4}}{s^2}$$

$$R_s = \frac{-(\langle 13\rangle[24])^{2s} \; [2|4|3\rangle^{4-2s}}{ut} \; g^2$$

$$R_u = \frac{-(\langle 13\rangle[24])^{2s} \; [2|4|3\rangle^{4-2s}}{st} \; g^2$$

$$R_t = \frac{-(\langle 13\rangle[24])^{2s} \; [2|4|3\rangle^{4-2s}}{su} \; g\kappa$$

We can make a consistent amp if $g = \kappa$ and

$$A_4 \propto \kappa^2 \frac{(\langle 13\rangle[24])^{2s} \; [2|4|3\rangle^{4-2s}}{stu}$$