Phys 731: String Theory - Assignment 4

Andrzej Pokraka

November 7, 2017

Problem 1

a) Find the Möbius transformation that takes three given points z_1, z_2, z_3 into specified new positions z_1', z_2', z_3'.

b) Show that the 4-tachyon integral

$$\int d^2z\, |z_{12}z_{13}z_{23}|^2 \prod_{i<j} |z_{ij}|^{\alpha' k_i \cdot k_j}$$

is independent of the choice $z_{1,2,3}$ by considering the effect of a Möbius transformation on the terms in the integral. You will need to use momentum conservation and the mass-shell condition.

Part (a)

To solve this problem it is easiest to make two Möbius transformations: $\{z_1, z_2, z_3\} \underset{f}{\rightarrow} \{0, 1, \infty\} \underset{g}{\rightarrow} \{z_1', z_2', z_3'\}$. The function

$$f(z) = \left(\frac{z_2 - z_3}{z_2 - z_1}\right)\left(\frac{z - z_1}{z - z_3}\right)$$

maps the points $\{z_1, z_2, z_3\}$ to $\{0, 1, \infty\}$. Next, the function

$$g(z) = \frac{(z_3' z_2' - z_3' z_1')\, z + (z_1' z_3' - z_1' z_2')}{(z_2' - z_1')\, z + (z_3' - z_2')}$$

maps the points $\{0, 1, \infty\}$ to $\{z_1', z_2', z_3'\}$. Therefore, the composite function $g \circ f$

$$\begin{aligned}(g \circ f)(z) &= \frac{(z_3' z_2' - z_3' z_1')\left(\frac{z_2 - z_3}{z_2 - z_1}\right)\left(\frac{z - z_1}{z - z_3}\right) + (z_1' z_3' - z_1' z_2')}{(z_2' - z_1')\left(\frac{z_2 - z_3}{z_2 - z_1}\right)\left(\frac{z - z_1}{z - z_3}\right) + (z_3' - z_2')} \\ &= \frac{(z_3' z_2' - z_3' z_1')(z_2 - z_3)(z - z_1) + (z_1' z_3' - z_1' z_2')(z_2 - z_1)(z - z_3)}{(z_2' - z_1')(z_2 - z_3)(z - z_1) + (z_3' - z_2')(z_2 - z_1)(z - z_3)}\end{aligned}$$

takes $\{z_1, z_2, z_3\} \underset{g\circ f}{\rightarrow} \{z_1', z_2', z_3'\}$.

Part (b)

The 4-point tachyon amplitude is

$$\mathcal{M} = \int d^2 z_4 \, |z_{12} z_{13} z_{23}|^2 \prod_{i<j} |z_{ij}|^{\alpha' k_i \cdot k_j} \, .$$

Using conservation of momentum, $k_1 + k_2 + k_3 + k_4 = 0$, and the mass shell condition, $k_i^2 = 4/\alpha'$, we obtain the kinematic relations

$$\begin{aligned}
k_1 \cdot k_2 &= k_3 \cdot k_4 \\
k_1 \cdot k_3 &= k_2 \cdot k_4 \\
k_2 \cdot k_3 &= k_1 \cdot k_4 \\
k_3 \cdot k_4 &= -\frac{4}{\alpha'} - k_1 \cdot k_4 - k_2 \cdot k_4.
\end{aligned}$$

