PHYS 643 Week 2: Cold Stars — White Dwarfs, Neutron Stars, and Planets

We start the course by discussing the topic of "cold stars", which encompasses white dwarfs, neutron stars, and planets. This is a good topic to start off with because we need only a couple of ideas: hydrostatic balance and the zero-temperature equation of state.

Hydrostatic balance

Stars and planets are in *hydrostatic balance* in which the pressure gradient from their interior to the surface balances their self-gravity[1]. Assuming spherical symmetry, the momentum equation in this situation is

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2} \tag{1}$$

where

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{2}$$

and $m(r)$ is the mass contained within radius r. The boundary conditions are $m = 0$ at $r = 0$ and $P = 0$ at $r = R$. To solve the equations, we just need a relation between P and ρ. Under the assumption $P \propto \rho^\gamma$, the solutions are known as polytropes. A polytrope of index n has $\gamma = 1 + \frac{1}{n}$.

Stars and planets on the back of the envelope

A rough estimate of the structure is to write the two sides of the hydrostatic balance equation as

$$\frac{dP}{dr} \approx \frac{P_c}{R} \qquad \rho g \approx \frac{M}{R^3}\frac{GM}{R^2},$$

where P_c is the central pressure and R is the radius. This gives a formula for the central pressure in terms of the mass and radius of the object

$$P_c \approx \frac{GM^2}{R^4}.$$

For an ideal gas, we can get the central temperature also:

$$P_c \approx \frac{\rho_c k_B T}{m_p} \Rightarrow T_c \approx \frac{GMm_p}{k_B R}.$$

[1] To see that must be the case, look at the momentum equation

$$\frac{D\mathbf{u}}{Dt} = \frac{-\nabla P}{\rho} + \mathbf{g},$$

and imagine turning off the pressure gradients. The fluid would then accelerate in response to gravity. The time to collapse would be $\sim \sqrt{R/g} \sim \sqrt{R^3/GM}$, or about 30 minutes for the Sun, much less than its 5 billion year age. This implies the pressure gradient must balance gravity to a high degree of accuracy!

Plugging in numbers for the Sun gives $T_c \approx 2 \times 10^7$ K, pretty close to the central temperature of the Sun, 1.5×10^7 K.

For a polytropic relation $P \propto \rho^\gamma$, we can get the mass-radius scaling

$$P_c \approx \frac{GM^2}{R^4} \propto \rho^\gamma \propto \left(\frac{M}{R^3}\right)^\gamma \Rightarrow M^{\gamma-2} \propto R^{3\gamma-4}.$$

Interesting cases:

- *White dwarfs*. For non-relativistic degenerate electrons, $\gamma = 5/3 \Rightarrow R \propto M^{-1/3}$. As the white dwarf mass increases, the electrons become relativistic and $\gamma \to 4/3$. Then M becomes independent of R! The corresponding mass is the Chandrasekhar mass $M_{Ch} \approx 1.4\ M_\odot$, a maximum mass for white dwarfs.

- *Neutron star*. Degenerate neutrons hold up the star, but interactions between neutrons stiffen the EOS, giving $\gamma = 2$. Then R is independent of M, as seen in realistic calculations.

- *Incompressible material*. $\gamma \to \infty \Rightarrow M \propto R^3$, we expect this to hold for small "rocky" bodies such as moons or rocky planets. (We'll see later that gas giant planets like Jupiter lie between the $M \propto R^3$ and $M \propto R^{-3}$ limits.)

- *Isothermal sphere*. This has $\gamma = 1$ so that $P \propto \rho$. It is often used as a model of stellar systems such as globular clusters, although it has the property that $\rho \propto 1/r^2$ and therefore the mass contained within radius r grows $\propto r$, so it must be truncated to give the system a finite mass.

Equations of state

The following table summarizes the chemical potential μ, pressure P and internal energy density U for four cases of interest:

	μ	P	U
Ideal gas	$k_BT\ln\left(\frac{n}{n_Q}\right) \ll -1$ $n_Q = (2\pi m k_B T/h^2)^{3/2}$	nk_BT	$\frac{3}{2}P = \frac{3}{2}nk_BT$
Non-relativistic degenerate	$E_F = \frac{p_F^2}{2m} \propto n^{2/3} \gg k_BT$ $p_F = \hbar k_F = \hbar(3\pi^2 n)^{1/3}$	$\frac{2}{5}nE_F \propto n^{5/3}$	$\frac{3}{2}P$
Relativistic degenerate	$E_F = p_F c \propto n^{1/3}$	$\frac{1}{4}nE_F \propto n^{4/3}$	$3P$
Radiation	0	$\frac{1}{3}aT^4$	$3P = aT^4$

Mixtures

Usually in astrophysics we are dealing with a plasma consisting of a mixture of different chemical species. There is a whole terminology for dealing with this which you'll see used a lot, so we'll go through this here in some detail.

