Phys 731: String Theory - Assignment 3

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Problem 1

a) Evaluate the OPE

$$T_{zz}(z) : e^{ik\cdot X(0,0)} :$$

using Wick's theorem. Expand the exponential, then sum over contractions between T and the sum (there will be zero, one, or two contractions), and then do the sum, which should simplify if you've done the combinatorics right. Finally, Taylor expand inside the normal ordering, and get all singular terms. Show that the exponential is a tensor, of the weight given in class.

b) Do the same for

$$: e^{ik_1\cdot X(z,\bar{z})} :: e^{ik_2\cdot X(0,0)} :$$

to recover the result given in class. Carry the Taylor expansion further, up to and including terms down by $z^2, \bar{z}^2, z\bar{z}$ in the OPE.

Part (a)

For the free scalar field, the holomorphic worldsheet energy-momentum function is

$$T_{zz}(z) = -\frac{1}{\alpha'} : \partial X^\mu(z) \partial_\mu X(z) : .$$

Therefore, we are looking for the OPE to

$$-\frac{1}{\alpha'} : \partial X^\mu(z) \partial_\mu X(z) :: e^{ik\cdot X(0,0)} : .$$

The product of two normal ordered operators is given by equation (2.2.10) of JBBS

$$: \mathcal{F} :: \mathcal{G} : =: \mathcal{F}\mathcal{G} : + \sum \text{cross-contractions.}$$

With $\mathcal{F} = \partial X^\mu(z) \partial X_\mu(z)$ and $\mathcal{G} = e^{ik\cdot X(0,0)}$ we obtain

$$T_{zz}(z) : e^{ik\cdot X(0,0)} := -\frac{1}{\alpha'} : \partial X^\mu(z) \partial X_\mu(z) e^{ik\cdot X(0,0)} : -\frac{1}{\alpha'} \sum \text{cross-contractions.}$$

The remaining unknown is just the sum of cross-contractions

$$\begin{aligned}\sum \text{cross-contractions} &\equiv \sum_{\mathrm{CC}} \left[\partial X^\mu(z)\partial X_\mu(z), e^{ik\cdot X(0,0)}\right]_{\mathrm{CC}} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathrm{CC}} \left[\partial X^\mu(z)\partial X_\mu(z), (ik\cdot X(0,0))^n\right]_{\mathrm{CC}} \end{aligned} \tag{1}$$

where $\sum_{\mathrm{CC}} [\mathcal{F},\mathcal{G}]_{\mathrm{CC}}$ denotes the sum of all cross-contractions of the operators $\mathcal{F}$ and $\mathcal{G}$. The last ingredient that we will need in for calculating the sum of all cross-contractions is the basic cross-contraction

$$[\partial X^\mu(z), X^\alpha(0,0)] = -\frac{\alpha'}{2}\eta^{\mu\alpha}\partial \ln |z|^2 = -\frac{\alpha'}{2z}\eta^{\mu\alpha}.$$

In equation (1) there will be terms with zero, one and two contractions:

1) For $n=0$, there are no contractions

$$\sum_{\mathrm{CC}} [\partial X^\mu(z)\partial X_\mu(z), 1]_{\mathrm{CC}} = 0.$$

2) For $n=1$, there are only terms with one contraction

$$\begin{aligned}\sum_{\mathrm{CC}} [\partial X^\mu(z)\partial X_\mu(z), ik\cdot X(0,0)]_{\mathrm{CC}} &= ik_\alpha\eta_{\mu\nu}\,[\partial X^\mu(z), X^\alpha(0,0)]_{\mathrm{CC}} :\partial X^\nu(z): \\ &\quad + ik_\alpha\eta_{\mu\nu}\,[\partial X^\nu(z), X^\alpha(0,0)]_{\mathrm{CC}} :\partial X^\mu(z): \\ &= \left(-\frac{\alpha'}{2z}\right) ik_\alpha :\partial X^\alpha(z): + \left(-\frac{\alpha'}{2z}\right) ik_\alpha :\partial X^\alpha(z): \\ &= \left(-\frac{\alpha'}{z}\right) ik_\alpha :\partial X^\alpha(z): . \end{aligned}$$

3) For $n \geq 2$, the cross-contraction sum is

$$\sum_{\mathrm{CC}} [\partial X^\mu(z)\partial X_\mu(z), (ik\cdot X(0,0))^n]_{\mathrm{CC}} = ik_{\alpha_1}\ldots ik_{\alpha_n} \sum_{\mathrm{CC}} [\partial X^\mu(z)\partial X_\mu(z), X^{\alpha_1}(0,0)\ldots X^{\alpha_n}(0,0)]_{\mathrm{CC}}.$$

Note that sum of cross-contractions is completely symmetric in the indices $\alpha_1\ldots\alpha_n$. Furthermore, there are terms with one contraction and terms with two:

(a) Single cross-contraction:

$$\begin{aligned}&\sum_{\mathrm{CC}} [\partial X^\mu(z)\partial X_\mu(z), (ik\cdot X(0,0))^n]_{\text{single CC}} \\ &= ik_{\alpha_1}\ldots ik_{\alpha_n} \sum_{\mathrm{CC}} [\partial X^\mu(z)\partial X_\mu(z), X^{\alpha_1}(0,0)\ldots X^{\alpha_n}(0,0)]_{\text{single CC}} \\ &= n ik_{\alpha_1}\ldots ik_{\alpha_n} :\partial X^\mu(z) X^{\alpha_2}(0,0)\ldots X^{\alpha_n}(0,0): [\partial X_\mu(z), X^{\alpha_1}(0,0)]_{\mathrm{CC}} \\ &\quad + n ik_{\alpha_1}\ldots ik_{\alpha_n} :\partial X_\mu(z) X^{\alpha_2}(0,0)\ldots X^{\alpha_n}(0,0): [\partial X^\mu(z), X^{\alpha_1}(0,0)]_{\mathrm{CC}} \\ &= n\left(-\frac{\alpha'}{z}\right) ik_\mu :\partial X^\mu(z)\,(ik\cdot X(0,0))^{n-1}: \end{aligned}$$

The factor of n comes from the fact that each ∂X can be paired up with X in n different ways.

(b) Double cross-contraction:

$$
\begin{aligned}
&\sum_{\mathrm{CC}} \left[\partial X^{\mu}(z)\partial X_{\mu}(z), (ik\cdot X(0,0))^{n}\right]_{\text{double CC}} \\
&= ik_{\alpha_1}\dots ik_{\alpha_n}\sum_{\mathrm{CC}}\left[\partial X^{\mu}(z)\partial X_{\mu}(z), X^{\alpha_1}(0,0)\dots X^{\alpha_n}(0,0)\right]_{\text{double CC}} \\
&= n\,(n-1)\,ik_{\alpha_1}\dots ik_{\alpha_n} : X^{\alpha_3}(0,0)\dots X^{\alpha_n}(0,0) : \left[\partial X_{\mu}(z), X^{\alpha_1}(0,0)\right]_{\mathrm{CC}}\left[\partial X^{\mu}(z), X^{\alpha_2}(0,0)\right]_{\mathrm{CC}} \\
&= n\,(n-1)\left(-k^2\right)\left(-\frac{\alpha'}{2z}\right)^2 : (ik\cdot X(0,0))^{n-2} :
\end{aligned}
$$