Substituting the above into the 4-point tachyon amplitude, we obtain

$$\begin{aligned}
\mathcal{M} &= \int d^2 z_4 \, |z_{12} z_{13} z_{23}|^2 \, |z_{12}|^{\alpha' k_1 \cdot k_2} \, |z_{13}|^{\alpha' k_1 \cdot k_3} \, |z_{23}|^{\alpha' k_2 \cdot k_3} \, |z_{14}|^{\alpha' k_1 \cdot k_4} \, |z_{24}|^{\alpha' k_2 \cdot k_4} \, |z_{34}|^{\alpha' k_3 \cdot k_4} \\
&= \int d^2 z_4 \, |z_{12} z_{13} z_{23}|^2 \, |z_{12}|^{\alpha' k_3 \cdot k_4} \, |z_{13}|^{\alpha' k_2 \cdot k_3} \, |z_{23}|^{\alpha' k_1 \cdot k_4} \, |z_{14}|^{\alpha' k_1 \cdot k_4} \, |z_{24}|^{\alpha' k_2 \cdot k_4} \, |z_{34}|^{\alpha' k_3 \cdot k_4} \\
&= \int d^2 z_4 \, |z_{12} z_{13} z_{23}|^2 \, |z_{13}|^{\alpha' k_2 \cdot k_4} \, |z_{23}|^{\alpha' k_1 \cdot k_4} \, |z_{14}|^{\alpha' k_1 \cdot k_4} \, |z_{24}|^{\alpha' k_2 \cdot k_4} \\
&\qquad \times |z_{12}|^{-4-\alpha'(k_1 \cdot k_4 + k_2 \cdot k_4)} \, |z_{34}|^{-4-\alpha'(k_1 \cdot k_4 + k_2 \cdot k_4)} \\
&= \int d^2 z_4 \left(\left| \frac{z_{13} z_{23}}{z_{12}} \right|^2 |z_{34}|^{-4} \left| \frac{z_{23}}{z_{12} z_{34}} \right|^{\alpha' k_1 \cdot k_4} \left| \frac{z_{13}}{z_{12} z_{34}} \right|^{\alpha' k_2 \cdot k_4} \right) |z_{24}|^{\alpha' k_2 \cdot k_4} \, |z_{14}|^{\alpha' k_1 \cdot k_4} \\
&= \lim_{z_3 \to \infty} \int d^2 z_4 \left(|1/z_3 - 1|^2 \left| \frac{z_3}{z_3} \right|^4 \left| \frac{1}{1 - z_4/z_3} \right|^4 \left| \frac{1/z_3 - 1}{1 - z_4/z_3} \right|^{\alpha' k_1 \cdot k_4} \left| \frac{1}{1 - z_4/z_3} \right|^{\alpha' k_2 \cdot k_4} \right) |1 - z_4|^{\alpha' k_2 \cdot k_4} \, |z_4|^{\alpha' k_1 \cdot k_4} \\
&= \int d^2 z_4 \, |1 - z_4|^{\alpha' k_2 \cdot k_4} \, |z_4|^{\alpha' k_1 \cdot k_4}
\end{aligned}$$

where we have used a Möbius transform to take $\{z_1, z_2, z_3\} \to \{0, 1, \infty\}$. Note that the Möbius transformation $f : \{z_1, z_2, z_3\} \to \{0, 1, \infty\}$ leaves all $z \notin \{0, 1, \infty\}$ invariant

$$\begin{aligned}
f(z) &= \lim_{\{z_1, z_2, z_3\} \to \{0,1,\infty\}} \left(\frac{z_2 - z_3}{z_2 - z_1} \right) \left(\frac{z - z_1}{z - z_3} \right) \\
&= \lim_{z_3 \to \infty} \left(\frac{1/z_3 - 1}{1} \right) \left(\frac{z}{z/z_3 - 1} \right) \\
&= z.
\end{aligned}$$

Problem 2

Show that the residue of the pole in the Virasoro-Shapiro amplitude at $M^2 = 4(N-1)/\alpha'$, for integer N, is a polynomial in $t - u \propto \cos\theta$. Compare the order of the polynomial with the maximum spin of a string state at that level (e.g., the maximum eigenvalue of the rotation J_{12}). If you plot the spin versus mass-squared, what is the slope? (This is the closed string version of problem 6.5a of JBBS and is done in Headrick).

The Virasoro-Shapiro amplitude is given by

$$\mathcal{M} = 2\pi \frac{\Gamma\left(-1-\alpha' s/4\right)\Gamma\left(-1-\alpha' t/4\right)\Gamma\left(-1-\alpha' u/4\right)}{\Gamma\left(2+\alpha' s/4\right)\Gamma\left(2+\alpha' t/4\right)\Gamma\left(2+\alpha' u/4\right)}$$

where

$$\begin{aligned} s &= -\left(p_1+p_2\right)^2 = E_{\text{cm}}^2 \\ t &= -\left(p_1+p_3\right)^2 = \left(4m^2 - E_{\text{cm}}^2\right)\left(1-\cos\theta\right) \\ u &= -\left(p_1+p_4\right)^2 = \left(4m^2 - E_{\text{cm}}^2\right)\left(1+\cos\theta\right) \\ s+t+u &= 4m^4 \end{aligned}$$

are the Mandelstam variables for $2 \to 2$ scattering with p_1, p_2 incoming and p_3, p_4 outgoing, and $m^2 = -4/\alpha'$ is the mass of the closed string tachyon. Since the inverse Gamma function is entire ($\Gamma^{-1}(z)$ is holomorphic for all $z \in \mathbb{C}$), the poles of $\mathcal{M}$ come from the factor $\Gamma\left(-1-\alpha' s/4\right)$ in the numerator (here, t is held fixed). The poles in $s$are when

$$-1 - \frac{4}{\alpha'}s = -N \in \mathbb{N} \quad \implies s = \frac{4}{\alpha'}\left(N-1\right).$$