The starting point is that in a mixture we add the different contributions to the pressure from each species of particle. These depend on the number densities of different species, which can be obtained from the mass density if we know their number fraction Y_i or mean molecular weight μ_i, defined by $\rho Y_i = n_i m_p$, $\rho = \mu_i n_i m_p$, $Y_i = 1/\mu_i$. For the ions, we also define the mass fraction X_i as $\rho X_i = n_i A_i m_p$. Then $Y_i = X_i/A_i$.

As an example, consider a fully-ionized solar composition gas with hydrogen mass fraction $X_H = 0.7$ and helium mass fraction $X_{He} = 0.3$. The ion pressure is

$$P_{\rm ion} = n_H k_B T + n_{He} k_B T = \frac{\rho k_B T}{m_p}\left(X_H + \frac{X_{He}}{4}\right) = \frac{\rho k_B T}{\mu_i m_p},$$

which defines $\mu_{\rm ion} = (X_H + X_{He}/4)^{-1} \approx 1.3$. For a general mixture of ions,

$$Y_{\rm ion} = \frac{1}{\mu_{\rm ion}} = \sum Y_i = \sum \frac{X_i}{A_i}.$$

The electrons contribute $P_e = n_e k_B T$ to the pressure if they are non-degenerate. From charge neutrality, $n_e = \sum n_i Z_i$ and so

$$P_e = \frac{\rho k_B T}{m_p}\sum Y_i Z_i = \frac{\rho k_B T}{m_p}\sum \frac{X_i Z_i}{A_i} = \frac{\rho k_B T}{\mu_e m_p}.$$

For the H/He mixture, we infer $\mu_e = (X_H + X_{He}/2)^{-1} \approx 1.2$. The total pressure is

$$P = (n_e + n_H + n_{He})k_B T = \frac{\rho k_B T}{m_p}\left(\frac{1}{\mu_{\rm ion}} + \frac{1}{\mu_e}\right) = \frac{\rho k_B T}{\mu m_p}.$$

This defines the mean molecular weight $\mu^{-1} = \mu_e^{-1} + \mu_{\rm ion}^{-1}$. For the solar mixture, $\mu^{-1} = 2X_H + 3X_{He}/4 \approx 0.6$.

Pure H has $\mu_e = \mu_i = 1$ and $\mu = 1/2$. Pure He has $\mu_e = 2$, $\mu_i = 4$, and $\mu = 4/3$. Heavier elements than helium also have $\mu_e \approx 2$ since $A \approx 2Z$ for all nuclei except hydrogen.

The ρ–T plane

The figure on the next page shows the different regions of the ρ–T plane, assuming a composition of pure helium. Electrons become degenerate when $E_F \approx k_B T$ (dashed line). For non-relativistic electrons, this is

$$\frac{\hbar^2}{2m_e}\left(3\pi^2 n_e\right)^{2/3} \approx k_B T \Rightarrow T_{d,nr} \approx 3\times 10^5\ {\rm K}\ (\rho Y_e)^{2/3},$$

using $n_e = \rho Y_e/m_p$. (For ions to become degenerate, would need to lower the temperature by a factor $> m_p/m_e \sim 2000$.) Degenerate electrons become relativistic when

$$p_F = \hbar(3\pi^2 n_e)^{1/3} \approx m_e c \Rightarrow \rho Y_e \approx 10^6\ {\rm g\ cm^{-3}}$$