Summing up the single cross-contraction terms we obtain

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\mathrm{CC}}\left[\partial X^{\mu}(z)\partial X_{\mu}(z), (ik\cdot X(0,0))^{n}\right]_{\text{single CC}} \\
&= \left(-\frac{\alpha'}{z}\right) ik_{\alpha} : \partial X^{\alpha}(z) : + \sum_{n=2}^{\infty}\frac{n}{n!}\left(-\frac{\alpha'}{z}\right) ik_{\mu} : \partial X^{\mu}(z)\,(ik\cdot X(0,0))^{n-1} : \\
&= \left(-\frac{\alpha'}{z}\right)\left[ik_{\alpha} : \partial X^{\alpha}(z) : + \sum_{n=2}^{\infty}\frac{1}{(n-1)!} ik_{\mu} : \partial X^{\mu}(z)\,(ik\cdot X(0,0))^{n-1} :\right] \\
&= \left(-\frac{\alpha'}{z}\right)\left[ik_{\alpha} : \partial X^{\alpha}(z) : + \sum_{n=1}^{\infty}\frac{1}{n!} ik_{\alpha} : \partial X^{\alpha}(z)\,(ik\cdot X(0,0))^{n} :\right] \\
&= \left(-\frac{\alpha'}{z}\right) ik_{\alpha} : \partial X^{\alpha}(z) e^{ik\cdot X(0,0)} : .
\end{aligned}
$$

Summing the double cross-contractions yields

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\mathrm{CC}}\left[\partial X^{\mu}(z)\partial X_{\mu}(z), (ik\cdot X(0,0))^{n}\right]_{\text{double CC}} \\
&= -\frac{\alpha'^2 k^2}{4z^2}\sum_{n=2}^{\infty}\frac{1}{(n-2)!} : (ik\cdot X(0,0))^{n-2} : \\
&= -\frac{\alpha'^2 k^2}{4z^2} : e^{ik\cdot X(0,0)} : .
\end{aligned}
$$

Substituting the single and double cross-contraction sums into (1), we obtain

$$
\begin{aligned}
T_{zz}(z) : e^{ik\cdot X(0,0)} : &= -\frac{1}{\alpha'} : \partial X^{\mu}(z)\partial X_{\mu}(z) e^{ik\cdot X(0,0)} : -\frac{1}{\alpha'}\sum \text{cross-contractions} \\
&= -\frac{1}{\alpha'} : \partial X^{\mu}(z)\partial X_{\mu}(z) e^{ik\cdot X(0,0)} : \\
&\quad + \frac{ik_{\alpha}}{z} : \partial X^{\alpha}(z) e^{ik\cdot X(0,0)} : \\
&\quad + \frac{\alpha' k^2}{4z^2} : e^{ik\cdot X(0,0)} : .
\end{aligned} \tag{2}
$$

To finish calculating an the OPE we Taylor expand inside the normal ordering to obtain all singular terms. The only term that needs to be expanded is

$$
\partial X^{\alpha}(z) = \partial X^{\alpha}(0) + z\partial^2 X^{\alpha}(0) + \mathcal{O}\left(z^2\right).
$$

Then

$$: \partial X^\mu(z)\partial X_\mu(z)e^{ik\cdot X(0,0)} := \text{sum of non-singular terms},$$
$$\frac{ik_\alpha}{z} : \partial X^\alpha(z)e^{ik\cdot X(0,0)} := \frac{ik_\alpha}{z} : \partial X^\alpha(0)e^{ik\cdot X(0,0)} : +\text{non-singular terms},$$

and

$$T_{zz}(z) : e^{ik\cdot X(0,0)} := \frac{\alpha' k^2}{4z^2} : e^{ik\cdot X(0,0)} : + \frac{ik_\alpha}{z} : \partial X^\alpha(0)e^{ik\cdot X(0,0)} : +\text{non-singular terms}.$$

In equation (2.4.16) of JBBS, it is shown that any tensor operator of weight $(h,\tilde{h})$ will have an OPE with T_{zz} and $T_{\bar{z}\bar{z}}$ of the form

$$T_{zz}(z) : \mathcal{O}(0,0) = \frac{h}{z^2}\mathcal{O}(0,0) + \frac{1}{z}\partial\mathcal{O}(0,0) + \text{non-singular terms}$$

with a similar expression for the OPE with $T_{\bar{z}\bar{z}}$. Comparing the above to our result (2), we see that $h = \alpha' k^2/4$ and a similar calculation tells us that $\tilde{h} = h$. Therefore, the operator $: e^{ik\cdot X(0,0)} :$ is a tensor of weight $\left(\alpha' k^2/4, \alpha' k^2/4\right)$.

Part (b)

Since there are no derivatives of fields in the product $: e^{ik_1\cdot X(z,\bar{z})} :: e^{ik_2\cdot X(0,0)} :$, we can use formula (2.2.8) of JBBS

$$: \mathcal{F} :: \mathcal{G} := \exp\left(-\frac{\alpha'}{4}\int dz_1 \int dz_2 \, \ln|z_1 - z_2|^2 \frac{\delta}{\delta X_{\mathcal{F}}(z_1,\bar{z}_1)} \frac{\delta}{\delta X_{\mathcal{G}}(z_2,\bar{z}_2)}\right) : \mathcal{F}\mathcal{G} :$$

to evaluate this OPE. Thus,

$$\begin{aligned}
: e^{ik_1\cdot X(z,\bar{z})} :: e^{ik_2\cdot X(0,0)} : &=: \exp\left(-\frac{\alpha'}{4}\int dz_1 \int dz_2 \, \ln|z_1 - z_2|^2 \, \eta^{\mu\nu} \frac{\delta}{\delta X^\mu_{\mathcal{F}}(z_1,\bar{z}_1)} \frac{\delta}{\delta X^\nu_{\mathcal{G}}(z_2,\bar{z}_2)}\right) e^{ik_1\cdot X(z,\bar{z})} e^{ik_2\cdot X(0,0)} : \\
&=: e^{ik_1\cdot X(z,\bar{z})} e^{ik_2\cdot X(0,0)} \exp\left(-\frac{\alpha'}{4}\int d^2z_1 \int d^2z_2 \, \ln|z_1 - z_2|^2 \, \eta^{\mu\nu} \frac{\delta\left(ik_1\cdot X(z,\bar{z})\right)}{\delta X^\mu(z_1,\bar{z}_1)} \frac{\delta\left(ik_2\cdot X(0,0)\right)}{\delta X^\nu(z_2,\bar{z}_2)}\right) : \\
&=: e^{ik_1\cdot X(z,\bar{z})} e^{ik_2\cdot X(0,0)} \exp\left(\frac{\alpha'}{4} k_1\cdot k_2 \int d^2z_1 \int d^2z_2 \, \ln|z_1 - z_2|^2 \, \delta^{(2)}\left(z_1 - z; \bar{z}_1 - \bar{z}\right) \delta^{(2)}\left(z_2;\bar{z}_2\right)\right) : \\
&=: e^{ik_1\cdot X(z,\bar{z})} e^{ik_2\cdot X(0,0)} e^{\frac{\alpha'}{4}(k_1\cdot k_2)\ln|z|^2} : \\
&=: |z|^{\alpha'(k_1\cdot k_2)/2} \, e^{ik_1\cdot X(z,\bar{z})} e^{ik_2\cdot X(0,0)} :
\end{aligned}$$