This represents the exchange of a particle in the s-channel of invariant mass $M^2 = 4\left(N-1\right)/\alpha'$. Setting $s = 4(N-1)/\alpha'$, the Mandelstam variables satisfy

$$\begin{aligned} s+t+u &= 4m^2 \\ \implies t+u &= \left(4m^2 - s\right) = -\frac{4}{\alpha'}\left(N+3\right). \end{aligned}$$

Since the residue of the Gamma function at each negative integer is given by

$$\operatorname{Res}[\Gamma(z); z = -n \in \mathbb{N}] = \frac{(-1)^n}{\Gamma(n+1)},$$

the residue of $\mathcal{M}$ at $M^2 = 4\left(N-1\right)/\alpha'$ is

$$\begin{aligned} \operatorname{Res}\left[\mathcal{M}; s = \frac{4}{\alpha'}\left(N-1\right) : N \in \mathbb{N}\right] &\propto \frac{(-1)^N}{\Gamma\left(N+1\right)}\left[\frac{\Gamma\left(-1-\alpha' t/4\right)\Gamma\left(-1-\alpha' u/4\right)}{\Gamma\left(2+\alpha' s/4\right)\Gamma\left(2+\alpha' t/4\right)\Gamma\left(2+\alpha' u/4\right)}\right]_{s=\frac{4}{\alpha'}(N-1)} \\ &= \frac{(-1)^N}{\Gamma^2\left(N+1\right)}\frac{\Gamma\left(-1-\alpha' t/4\right)\Gamma\left(-1-\alpha' u/4\right)}{\Gamma\left(2+\alpha' t/4\right)\Gamma\left(2+\alpha' u/4\right)}. \end{aligned}$$

To show that the above is a polynomial in $t-u$, we want to write t and u in terms of $q = t-u$

$$\begin{aligned} t &= \frac{1}{2}\left(t+u\right) + \frac{1}{2}\left(t-u\right) = -\frac{4}{\alpha'}\frac{(N+3)}{2} + \frac{q}{2} \\ u &= \frac{1}{2}\left(t+u\right) - \frac{1}{2}\left(t-u\right) = -\frac{4}{\alpha'}\frac{(N+3)}{2} - \frac{q}{2}. \end{aligned}$$

Expanding the gamma functions we see that the amplitude is a polynomial in q

$$\begin{aligned} \operatorname{Res}\left[\mathcal{M}; s = \frac{4}{\alpha'}\left(N-1\right) : N \in \mathbb{N}\right] &\propto \frac{(-1)^N}{\Gamma^2\left(N+1\right)}\frac{\Gamma\left(-1-\left(-\frac{(N+3)}{2}+\frac{\alpha'}{4}\frac{q}{2}\right)\right)\Gamma\left(-1-\left(-\frac{(N+3)}{2}-\frac{\alpha'}{4}\frac{q}{2}\right)\right)}{\Gamma\left(2+\left(-\frac{(N+3)}{2}+\frac{\alpha'}{4}\frac{q}{2}\right)\right)\Gamma\left(2+\left(-\frac{(N+3)}{2}-\frac{\alpha'}{4}\frac{q}{2}\right)\right)} && (1) \\ &= \frac{(-1)^N}{\Gamma^2\left(N+1\right)}\frac{\Gamma\left(\frac{1}{2}N+\frac{1}{2}-\frac{\alpha'}{4}\frac{q}{2}\right)\Gamma\left(\frac{1}{2}N+\frac{1}{2}+\frac{\alpha'}{4}\frac{q}{2}\right)}{\Gamma\left(-\frac{1}{2}N+\frac{1}{2}+\frac{\alpha'}{4}\frac{q}{2}\right)\Gamma\left(-\frac{1}{2}N+\frac{1}{2}-\frac{\alpha'}{4}\frac{q}{2}\right)} && (2) \\ &\sim \frac{(-1)^N}{\left(N!\right)^2}q^{2N} + \text{terms of lower powers of } q \end{aligned}$$