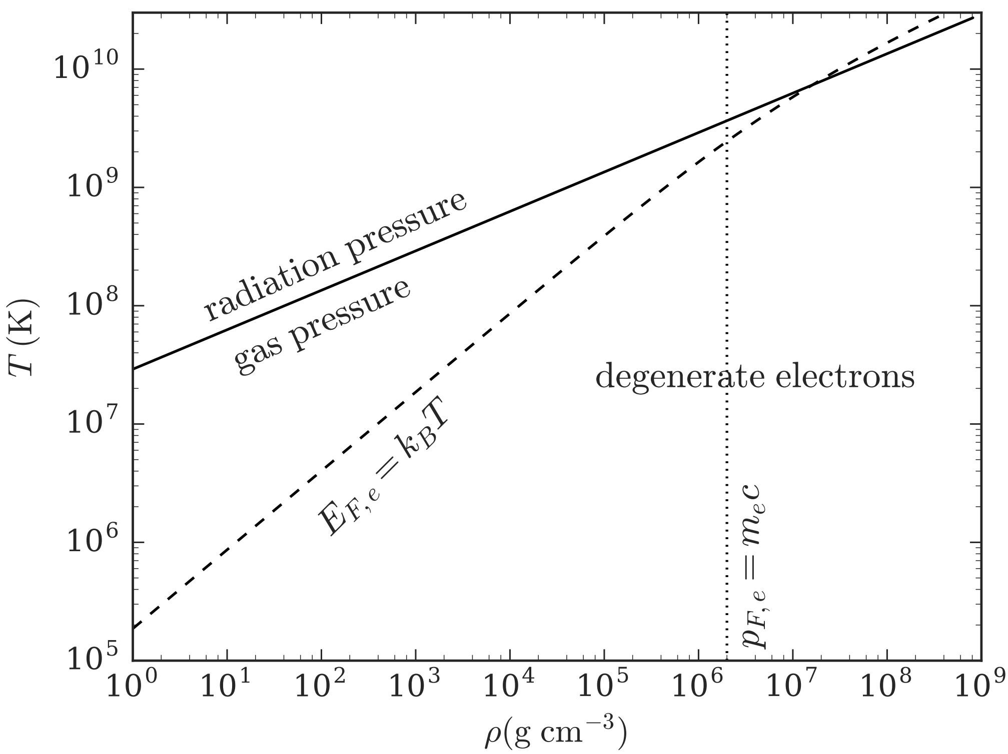
(vertical dotted line in the plot). The dashed line shown in the plot takes relativity into account by writing

$$E_F = m_e c^2 \left(\sqrt{1+x^2} - 1\right) \approx k_B T$$

where $x = p_F/m_e c \approx (\rho Y_e/10^6 \text{ g cm}^{-3})^{1/3}$; notice it changes slope at $\rho \gtrsim 10^6 \text{ g cm}^{-3}$ once the electrons become relativistic.

The solid curve shows the boundary between radiation pressure and gas pressure, assuming the gas pressure is ideal:

$$\frac{1}{3} a T^4 = \frac{\rho k_B T}{\mu m_p} \Rightarrow T_{\rm rad} = \left(\frac{3\rho k_B}{\mu m_p a}\right)^{1/3} \approx 3 \times 10^7 \text{ K} \left(\frac{\rho}{\mu}\right)^{1/3}.$$



White dwarf mass-radius relation

White dwarfs are stars held up by degenerate electron pressure. For low masses, the electrons are non-relativistic so that $P \propto \rho^{5/3}$, but as the mass approaches the Chandrasekhar mass the electrons become more and more relativistic and $\gamma \to 4/3$. (The positive ions also have a pressure, but it is much smaller than the electrons. That is because the ions are non-degenerate, so their pressure is a factor $\sim k_B T/E_F$ times smaller.)

As we mentioned earlier, the solutions of the stellar structure equations (1) and (2) for $P \propto \rho^\gamma \propto \rho^{1+1/n}$ are known as polytropes. You can look up the properties of polytropes for different values of polytropic index n, in particular the numerical solutions give the values of

$$\alpha_n = \frac{P_c}{GM^2/R^4} \qquad \beta_n = \frac{\rho_c}{\langle \rho \rangle},$$

where $\langle \rho \rangle = 3M/4\pi R^3$ is the mean density. For $\gamma = 5/3$, $n = 3/2$, $\alpha = 0.77$ and $\beta = 5.99$. For $\gamma = 4/3$, $n = 3$, $\alpha = 11.1$ and $\beta = 54.2$.