The factor $|z|^{\alpha'(k_1\cdot k_2)/2}$ could be singular. We therefore expand the exponential to get a few of the less singular terms

$$\begin{aligned}
: e^{ik_1\cdot X(z,\bar{z})} :: e^{ik_2\cdot X(0,0)} : &= |z|^{\alpha'(k_1\cdot k_2)/2} : e^{ik_1\cdot X(0,0) + zik_1\cdot\partial X(0) + \bar{z}ik_1\cdot\bar{\partial}X(0) + \mathcal{O}\left(z^2,\bar{z}^2\right)} e^{ik_2\cdot X(0,0)} : \\
&= |z|^{\alpha'(k_1\cdot k_2)/2} : e^{i(k_1+k_2)\cdot X(0,0)} \sum_{n=0}^{\infty} \left(zik_1\cdot\partial X(0) + \bar{z}ik_1\cdot\bar{\partial}X(0) + \mathcal{O}(z^2,\bar{z}^2)\right)^n : \\
&= z^{\frac{\alpha'(k_1\cdot k_2)}{4}} \bar{z}^{\frac{\alpha'(k_1\cdot k_2)}{4}} : e^{i(k_1+k_2)\cdot X(0,0)} : \\
&\quad + iz^{\frac{\alpha'(k_1\cdot k_2)}{4}+1} \bar{z}^{\frac{\alpha'(k_1\cdot k_2)}{4}} : (k_1\cdot\partial X(0))\, e^{i(k_1+k_2)\cdot X(0,0)} : \\
&\quad + iz^{\frac{\alpha'(k_1\cdot k_2)}{4}} \bar{z}^{\frac{\alpha'(k_1\cdot k_2)}{4}+1} : \left(k_1\cdot\bar{\partial}X(0)\right) e^{i(k_1+k_2)\cdot X(0,0)} : \\
&\quad - z^{\frac{\alpha'(k_1\cdot k_2)}{4}+2} \bar{z}^{\frac{\alpha'(k_1\cdot k_2)}{4}} : (k_1\cdot\partial X(0))^2 \, e^{i(k_1+k_2)\cdot X(0,0)} : \\
&\quad - z^{\frac{\alpha'(k_1\cdot k_2)}{4}} \bar{z}^{\frac{\alpha'(k_1\cdot k_2)}{4}+2} : \left(k_1\cdot\bar{\partial}X(0)\right) e^{i(k_1+k_2)\cdot X(0,0)} : \\
&\quad - 2z^{\frac{\alpha'(k_1\cdot k_2)}{4}+1} \bar{z}^{\frac{\alpha'(k_1\cdot k_2)}{4}+1} : \left(k_1\cdot\bar{\partial}X(0)\right)(k_1\cdot\partial X(0))\, e^{i(k_1+k_2)\cdot X(0,0)} : + \mathcal{O}\left(|z|^3\right)
\end{aligned}$$

Problem 2 (2.8 of JBBS)

What is the weight of $f_{\mu\nu} : \partial X^\mu \partial X^\nu e^{ik\cdot X}$:? What are the conditions of $f_{\mu\nu}$ and $k_\mu$in order for it to be a tensor?

We could take the OPE of $f_{\mu\nu} : \partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X}$: with T and $\bar{T}$ to find its weight. Or we could make a rigid rotation and see how $f_{\mu\nu} : \partial X^\mu \partial X^\nu e^{ik\cdot X}$: transforms. Since we already know the weights of ∂X^μ and $: e^{ik\cdot X} :$, $(1,0)$ and $\left(\frac{\alpha' k^2}{4}, \frac{\alpha' k^2}{4}\right)$ respectively, under $z \to \zeta z$ and $\bar{z} \to \bar{\zeta}\bar{z}$ we obtain

$$f_{\mu\nu} : \partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X} :\to \zeta^{1+\frac{\alpha' k^2}{4}} \bar{\zeta}^{1+\frac{\alpha' k^2}{4}} f_{\mu\nu} : \partial X^\mu \partial X^\nu e^{ik\cdot X} : .$$

Thus, the weight is $\left(1 + \frac{\alpha' k^2}{4}, 1 + \frac{\alpha' k^2}{4}\right)$.

The weight is always the coefficient of the z^{-2} or $\bar{z}^{-2}$ terms in the OPE expansion of $T(z)\mathcal{O}(0,0)$ or $\bar{T}(\bar{z})\mathcal{O}(0,0)$. For a general operator, the OPE expansion may have more singular terms than z^{-2} or $\bar{z}^{-2}$. However, the OPE of a tensor with T or $\bar{T}$ does not have terms more singular than z^{-2} or $\bar{z}^{-2}$.

We must work out the OPE with T and $\bar{T}$ to find conditions on $f_{\mu\nu}$ and k_μ.

$$\begin{aligned}
T(z) : \partial X^\mu(0)\bar{\partial} X^\nu(0) e^{ik\cdot X(0,0)} :&= -\frac{1}{\alpha'} : \partial X^\alpha(z)\partial X_\alpha(z) :: \partial X^\mu(0)\bar{\partial} X^\nu(0) e^{ik\cdot X(0,0)} : \\
&= -\frac{1}{\alpha'} : \partial X^\alpha(z)\partial X_\alpha(z)\partial X^\mu(0)\bar{\partial} X^\nu(0) e^{ik\cdot X(0,0)} : + \sum \text{cross-contractions} \\
&= -\frac{\alpha'}{2z^3} ik^\mu : \bar{\partial} X^\nu(0) e^{ik\cdot X(0,0)} : + \frac{\alpha' k^2/4 + 1}{z^2} : \partial X^\mu(0)\bar{\partial} X^\nu(0) e^{ik\cdot X(0,0)} : \\
&\quad + \frac{ik_\alpha}{z} : \partial X^\alpha(z)\partial X^\mu(0)\bar{\partial} X^\nu(0) e^{ik\cdot X(0,0)} : + \text{non-singular terms.}
\end{aligned}$$

By symmetry we also have the OPE with $\bar{T}$

$$\begin{aligned}
\bar{T}(\bar{z}) : \partial X^\mu(0)\bar{\partial} X^\nu(0) e^{ik\cdot X(0,0)} : &= -\frac{\alpha'}{2\bar{z}^3} ik^\nu : \partial X^\mu(0) e^{ik\cdot X(0,0)} : + \frac{\alpha' k^2/4 + 1}{\bar{z}^2} : \partial X^\mu(0)\bar{\partial} X^\nu(0) e^{ik\cdot X(0,0)} : \\
&\quad + \frac{ik_\alpha}{\bar{z}} : \bar{\partial} X^\alpha(\bar{z})\partial X^\mu(0)\bar{\partial} X^\nu(0) e^{ik\cdot X(0,0)} : + \text{non-singular terms.}
\end{aligned}$$

From the above OPEs we see that $f_{\mu\nu}k^\mu = f_{\mu\nu}k^\nu = 0$.

The intermediate results used to calculate this OPE are listed below:

- The sum over all contractions factorizes into sums with zero, one or two contractions:

$$\begin{aligned}
\sum \text{cross-contractions} &= -\frac{\eta_{\alpha\beta}}{\alpha'} \sum_{CC} \left[\partial X^\alpha(z)\partial X^\beta(z), \partial X^\mu(0)\bar{\partial} X^\nu(0) e^{ik\cdot X(0,0)}\right]_{CC} \\
&= -\frac{\eta_{\alpha\beta}}{\alpha'} \sum_{\text{single-CC}} \left[\partial X^\alpha(z)\partial X^\beta(z), \partial X^\mu(0)\bar{\partial} X^\nu(0) e^{ik\cdot X(0,0)}\right]_{\text{single-CC}} \\
&\quad - \frac{\eta_{\alpha\beta}}{\alpha'} \sum_{\text{double-CC}} \left[\partial X^\alpha(z)\partial X^\beta(z), \partial X^\mu(0)\bar{\partial} X^\nu(0) e^{ik\cdot X(0,0)}\right]_{\text{double-CC}}
\end{aligned}$$