where

$$\frac{\Gamma\left(\frac{1}{2}N+\frac{1}{2}-\frac{\alpha'}{4}\frac{q}{2}\right)}{\Gamma\left(-\frac{1}{2}N+\frac{1}{2}-\frac{\alpha'}{4}\frac{q}{2}\right)} = \left(\frac{1}{2}(N-1)+\frac{1}{2}-\frac{\alpha'}{4}\frac{q}{2}\right)\left(\frac{1}{2}(N-2)+\frac{1}{2}-\frac{\alpha'}{4}\frac{q}{2}\right)\cdots\left(-\frac{1}{2}N+\frac{1}{2}-\frac{\alpha'}{4}\frac{q}{2}\right)$$
$$\frac{\Gamma\left(\frac{1}{2}N+\frac{1}{2}+\frac{\alpha'}{4}\frac{q}{2}\right)}{\Gamma\left(-\frac{1}{2}N+\frac{1}{2}+\frac{\alpha'}{4}\frac{q}{2}\right)} = \left(\frac{1}{2}(N-1)+\frac{1}{2}+\frac{\alpha'}{4}\frac{q}{2}\right)\left(\frac{1}{2}(N-2)+\frac{1}{2}+\frac{\alpha'}{4}\frac{q}{2}\right)\cdots\left(-\frac{1}{2}N+\frac{1}{2}+\frac{\alpha'}{4}\frac{q}{2}\right).$$

Since the exchange of an s-channel particle of spin-J will give a spin-J spherical harmonic (a degree J polynomial of $\cos\theta \propto q$) the amplitude will be proportional to t^J. Comparing with equation (1), we see that $J = 2N$. As a function of J, the invariant mass of the s-channel exchange particle is a straight line with slope of $2/\alpha'$ and intercept of the invariant mass squared of the tachyon

$$M^2 = \frac{2}{\alpha'}J - \frac{4}{\alpha'} = \frac{2}{\alpha'}J + m^2.$$

Problem 3

a) There are three terms in the Veneziano amplitude; focus on the one that has poles in s and t. Identify the residue of the pole at $t = 0$. Show that this residue is the same as you would get in field theory from exchange of a photon between two charged scalars, and use this to determine the gauge coupling in terms of g_o.

b) Show that the term that has poles in u and t makes an equal and opposite contribution to this residue.

c) So far this is without Chan-Paton factors; show that the cancelation is no longer present with these, and find the result.

Part (a)

The Veneziano amplitude is given by

$$\mathcal{M}_{\text{Veneziano}} = \frac{2ig_o^2}{\alpha'}\left(\frac{\Gamma(-1-\alpha' u)\,\Gamma(-1-\alpha' t)}{\Gamma(2+\alpha' s)} + \frac{\Gamma(-1-\alpha' s)\,\Gamma(-1-\alpha' t)}{\Gamma(2+\alpha' u)} + \frac{\Gamma(-1-\alpha' s)\,\Gamma(-1-\alpha' u)}{\Gamma(2+\alpha' t)}\right).$$

The term with s, t poles is

$$\mathcal{M}_{st} = \frac{2ig_o^2}{\alpha'}\frac{\Gamma(-1-\alpha' s)\,\Gamma(-1-\alpha' t)}{\Gamma(2+\alpha' u)}.$$

To extract the residue at $t = 0$, it pays to rewrite $\mathcal{M}$ in order to make the $t = 0$ pole explicit

$$\mathcal{M}_{st} = \frac{2ig_o^2}{\alpha'}\frac{\Gamma(-1-\alpha' s)\,\Gamma(\alpha' t)}{\Gamma(-2-\alpha' s-\alpha' t)\,(-1-\alpha' t)}.$$

It is now clear that the pole at $t = 0$ comes from $\Gamma(\alpha' t)$. The residue at $t = 0$ is therefore given by

$$\begin{aligned}\text{Res}\left[\mathcal{M}_{st}; t=0\right] &= \frac{2ig_o^2}{\alpha'}\text{Res}\left[\Gamma(\alpha' t); t=0\right]\left[\frac{\Gamma(-1-\alpha' s)}{\Gamma(-2-\alpha' s-\alpha' t)\,(-1-\alpha' t)}\right]_{t=0} \\ &= \frac{2ig_o^2}{\alpha'}\left(-\frac{1}{\alpha'}\right)\frac{\Gamma(-1-\alpha' s)}{\Gamma(-2-\alpha' s)} \\ &= \frac{2ig_o^2}{\alpha'}(-2-\alpha' s) \\ &= \frac{2ig_o^2}{\alpha'}\left(\frac{2}{\alpha'}+s\right).\end{aligned}$$

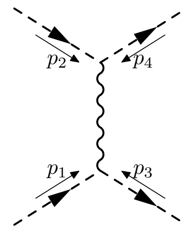

Figure 1: t-channel single photon exchange amplitude.