To get the white dwarf mass–radius relation, we write the equation of state at the center as $P_c = K_{nr}\rho_c^{5/3}$, where

$$K_{nr} = \frac{P}{\rho^{5/3}} = \frac{2}{5}\frac{nE_F}{\rho^{5/3}} = \frac{2}{5}\frac{n}{\rho^{5/3}}\frac{p_F^2}{2m} = \frac{2}{5}\frac{\hbar^2(3\pi^2)^{2/3}}{2m}\left(\frac{n}{\rho}\right)^{5/3} = 9.9\times 10^{12}\ \mathrm{cgs}\ Y_e^{5/3}.$$

Then using the $n = 3/2$ polytrope results for α and β gives the white dwarf mass-radius relation at low masses

$$R_{5/3} = M^{-1/3}\left(\frac{K_{nr}}{\alpha_{3/2}G}\right)\left(\frac{3\beta_{3/2}}{4\pi}\right)^{5/3} \approx 9\times 10^8\ \mathrm{cm}\ \left(\frac{M}{M_\odot}\right)^{-1/3}\left(\frac{Y_e}{0.5}\right)^{5/3}.$$

(We write $R_{5/3}$ to indicate that this is the white dwarf radius assuming $\gamma = 5/3$). As the star gets more massive, the radius shrinks. The central density increases rapidly with mass, $\rho_c \propto M/R^3 \propto M^2$.

Doing the same thing for the equation of state $P_c = K_r\rho_c^{4/3}$, the radius drops out and we get an expression for the Chandrasekhar mass

$$M_{Ch} = \left(\frac{K_r}{\alpha_3 G}\right)^{3/2}\left(\frac{3\beta_3}{4\pi}\right)^2 = 1.45\ M_\odot\ \left(\frac{Y_e}{0.5}\right)^2.$$

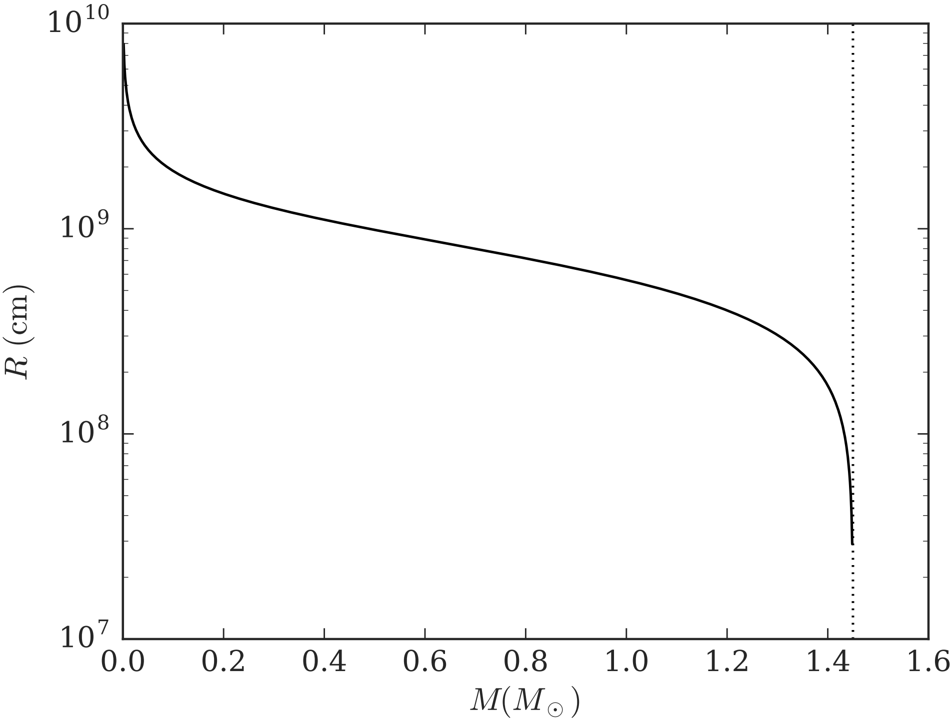
We can interpolate between the two limits by using the fitting formula obtained by Paczynski (1983) for the pressure of degenerate electrons

$$P_e^{-2} \approx P_{e,nr}^{-2} + P_{e,r}^{-2}, \tag{3}$$

which interpolates between non-relativistic and relativistic electrons (and Pacynski found was accurate to a few percent). If you use this formula for the central pressure, you will find

$$R \approx R_{5/3}\left[1 - \left(\frac{M}{M_{Ch}}\right)^{4/3}\right]^{1/2}.$$

Here is a plot of this $M(R)$ relation:



As the mass approaches the Chandrasekhar mass, the central density increases dramatically (because of decreasing radius but also the increasing value of β_n as $\gamma \rightarrow 4/3$, see above). Once it gets to $\rho_c \sim 10^9$ g cm^{-3}, interesting things can happen. One possibility is carbon fusion leading to a Type Ia supernova. The other is that electrons can capture into the nuclei, removing pressure support and leading to collapse to a neutron star. (White dwarfs can reach these large masses either through merging or accretion, or through stellar evolution, e.g. the iron core of a massive star).