- Single cross-contraction sum:

$$-\frac{\eta_{\alpha\beta}}{\alpha'}\sum_{\text{single-CC}}\left[\partial X^\alpha(z)\partial X^\beta(z),\partial X^\mu(0)\bar\partial X^\nu(0)e^{ik\cdot X(0,0)}\right]_{\text{single-CC}}$$

$$= -\frac{2}{\alpha'}\left[\partial X^\alpha(z),\partial X^\mu(0)\right]_{CC} :\partial X_\alpha(z)\bar\partial X^\nu(0)e^{ik\cdot X(0,0)}: -\frac{2}{\alpha'}\sum_{CC}\left[\partial X^\alpha(z),e^{ik\cdot X(0,0)}\right]_{CC} :\partial X_\alpha(z)\partial X^\mu(0)\bar\partial X^\nu(0):$$

$$= \frac{1}{z^2}:\partial X^\mu(z)\bar\partial X^\nu(0)e^{ik\cdot X(0,0)}: -\frac{2}{\alpha'}:\sum_{n=1}^\infty\frac{1}{n!}\left(nik_\rho\left[\partial X^\alpha(z),X^\rho(0,0)\right]_{CC}\right)\partial X_\alpha(z)\partial X^\mu(0)\bar\partial X^\nu(0)\left(ik\cdot X(0,0)\right)^{n-1}:$$

$$= \frac{1}{z^2}:\partial X^\mu(z)\bar\partial X^\nu(0)e^{ik\cdot X(0,0)}: +\frac{ik_\alpha}{z}:\partial X^\alpha(z)\partial X^\mu(0)\bar\partial X^\nu(0)e^{ik\cdot X(0,0)}:$$

- Double cross-contraction sum

$$-\frac{\eta_{\alpha\beta}}{\alpha'}\sum_{\text{double-CC}}\left[\partial X^\alpha(z)\partial X^\beta(z),\partial X^\mu(0)\bar\partial X^\nu(0)e^{ik\cdot X(0,0)}\right]_{\text{double-CC}}$$

$$= -\frac{2\eta_{\alpha\beta}}{\alpha'}\sum_{\text{CC}}\left[\partial X^\alpha(z),\partial X^\mu(0)\right]_{\text{CC}}\left[\partial X^\beta(z),e^{ik\cdot X(0,0)}\right]_{\text{CC}} :\bar\partial X^\nu(0):$$

$$-\frac{\eta_{\alpha\beta}}{\alpha'}\sum_{\text{CC}}\left[\partial X^\alpha(z)\partial X^\beta(z),e^{ik\cdot X(0,0)}\right]_{\text{double-CC}} :\partial X^\mu(0)\bar\partial X^\nu(0):$$

$$= -\frac{2\eta_{\alpha\beta}}{\alpha'}\sum_{\text{CC}}\left[\partial X^\alpha(z),\partial X^\mu(0)\right]_{\text{CC}}\left[\partial X^\beta(z),e^{ik\cdot X(0,0)}\right]_{\text{CC}} :\bar\partial X^\nu(0):$$

$$-\frac{\eta_{\alpha\beta}}{\alpha'}\sum_{n=0}^\infty\frac{1}{n!}\sum_{\text{CC}}\left[\partial X^\alpha(z)\partial X^\beta(z),\left(ik\cdot X(0,0)\right)^n\right]_{\text{double-CC}} :\partial X^\mu(0)\bar\partial X^\nu(0):$$

$$= -\frac{2\eta_{\alpha\beta}}{\alpha'}ik_\rho\left[\partial X^\alpha(z),\partial X^\mu(0)\right]_{\text{CC}}\left[\partial X^\beta(z),X^\rho(0,0)\right]_{\text{CC}} :\bar\partial X^\nu(0)\sum_{n=1}^\infty\frac{1}{n!}n\left(ik\cdot X(0,0)\right)^{n-1}:$$

$$-\frac{\eta_{\alpha\beta}}{\alpha'}ik_{\rho_1}ik_{\rho_2}\left[\partial X^\alpha(z),X^{\rho_1}(0,0)\right]_{\text{CC}}\left[\partial X^\beta(z),X^{\rho_2}(0,0)\right]_{\text{CC}} :\partial X^\mu(0)\bar\partial X^\nu(0)\sum_{n=2}^\infty\frac{1}{n!}n\left(n-1\right)\left(ik\cdot X(0,0)\right)^{n-2}:$$

$$= -\frac{\alpha'}{2z^3}ik^\mu:\bar\partial X^\nu(0)e^{ik\cdot X(0,0)}: +\frac{\alpha'}{4z^2}k^2:\partial X^\mu(0)\bar\partial X^\nu(0)e^{ik\cdot X(0,0)}:$$

- $\left[\partial X^\mu(z),X^\alpha(0,0)\right] = -\frac{\alpha'}{2}\eta^{\mu\alpha}\partial\ln|z|^2 = -\frac{\alpha'}{2z}\eta^{\mu\alpha}$
- $\left[\partial X^\mu(z),\partial X^\alpha(0)\right] = -\frac{\alpha'}{2}\eta^{\mu\alpha}\lim_{z'\to 0}\left[\partial\partial'\ln|z-z'|^2\right] = -\frac{\alpha'}{2z^2}\eta^{\mu\alpha}$
- $\left[\partial X^\mu(z),\bar\partial X^\alpha(0)\right] = 0$

Problem 3 (4.1 of JBBS)

Extend the old covariant quantization, for $A=-1$ and general D, to the second excited level of the **open** string. Verify the assertions made in the text about extra positive- and negative-norm states.

Basically, the covariantly quantized Hilbert space is larger than the physical Hilbert space. The process of old covariant quantization is the identification of the physical Hilbert space.

The Virasoro generators generate the left-over gauge symmetry. Therefore, the Virasoro generators must act trivially on physical states $|\psi\rangle, |\psi'\rangle$

$$\langle\psi|L_m|\psi\rangle = 0. \tag{3}$$

The weakest condition that we can impose that satisfies the above constraint is

$$L_m|\psi\rangle = 0 \text{ for } m > 0. \tag{4}$$

Then, for $m < 0$ the matrix element (3) vanishes by action on the left: $L_m^\dagger = L_{-m}$. Also, for L_0 we need

$$L_0|\psi\rangle = -A|\psi\rangle \tag{5}$$

where A is a possible ordering constant. Any state satisfying equations (4) and (5) is called *physical*. Note that any state of the form $L_{-n}|\chi\rangle$ for $n > 0$ is orthogonal to all physical states

$$\langle\psi|L_{-n}|\chi\rangle = \langle L_n\psi|\chi\rangle = 0.$$

Such states are called *spurious*. A spurious state that is physical is called *null* – a state that is orthogonal to all physical states including itself (such states are essentially zero). We therefore need an equivalence class relation. Two physical states, $|\psi\rangle$ and $|\psi'\rangle$, are equivalent $|\psi\rangle \cong |\psi'\rangle$ if $|\psi\rangle - |\psi'\rangle$ is null. The set of equivalence classes is called the *observable spectrum*.

- *Virasoro generators:*
  - $L_{m\neq 0} = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha^\mu_{m-n}\alpha_{n\mu}$
  - $L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha^\mu_{-n}\alpha_{n\mu}$
  - Note that we have not included the ordering constant A in L_0. Instead we have choose to make the A dependence of explicit in the condition (5).
- *Level* 0:
  - The only states at this level are $|0;k\rangle$. They are annihilated by $L_{m>0}$ because there are no lower states.
  - There are also no spurious states because there are not any lower states to raise.
  - The physical condition becomes:

    $$(L_0 + A)\,|0,k\rangle = 0.$$

    Using the fact that $L_0|\psi\rangle = \alpha' k^2$, we must have $A = -\alpha' k^2$.
  - $m_0^2 = -k^2 = A/\alpha'$
- *Level* 1:
  - At level 1 we have D possible states: $|e,k\rangle = e_\mu\alpha^\mu_{-1}|0;k\rangle$.
  - The condition (4) yields

    $$\begin{aligned} L_1|e,k\rangle &= \sqrt{2\alpha'}k\cdot e|0,k\rangle = 0 \\ &\implies e_\mu k^\mu = 0 \end{aligned}$$

    where we have used

    $$\begin{aligned} L_1|e,k\rangle &= \frac{e_\nu}{2}\left(\sum_{n=-\infty}^{\infty}\alpha^\mu_{m-n}\alpha_{n\mu}\right)\alpha^\nu_{-1}|0,k\rangle \\ &= \frac{e_\nu}{2}\left(\alpha^\mu_1\alpha_{0\mu} + \alpha^\mu_0\alpha_{1\mu}\right)\alpha^\nu_{-1}|0,k\rangle \\ &= e_\nu\left(\alpha_{1\mu}\alpha^\nu_{-1}\right)\alpha^\mu_0|0,k\rangle \\ &= \sqrt{2\alpha'}k_\mu e_\nu\left(\eta^{\mu\nu} + \alpha^\nu_{-1}\alpha^\mu_1\right)|0,k\rangle \\ &= \sqrt{2\alpha'}k\cdot e|0,k\rangle. \end{aligned}$$