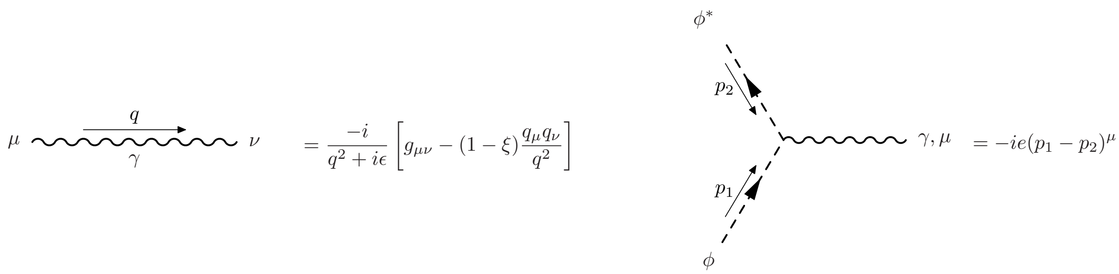

Figure 2: Relevant Feynman rules for calculating the single photon exchange amplitude.

Next, we look at t-channel single photon exchange in scalar QED (Figure 1). In the Feynman gauge $\xi = 1$, the amplitude of Figure 1 becomes

$$
\begin{aligned}
\mathcal{M}_{\text{t-channel } \phi\phi\to\phi\phi} &= (-ie)\,(p_1 - p_3)^\mu \left[\frac{-ig_{\mu\nu}}{(p_1+p_3)^2}\right](-ie)\,(p_2 - p_4)^\nu \\
&= -\frac{ie^2}{t}\,(p_1 - p_3)\cdot(p_2 - p_4) \\
&= -\frac{ie^2}{t}\,(p_1\cdot p_2 - p_3\cdot p_2 - p_1\cdot p_4 + p_3\cdot p_4) \\
&= -\frac{ie^2}{t}\,(2p_1\cdot p_2 - 2p_1\cdot p_4) \\
&= -\frac{ie^2}{t}\,\big((2m^2 - s) - (2m^2 - u)\big) \\
&= -ie^2\frac{(u-s)}{t}
\end{aligned}
$$

where the relevant Feynman rules are given in Figure 2. The $t = 0$ residue of the above amplitude is

$$\mathcal{M}_{\text{t-channel } \phi\phi\to\phi\phi} = -ie^2\,(u-s) = \big(4m^2 - t - s\big)_{t\to 0} - s = 4m^2 - 2s$$

where m is the mass of the scalars. Setting the scalar mass equal to the (open string) tachyon mass, $m^2 = -1/\alpha'$ we obtain

$$\mathcal{M}_{\text{t-channel } \phi\phi\,\to\,\phi\phi} = -ie^2\,(u-s) = -ie^2\left(-\frac{4}{\alpha'} - 2s\right) = 2ie^2\left(\frac{2}{\alpha'} + s\right).$$

Evidently, the $t = 0$ residue of the t-channel scalar QED amplitude, $\mathcal{M}_{\text{t-channel } \phi\phi\,\to\,\phi\phi}$, and the $t = 0$ residue of $\mathcal{M}_{st}$ match if

$$\frac{2ig_o^2}{\alpha'} = 2ie \qquad \Longrightarrow\; g_o = \sqrt{e\alpha'}.$$

Part (b)

The term with u, t poles is

$$\begin{aligned}\mathcal{M}_{ut} &= \frac{2ig_o^2}{\alpha'}\frac{\Gamma(-1-\alpha' u)\,\Gamma(-1-\alpha' t)}{\Gamma(2+\alpha' s)} \\ &= \frac{2ig_o^2}{\alpha'}\frac{\Gamma(-1-\alpha' u)\,\Gamma(\alpha' t)}{\Gamma(-2-\alpha' u-\alpha' t)\,(-1-\alpha' t)}.\end{aligned}$$