- The L_0 condition (5) becomes

$$(L_0 + A)\,|e,k\rangle = \left(\alpha' k^2 + 1 + A\right)|e,k\rangle = 0$$
$$\implies A = -\alpha' k^2 - 1$$

  where we have used

$$\begin{aligned} L_0|e,k\rangle &= \left(\alpha_0^2 + \sum_{n>0} \alpha^\mu_{-n}\alpha_{n\mu}\right) e_\mu \alpha^\mu_{-1}|0,k\rangle \\ &= \left(\alpha_0^2 + \alpha^\nu_{-1}\alpha_{1\nu}\right) e_\mu \alpha^\mu_{-1}|0,k\rangle \\ &= e_\mu \left(\alpha' \alpha^\mu_{-1}\alpha_0^2|0,k\rangle + \alpha^\nu_{-1}\left(\eta^\mu_\nu + \alpha^\mu_{-1}\alpha_{1\nu}\right)|0,k\rangle\right) \\ &= e_\mu \left(\alpha' k^2 + 1\right)\alpha^\mu_{-1}|0,k\rangle \\ &= \left(\alpha' k^2 + 1\right)|e,k\rangle. \end{aligned}$$

- The mass of the $|e,k\rangle$ sate is $m_1^2 = -k^2 = (A+1)/\alpha'$. There are three cases for invariant mass:
  - For $A > -1$ the invariant mass is positive. However, a consistent theory of interactions with $A > -1$ is not known.
  - For $A < -1$ the invariant mass is negative. Clearly, this is not physical.
  - Hence, we concentrate on the case where $A = -1$. Here, the level 1 invariant mass is zero, $m_1^2 = 0$.
- Spurious states are of the form $L_{-1}|0,k\rangle = \sqrt{2\alpha'}k \cdot \alpha_{-1}|0,k\rangle$. Spurious states have $e \propto k$ and are null if $k^2 = 0$. Since $m_1^2 = 0$, $k^2 = 0$ for all level 1 states and the spurious states become physical (null).
- The observable spectrum has

$$k^2 = 0, \qquad e \cdot k = 0, \qquad e^\mu \cong e^\mu + \gamma k^\mu.$$

  Going to a reference frame for which $k_\mu = (1,1,0,\dots)$ or equivalently $k^\mu = (-1,1,0,\dots)$ implies that $e^0 = -e^1$ and $\left(e^0, -e^0, e^2, \dots\right) \cong \left(e^0 - \gamma, -e^0 + \gamma, e^2, \dots\right)$. Choosing $\gamma = e^0$ we obtain $e^\mu = \left(0,0,e^2,\dots\right)$. Therefore, there are a total of $D-2$ physical states at level 1. This is the same as in light-cone quantization.

- *Level* 2:
  - At level 2 we have the states $|f,e,k\rangle = \left(f_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + e_\mu\alpha^\mu_{-2}\right)|0,k\rangle$ where $f_{\mu\nu}$ is symmetric. Since $f_{\mu\nu}$ is symmetric, there are $(D+1)\,D/2{+}D$ states. The norm of a general state is $\langle f,e,k|f,e,k'\rangle = 2\left(f^*_{\mu\nu}f^{\mu\nu} + e^*_\mu e^\mu\right)\langle 0,k|0,k'\rangle$.
  - The L_0 condition

$$(L_0 - 1)\,|f,e,k\rangle = \left(\alpha' k^2 + 1\right)|f,e,k\rangle = 0$$
$$\implies m_2^2 = -k^2 = 1/\alpha'$$

  where

$$\begin{aligned} L_0|f,e,k\rangle &= \left(\frac{1}{2}\alpha_0^2 + \sum_{n=1}^\infty \alpha^\rho_{-n}\alpha_{n\rho}\right)\left(f_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + e_\mu\alpha^\mu_{-2}\right)|0,k\rangle \\ &= \left(\frac{1}{2}\alpha_0^2 + \alpha^\rho_{-1}\alpha_{1\rho} + \alpha^\rho_{-2}\alpha_{2\rho}\right)\left(f_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + e_\mu\alpha^\mu_{-2}\right)|0,k\rangle \\ &= \alpha' k^2|f,e,k\rangle + f_{\mu\nu}\alpha^\rho_{-1}\alpha_{1\rho}\alpha^\mu_{-1}\alpha^\nu_{-1}|0,k\rangle + e_\mu\alpha^\rho_{-2}\alpha_{2\rho}\alpha^\mu_{-2}|0,k\rangle \\ &= \alpha' k^2|f,e,k\rangle + f_{\mu\nu}\left(\alpha^\mu_{-1}\alpha^\nu_{-1} + \alpha^\nu_{-1}\alpha^\mu_{-1} + \alpha^\rho_{-1}\alpha^\mu_{-1}\alpha^\nu_{-1}\alpha_{1\rho}\right)|0,k\rangle \\ &\quad + e_\mu\left(2\alpha^\mu_{-2} + \alpha^\rho_{-2}\alpha^\mu_{-2}\alpha_{2\rho}\right)|0,k\rangle \\ &= \alpha' k^2|f,e,k\rangle + 2f_{\mu\nu}\left(\alpha^\mu_{-1}\alpha^\nu_{-1}\right)|0,k\rangle + 2e_\mu\alpha^\mu_{-2}|0,k\rangle \\ &= \left(\alpha' k^2 + 2\right)|f,e,k\rangle. \end{aligned}$$

– The L_1 condition gives e in terms of f and removes D degrees of freedom

$$\begin{aligned} L_1|f,e,k\rangle &= 2\left(e_\nu + \sqrt{2\alpha'}f_{\mu\nu}k^\mu\right)\alpha^\nu_{-1}|0,k\rangle = 0 \\ &\Longrightarrow e_\nu + \sqrt{2\alpha'}f_{\mu\nu}k^\mu = 0 \end{aligned} \tag{6}$$

where we have used

$$\begin{aligned} L_1|f,e,k\rangle &= \frac{1}{2}\left(\sum_{n=-\infty}^{\infty}\alpha^\rho_{1-n}\alpha_{n\rho}\right)\left(f_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + e_\mu\alpha^\mu_{-2}\right)|0,k\rangle \\ &= \frac{1}{2}\left(\alpha^\rho_2\alpha_{-1\rho} + \alpha^\rho_1\alpha_{0\rho} + \alpha^\rho_0\alpha_{1\rho} + \alpha^\rho_{-1}\alpha_{2\rho}\right)\left(f_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + e_\mu\alpha^\mu_{-2}\right)|0,k\rangle \\ &= \left(\alpha_{-1\rho}\alpha^\rho_2 + \sqrt{2\alpha'}k_\rho\alpha^\rho_1\right)\left(f_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + e_\mu\alpha^\mu_{-2}\right)|0,k\rangle \\ &= e_\mu\alpha_{-1\rho}\alpha^\rho_2\alpha^\mu_{-2}|0,k\rangle + \sqrt{2\alpha'}f_{\mu\nu}k_\rho\alpha^\rho_1\alpha^\mu_{-1}\alpha^\nu_{-1}|0,k\rangle \\ &= e_\mu\left(2\alpha^\mu_{-1} + \alpha_{-1\rho}\alpha^\mu_{-2}\alpha^\rho_2\right)|0,k\rangle + \sqrt{2\alpha'}f_{\mu\nu}k_\rho\left(\eta^{\rho\mu}\alpha^\nu_{-1} + \eta^{\rho\nu}\alpha^\mu_{-1} + \alpha^\mu_{-1}\alpha^\nu_{-1}\alpha^\rho_1\right)|0,k\rangle \\ &= 2\left(e_\nu + \sqrt{2\alpha'}f_{\mu\nu}k^\mu\right)\alpha^\nu_{-1}|0,k\rangle. \end{aligned}$$

– The L_2 condition gives a condition on the trace of f

$$\begin{aligned} L_2|f,e,k\rangle &= 2\left(\sqrt{2\alpha'}k\cdot e + f^\mu_\mu\right)|0,k\rangle = 0 \\ &\Longrightarrow \sqrt{2\alpha'}k\cdot e + f^\mu_\mu = 0 \\ &\Longrightarrow f^\mu_\mu = 2\alpha' f_{\mu\nu}k^\mu k^\nu \end{aligned} \tag{7}$$

where we have used

$$\begin{aligned} L_2|f,e,k\rangle &= \frac{1}{2}\left(\sum_{n=-\infty}^{\infty}\alpha^\rho_{2-n}\alpha_{n\rho}\right)\left(f_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + e_\mu\alpha^\mu_{-2}\right)|0,k\rangle \\ &= \left(\alpha^\rho_0\alpha_{2\rho} + \frac{1}{2}\alpha^\rho_1\alpha_{1\rho}\right)\left(f_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + e_\mu\alpha^\mu_{-2}\right)|0,k\rangle \\ &= e_\mu\alpha_{0\rho}\alpha^\rho_2\alpha^\mu_{-2}|0,k\rangle + \frac{1}{2}f_{\mu\nu}\alpha^\rho_1\alpha_{1\rho}\alpha^\mu_{-1}\alpha^\nu_{-1}|0,k\rangle \\ &= \sqrt{2\alpha'}k_\rho e_\mu\left(2\eta^{\rho\mu} + \alpha^\mu_{-2}\alpha^\rho_2\right)|0,k\rangle + \frac{1}{2}f_{\mu\nu}\left(\eta^\mu_\rho\eta^{\rho\nu} + \eta^\mu_\rho\alpha^\nu_{-1}\alpha^\rho_1 + \eta^\nu_\rho\eta^{\rho\mu} + \eta^\nu_\rho\alpha^\mu_{-1}\alpha^\rho_1 + \alpha^\rho_1\alpha^\mu_{-1}\alpha^\nu_{-1}\alpha_{1\rho}\right)|0,k\rangle \\ &= \left(2\sqrt{2\alpha'}k\cdot e + f^\mu_\mu\right)|0,k\rangle. \end{aligned}$$

– In the rest frame, $k_0 = 1/\sqrt{\alpha'}$ and $k_i = 0$, equations (6) and (7) imply $e_\nu = \sqrt{2}f_{0\nu}$ and $f_{ii} = 5f_{00}$.

– There are $D+1$ spurious states at this level

$$\begin{aligned} |a,b;k\rangle &= L_{-1}\underbrace{a_\mu\alpha^\mu_{-1}|0,k\rangle}_{|a,k\rangle} + bL_{-2}|0,k\rangle \\ &= a_\mu\left(\alpha_{-1}\cdot\alpha_0\,\alpha^\mu_{-1} + \alpha_{-2}\cdot\alpha_1\,\alpha^\mu_{-1}\right)|0,k\rangle + b\left(\alpha_0\cdot\alpha_{-2} + \frac{1}{2}\alpha_{-1}\cdot\alpha_{-1}\right)|0,k\rangle \\ &= a_\mu\left(\sqrt{2\alpha'}k_\nu\alpha^\nu_{-1}\alpha^\mu_{-1} + \left(\alpha^\mu_{-2} + \alpha^\mu_{-1}\,\alpha_{-2}\cdot\alpha_1\right)\right)|0,k\rangle + b\left(\sqrt{2\alpha'}k\cdot\alpha_{-2} + \frac{1}{2}\alpha_{-1}\cdot\alpha_{-1}\right)|0,k\rangle \\ &= \sqrt{2\alpha'}k_\nu a_\mu\alpha^\nu_{-1}\alpha^\mu_{-1} + a_\mu\alpha^\mu_{-2}|0,k\rangle + \sqrt{2\alpha'}bk\cdot\alpha_{-2}|0,k\rangle + \frac{b}{2}\alpha_{-1}\cdot\alpha_{-1}|0,k\rangle \\ &= \left(\sqrt{2\alpha'}k_\nu a_\mu + \frac{b}{2}\eta_{\mu\nu}\right)\alpha^\nu_{-1}\alpha^\mu_{-1}|0,k\rangle + \left(a_\mu + \sqrt{2\alpha'}bk_\mu\right)\alpha^\mu_{-2}|0,k\rangle \\ &\equiv |f',e',k\rangle \end{aligned}$$

where

$$\begin{aligned}f'_{\mu\nu} &= \sqrt{2\alpha'}k_\nu a_\mu + \frac{b}{2}\eta_{\mu\nu} \\ e'_\mu &= a_\mu + \sqrt{2\alpha'}bk_\mu.\end{aligned}$$

– In the rest frame,

$$\begin{aligned}e'_\mu &= a_\mu + \sqrt{2\alpha'}bk_\mu = a_\mu + \sqrt{2}b\delta_{0,\mu} \\ e'_\mu &= \sqrt{2}f'_{0\mu} = \left(2a_0 - \sqrt{2}b\right)\delta_{0,\mu} \\ f_{00} &= \frac{1}{5}f_{ii} = \frac{1}{5}\left(\frac{b}{2}\eta_{ii}\right) = \frac{b}{10}\left(D-1\right)\end{aligned}$$

$$e'^{*}\cdot e' = \left|2a_0 - \sqrt{2}b\right|^2 = \left|a\right|^2 + \sqrt{2}a^{0*}b + \sqrt{2}a_0 b^* + 2\left|b\right|^2$$

– In the rest frame, the states with non-negative norm are those for which $a_0 = b = 0$.

Problem 4

For the X^μ CFT, show that the operators $L'_m = L_m + (m+1)v_\mu\alpha^\mu_m$ also satisfy a Virasoro algebra, and finds its central charge (this should be relatively simple using the known result for the algebra of the L_m's). This has an interpretation in terms of strings moving in a certain nontrivial spacetime.

We wish to show that the operators L'_m satisfy a Virasoro algebra:

$$[L'_m, L'_n] = (m-n)\,L'_{m+n} + \frac{c'}{12}\left(m^3 - m\right)\delta_{m,-n}$$

for some central charge c'. To show that L'_m satisfies the above algebra, we calculate the L'_m commutator using the properties of the L_m and α^μ_n

$$\begin{aligned}[L'_m, L'_n] &= [L_m + (m+1)v_\mu\alpha^\mu_m, L_n + (n+1)v_\mu\alpha^\mu_n] \\ &= [L_m, L_n] + (n+1)v_\mu\,[L_m, \alpha^\mu_n] + (m+1)v_\mu\,[\alpha^\mu_m, L_n] + (m+1)v_\mu(n+1)v_\nu\,[\alpha^\mu_m, \alpha^\nu_n] \\ &= \left((m-n)\,L_{m+n} + \frac{c}{12}\left(m^3-m\right)\delta_{m,-n}\right) + (n+1)v_\mu\,[L_m, \alpha^\mu_n] - (m+1)v_\mu\,[L_n, \alpha^\mu_m] - (m^3-m)v^2\delta_{m,-n} \\ &= (m-n)\,L_{m+n} - n(n+1)v_\mu\alpha^\mu_{m+n} + m(m+1)v_\mu\alpha^\mu_{m+n} + \left(m^3-m\right)\left(\frac{c}{12} - v^2\right)\delta_{m,-n} \\ &= (m-n)\,L_{m+n} + \left(m^2+m-n^2-n\right)v_\mu\alpha^\mu_{m+n} + \left(m^3-m\right)\left(\frac{c}{12} - v^2\right)\delta_{m,-n} \\ &= (m-n)\,L_{m+n} + (m-n)\,(m+n+1)\,v_\mu\alpha^\mu_{m+n} + \left(m^3-m\right)\frac{c-12v^2}{12}\delta_{m,-n} \\ &= (m-n)\,L'_{m+n} + \frac{c'}{12}\left(m^3-m\right)\delta_{m,-n}.\end{aligned}$$

Thus, we see that L'_{m+n} satisfies a Virasoro algebra with central charge $c' = c - 12v^2$.