Its residue at $t=0$ is

$$\begin{aligned}\operatorname{Res}[\mathcal{M}_{ut}; t=0] &= \frac{2ig_o^2}{\alpha'}\left(-\frac{1}{\alpha'}\right)\left[\frac{\Gamma(-1-\alpha' u)}{\Gamma(-2-\alpha' u-\alpha' t)}\right]_{t=0} \\ &= -\frac{2ig_o^2}{\alpha'}\left(\frac{1}{\alpha'}\right)\frac{\Gamma(-1-\alpha' u)}{\Gamma(-2-\alpha' u)} \\ &= -\frac{2ig_o^2}{\alpha'}\left(\frac{1}{\alpha'}\right)(-2-\alpha' u) \\ &= \frac{2ig_o^2}{\alpha'}\left(\frac{2}{\alpha'}+u\right).\end{aligned}$$

At $t=0$, the Mandelstam variables satisfy $s+u=4m^2=-4/\alpha'$. Writing the residue of $\mathcal{M}_{ut}$ in terms of s, obtain

$$\operatorname{Res}[\mathcal{M}_{ut}; t=0] = -\frac{2ig_o^2}{\alpha'}\left(\frac{2}{\alpha'}+s\right) = -\operatorname{Res}[\mathcal{M}_{st}; t=0].$$

Part (c)

Recall that there were six cyclic ordering in the computation of the Veneziano amplitude given in Figure 6.3 of JBBS. If one includes Chan-Paton factors, the overall factor of 2 in the Veneziano amplitude from the addition of the six cycles is replaced by the trace over a sum of Chan-Paton factors. With a_i the Chan-Paton index for the i^{th} tachyon, the Veneziano amplitude becomes

$$\begin{aligned}\mathcal{M}_{\text{Veneziano}} = \frac{2ig_o^2}{\alpha'}\Bigg[&\operatorname{Tr}\left(\lambda^{a_1}\lambda^{a_2}\lambda^{a_4}\lambda^{a_3}+\lambda^{a_1}\lambda^{a_3}\lambda^{a_4}\lambda^{a_2}\right)\frac{\Gamma(-1-\alpha' u)\,\Gamma(-1-\alpha' t)}{\Gamma(2+\alpha' s)} \\ &+\operatorname{Tr}\left(\lambda^{a_1}\lambda^{a_3}\lambda^{a_2}\lambda^{a_4}+\lambda^{a_1}\lambda^{a_4}\lambda^{a_2}\lambda^{a_3}\right)\frac{\Gamma(-1-\alpha' s)\,\Gamma(-1-\alpha' t)}{\Gamma(2+\alpha' u)} \\ &+\operatorname{Tr}\left(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}\lambda^{a_4}+\lambda^{a_1}\lambda^{a_4}\lambda^{a_3}\lambda^{a_2}\right)\frac{\Gamma(-1-\alpha' s)\,\Gamma(-1-\alpha' u)}{\Gamma(2+\alpha' t)}\Bigg].\end{aligned}$$

The $t=0$ residues are modified to be

$$\begin{aligned}\operatorname{Res}[\mathcal{M}_{st}; t=0] &= \frac{2ig_o^2}{\alpha'}\left(\frac{2}{\alpha'}+s\right)\operatorname{Tr}\left(\lambda^{a_1}\lambda^{a_3}\lambda^{a_2}\lambda^{a_4}+\lambda^{a_1}\lambda^{a_4}\lambda^{a_2}\lambda^{a_3}\right) \\ \operatorname{Res}[\mathcal{M}_{ut}; t=0] &= -\frac{2ig_o^2}{\alpha'}\left(\frac{2}{\alpha'}+s\right)\operatorname{Tr}\left(\lambda^{a_1}\lambda^{a_2}\lambda^{a_4}\lambda^{a_3}+\lambda^{a_1}\lambda^{a_3}\lambda^{a_4}\lambda^{a_2}\right).\end{aligned}$$

The total residue at $t=0$ is therefore

$$\begin{aligned}\operatorname{Res}[\mathcal{M}_{\text{Veneziano}}; t=0] &= \frac{2ig_o^2}{\alpha'}\left(\frac{2}{\alpha'}+s\right)\operatorname{Tr}\left(\lambda^{a_1}\lambda^{a_3}\left[\lambda^{a_2},\lambda^{a_4}\right]+\lambda^{a_1}\left[\lambda^{a_4},\lambda^{a_2}\right]\lambda^{a_3}\right) \\ &= \frac{2ig_o^2}{\alpha'}\left(\frac{2}{\alpha'}+s\right)\operatorname{Tr}\left(\left\{\lambda^{a_1},\lambda^{a_3}\right\}\left[\lambda^{a_2},\lambda^{a_4}\right]\right) \\ &\neq 0.\end{aligned}$$