We have used the following facts in the derivation of $[L'_m, L'_n]$:

- The normal ordering of the creation and annihilation operators is defined so that all annihilation operators (α_m^μ with $m > 0$) are to the left of the creations operators (α_m^μ with $m < 0$)

$$
\begin{aligned}
L_m &= \frac{1}{2} \sum_{l=-\infty}^{\infty} {}^{\circ}_{\circ}\alpha_{m-l}^{\nu}\alpha_{l\nu}{}^{\circ}_{\circ} \\
&= \begin{cases} \frac{1}{2}\sum_{l=-\infty}^{\infty} \alpha_{m-l}^{\nu}\alpha_{l\nu} & \text{if } m \neq 0 \\ \frac{1}{2}\alpha_0^\nu\alpha_{0\nu} + \sum_{l=1}^{\infty} \alpha_{-l}^{\nu}\alpha_{l\nu} & \text{if } m = 0 \end{cases}
\end{aligned}
$$

- The commutator of L_m with α_n^μ is

$$
\begin{aligned}
[L_m, \alpha_n^\mu] &= \left[\frac{1}{2}\sum_{l=-\infty}^{\infty} {}^{\circ}_{\circ}\alpha_{m-l}^{\nu}\alpha_{l\nu}{}^{\circ}_{\circ}, \alpha_n^\mu\right] \\
&= \begin{cases} \frac{1}{2}\sum_{l=-\infty}^{\infty} \left[\alpha_{m-l}^{\nu}\alpha_{l\nu}, \alpha_n^\mu\right] & \text{if } m \neq 0 \\ \left[\frac{1}{2}\alpha_0^\nu\alpha_{0\nu} + \sum_{l=1}^{\infty} \alpha_{-l}^{\nu}\alpha_{l\nu}, \alpha_n^\mu\right] & \text{if } m = 0 \end{cases} \\
&= \begin{cases} \frac{1}{2}\sum_{l=-\infty}^{\infty} \left(\alpha_{m-l}^{\nu}\left[\alpha_{l\nu}, \alpha_n^\mu\right] + \left[\alpha_{m-l}^{\nu}, \alpha_n^\mu\right]\alpha_{l\nu}\right) & \text{if } m \neq 0 \\ \frac{1}{2}\alpha_{0\mu}\cancel{\left[\alpha_0^\nu, \alpha_n^\mu\right]} + \frac{1}{2}\cancel{\left[\alpha_0^\nu, \alpha_n^\mu\right]}\alpha_{0\nu} + \sum_{l=1}^{\infty}\alpha_{-l}^{\nu}\left[\alpha_{l\nu}, \alpha_n^\mu\right] + \sum_{l=1}^{\infty}\left[\alpha_{-l}^{\nu}, \alpha_n^\mu\right]\alpha_{l\nu} & \text{if } m = 0 \end{cases} \\
&= \begin{cases} \frac{1}{2}\sum_{l=-\infty}^{\infty} \left(l\delta_{l,-n}\eta_\nu^\mu\alpha_{m-l}^\nu + (m-l)\,\delta_{m-l,-n}\eta^{\mu\nu}\alpha_{l\nu}\right) & \text{if } m \neq 0 \\ \sum_{l=1}^{\infty} l\delta_{l,-n}\eta_\nu^\mu\alpha_{-l}^\nu + \sum_{l=1}^{\infty}(-l)\,\delta_{-l,-n}\eta^{\mu\nu}\alpha_{l\nu} & \text{if } m = 0 \end{cases} \\
&= \begin{cases} -n\alpha_{m+n}^\mu & \text{if } m \neq 0 \\ -n\alpha_n^\mu & \text{if } m = 0 \end{cases} \\
&= -n\alpha_{m+n}^\mu
\end{aligned}
$$

Problem 5

Consider a CFT with the mode algebra

$$
\begin{aligned}
\{b_m, c_n\} &= \delta_{m+n,0}, \\
\{b_m, b_n\} &= \{c_m, c_n\} = 0.
\end{aligned}
$$

Show that the operators

$$
L_m = \sum_{n=-\infty}^{\infty} (2m - n)\, b_n c_{m-n} \text{ for } m \neq 0
$$

$$
L_0 = -1 + \sum_{n=1}^{\infty} n\left(b_{-n}c_n + c_{-n}b_n\right),
$$

satisfy a Virasoro algebra, and determine its central charge. I suggest that, as in class, you pick out the operator (single-commutator) terms first, and get the constant by acting with $[L_m, L_{-m}]$ on some particular state, e.g., the one annihilated by b_n and c_n for $n > 0$ and by c_0 (though any state will give the same answer).

Note that the mode operators b_n, c_n belong the the free bc CFT. A more general expression for the Virasoro generators in this theory can be found in (2.7.19) of JBBS

$$
L_m = \sum_{n=-\infty}^{\infty} (\lambda m - n)\, {}^{\circ}_{\circ} b_n c_{m-n} {}^{\circ}_{\circ} + \frac{1}{2}\lambda\,(1-\lambda)\,\delta_{m,0}
$$

(note that b_n and c_{-n} anti-commute inside the normal ordering in the expression for L_0 above). In this problem, we focus on the special case where $\lambda = 2$.

Since $L_{m\neq 0}$ and L_0 cannot be expressed in the same format., we must treat several cases of commutator $[L_m, L_n]$:

- $m \neq 0, n \neq 0$ and $m \neq n$

$$
\begin{aligned}
[L_m, L_n] &= \sum_{i,j=-\infty}^{\infty} (2m-i)(2n-j)\,[b_i c_{m-i}, b_j c_{n-j}] \\
&= \sum_{i,j=-\infty}^{\infty} (2m-i)(2n-j)\,[b_i c_{m-i} b_j c_{n-j} - b_j c_{n-j} b_i c_{m-i}] \\
&= \sum_{i,j=-\infty}^{\infty} (2m-i)(2n-j)\,[b_i(\delta_{m-i+j,0} - b_j c_{m-i})\,c_{n-j} - b_j(\delta_{n-j+i,0} - b_i c_{n-j})\,c_{m-i}] \\
&= \sum_{i,j=-\infty}^{\infty} (2m-i)(2n-j)\,[(\delta_{m-i+j,0} b_i c_{n-j} - b_i b_j c_{m-i} c_{n-j}) - (\delta_{n-j+i,0} b_j c_{m-i} - b_j b_i c_{n-j} c_{m-i})] \\
&= \sum_{i,j=-\infty}^{\infty} (2m-i)(2n-j)\,[(\delta_{m-i+j,0} b_i c_{n-j} - \delta_{n-j+i,0} b_j c_{m-i})] \\
&= \sum_{i=-\infty}^{\infty} [(m-i)(2n-i)\,b_{m+i} c_{n-i} - (2m-i)(n-i)\,b_{n+i} c_{m-i}] \\
&= \sum_{k=-\infty}^{\infty} [(2m-k)(2n+m-k) - (2m+n-k)(2n-k)]\,b_k c_{(n+m)-k} \\
&= (m-n) \sum_{k=-\infty}^{\infty} [2(m+n)-k]\,b_k c_{(n+m)-k} \\
&= (m-n)\,L_{m+n}
\end{aligned}
$$

- $m \neq 0, n = 0$

$$
\begin{aligned}
[L_m, L_0] &= \sum_{i=-\infty}^{\infty} \sum_{j=1}^{\infty} (2m-i)\,j\,([b_i c_{m-i}, b_{-j} c_j] + [b_i c_{m-i}, c_{-j} b_j]) \\
&= \sum_{i=-\infty}^{\infty} \sum_{j=1}^{\infty} (2m-i)\,j\,(\delta_{m-i-j,0} b_i c_j - \delta_{i+j,0} b_{-j} c_{m-i} + \delta_{i-j,0} b_j c_{m-i} - \delta_{j-i+m,0} b_i c_{-j}) \\
&= \sum_{j=1}^{\infty} j\,((m+j)\,b_{m-j} c_j - (2m+j)\,b_{-j} c_{m+j} + (2m-j)\,b_j c_{m-j} - (m-j)\,b_{m+j} c_{-j}) \\
&= \sum_{j>0} j\,(m+j)\,b_{m-j} c_j + \sum_{j<0} j\,(m+j)\,b_{m-j} c_j + \sum_{j>0} j\,(2m-j)\,b_j c_{m-j} + \sum_{j<0} j\,(2m-j)\,b_j c_{m-j} \\
&= \sum_{j=-\infty}^{\infty} j\,(m+j)\,b_{m-j} c_j + \sum_{j=-\infty}^{\infty} j\,(2m-j)\,b_j c_{m-j} \\
&= \sum_{k=-\infty}^{\infty} [(m-k)(2m-k)\,b_k c_{m-k} + k\,(2m-k)\,b_k c_{m-k}] \\
&= m \sum_{k=-\infty}^{\infty} (2m-k)\,b_k c_{m-k} \\
&= m L_m
\end{aligned}
$$

where we have used the intermediate results

$$\begin{aligned}
[b_i c_{m-i}, b_{-j} c_j] &= b_i c_{m-i} b_{-j} c_j - b_{-j} c_j b_i c_{m-i} \\
&= b_i \left(\delta_{m-i-j,0} - b_{-j} c_{m-i}\right) c_j - b_{-j} \left(\delta_{i+j,0} - b_i c_j\right) c_{m-i} \\
&= \left(b_{-j} b_i c_j c_{m-i} - b_i b_{-j} c_{m-i} c_j\right) + \left(\delta_{m-i-j,0} b_i c_j - \delta_{i+j,0} b_{-j} c_{m-i}\right) \\
&= \delta_{m-i-j,0} b_i c_j - \delta_{i+j,0} b_{-j} c_{m-i},
\end{aligned}$$

$$\begin{aligned}
[b_i c_{m-i}, c_{-j} b_j] &= b_i c_{m-i} c_{-j} b_j - c_{-j} b_j b_i c_{m-i} \\
&= b_i c_{m-i} \left(1 - b_j c_{-j}\right) - \left(1 - b_j c_{-j}\right) b_i c_{m-i} \\
&= \cancel{b_i c_{m-i}} - b_i c_{m-i} b_j c_{-j} - \cancel{b_i c_{m-i}} + b_j c_{-j} b_i c_{m-i} \\
&= -b_i \left(\delta_{j+m-i,0} - b_j c_{m-i}\right) c_{-j} + b_j \left(\delta_{i-j,0} - b_i c_{-j}\right) c_{m-i} \\
&= b_i b_j c_{m-i} c_{-j} - b_j b_i c_{-j} c_{m-i} + \delta_{i-j,0} b_j c_{m-i} - \delta_{j+m-i,0} b_i c_{-j} \\
&= \delta_{i-j,0} b_j c_{m-i} - \delta_{j-i+m,0} b_i c_{-j}.
\end{aligned}$$

- The results above are only valid only up to a constant, $A\delta_{m,-n}$. Therefore, the L_m satisfy the algebra

$$[L_m, L_n] = (m-n) L_{m+n} + A\delta_{m,-n}$$

for an undetermined constant A.

To determine A, we act with $[L_m, L_{-m}]$ on the physical state $|\psi\rangle$ that is annihilated by $b_{n>0}$ and $c_{n\geq 0}$. Taking $m > 0$, $L_{-m} L_m |\psi\rangle = 0$ and

$$\begin{aligned}
A|\psi\rangle &= [L_m, L_{-m}] |\psi\rangle - 2m L_0 |\psi\rangle \\
&= L_m L_{-m} |\psi\rangle + 2m|\psi\rangle \\
&= \left(\frac{m - 13m^3}{6} + 2m\right) |\psi\rangle \\
&= \frac{13}{6} \left(m - m^3\right) |\psi\rangle \\
&= \frac{c}{12} \left(m^3 - m\right)
\end{aligned}$$

where the central charge is $c = -26$. We have also used the following:

$$L_0 |\psi\rangle = -|\psi\rangle + \sum_{i\geq 1}^{\infty} i \left(b_{-i} c_i + c_{-i} b_i\right) |\psi\rangle = -|\psi\rangle + \cancel{\sum_{i\geq 1}^{\infty} i b_{-i} c_i |\psi\rangle} + \cancel{\sum_{i\geq 1}^{\infty} i c_{-i} b_i |\psi\rangle} = -|\psi\rangle$$

$$
\begin{aligned}
L_m L_{-m}|\psi\rangle &= \sum_{i,j} (2m-i)(-2m-j)\, b_i c_{m-i} \underbrace{b_j c_{-m-j}|\psi\rangle}_{\neq 0 \text{ if } -m+1\leq j\leq 0} \\
&= \sum_{-m+1\leq j\leq 0} \sum_i (2m-i)(-2m-j)\, b_i c_{m-i} b_j c_{-m-j}|\psi\rangle \\
&= \sum_{-m+1\leq j\leq 0} \sum_i (2m-i)(-2m-j)\left(\delta_{m-i+j,0}\left(\delta_{i-m-j,0} - c_{-m-j}b_i\right) - (-b_j b_i)(-c_{-m-j}c_{m-i})\right)|\psi\rangle \\
&= \sum_{-m+1\leq j\leq 0} \sum_i (2m-i)(-2m-j)\left(\left(\delta_{i,m+j}\right)^2 - \delta_{i,j+m}c_{-m-j}b_i - \delta_{i,j+m}b_j c_{m-i} + b_j c_{-m-j} b_i c_{m-i}\right)|\psi\rangle \\
&= \sum_{-m+1\leq j\leq 0} (m-j)(-2m-j)\,\delta_{00}|\psi\rangle - \sum_{-m+1\leq j\leq 0} (m-j)(-2m-j)\left(c_{-m-j}b_{j+m} + b_j c_{-j}\right)|\psi\rangle \\
&\quad + \sum_{-m+1\leq j\leq 0} \sum_i (2m-i)(-2m-j)\, b_j c_{-m-j} \underbrace{b_i c_{m-i}|\psi\rangle}_{0} \\
&= \sum_{-m+1\leq j\leq 0} (m-j)(-2m-j)|\psi\rangle - \sum_{-m+1\leq j\leq 0} (m-j)(-2m-j)\left(\underbrace{c_{-m-j}b_{j+m}|\psi\rangle}_{0 \text{ for } -m+1\leq j\leq 0} + \underbrace{b_j c_{-j}|\psi\rangle}_{0 \text{ for } -m+1\leq j\leq 0}\right) \\
&= \sum_{-m+1\leq j\leq 0} (m-j)(-2m-j)|\psi\rangle \\
&= \frac{m-13m^3}{6}
\end{aligned}
$$