Neutron stars

We saw that the radius of a $\gamma = 5/3$ star is $R \propto M^{-1/3} K_{nr}$. The key point for neutron stars is that $K_{nr} \propto 1/m$ where m is the mass of the degenerate particle. For white dwarfs this is the electron mass; for neutron stars, the star is held up by degenerate neutron pressure and we should take $m = m_n$ the neutron mass. We expect the radius of a neutron star to be smaller than a white dwarf by a factor of $m_n/m_e \approx 2000$, or $R_{NS} \sim 10^9$cm$/2000 \approx 5$ km. This is about right. Detailed models give neutron star radii $\approx$ 10–13 km. They are a little larger because the neutrons repel each other when they are very close, so that the equation of state is stiffer than $\gamma = 5/3$, in fact closer to $\gamma \approx 2$. As we argued in the beginning, this gives radius almost independent of mass, which is seen in detailed calculations of mass–radius relations.

Coulomb pressure and planets

If you plug in Jupiter's mass $M_J \approx 10^{-3} M_\odot$ into the white dwarf mass-radius relation, you'll get a radius $\approx 10^{10}$ cm which is not too far off ($1\ R_J \approx 0.1\ R_\odot \approx 7 \times 10^9$ cm). But clearly, as we reduce mass further something else must happen: eventually, we expect to see radius get smaller with decreasing mass. For example, if we scale up from the Earth assuming the same density ($M \propto R^3$) then that is also not so far off — Jupiter is about 10 times the radius of Earth and about 300 times the mass[2]. Somehow the mass–radius relation must turnover and change from $M \propto R^{-3}$ to $M \propto R^3$ at low masses.

What happens is that the Coulomb attraction of the positive ions and electrons in the plasma becomes important, leading to a negative contribution to the pressure, the *Couloumb pressure*. To calculate the size of this effect, first note that it is a good approximation to assume the electrons are uniformly distributed in space because $E_F \gg Ze^2/a$ where a is the interion spacing, so the electrons barely notice the ions. Then we can use the Wigner-Seitz approximation to calculate the energy associated with each ion. We consider an electrically-neutral sphere of radius R_Z around each ion that contains Z electrons, ie. $(4\pi R_Z^3/3)n_e = Z$. The electrostatic energy of the sphere has two contributions:

$$U_{ee} = \frac{3}{5}\frac{(Ze)^2}{R_Z} \qquad \text{electron} - \text{electron repulsion}$$

$$U_{ei} = -\frac{3}{2}\frac{(Ze)^2}{R_Z} \qquad \text{electron} - \text{ion attraction.}$$

The total energy per unit volume is then

$$U_C = -n_e \frac{9}{10}\frac{Ze^2}{R_Z} = -\frac{9}{10}\left(\frac{4\pi}{3}\right)^{1/3} Z^{2/3} e^2 n_e^{4/3}$$

(where I used $n_i = n_e/Z$). Notice that U_C becomes more negative as density increases, giving a negative pressure! The pressure is $-\partial(U_C V)/\partial V$ for volume V, giving $P_C = (1/3)U_C$ or

$$P_C \approx -6 \times 10^{12}\ \mathrm{erg\ cm^{-3}}\ (\rho Y_e)^{4/3} Z^{2/3}.$$

We can do two interesting things with this. The first is the *zero pressure solid*. We write down the total pressure from electrons and Coulomb:

$$P_{\mathrm{tot}} = K_e \rho^{5/3} - K_C \rho^{4/3}. \tag{4}$$

There is a zero-pressure solution with density

$$\rho_0 = \left(\frac{K_C}{K_e}\right)^3 \approx 0.2\ \mathrm{g\ cm^{-3}}\ ZA.$$

[2] I'm ignoring factors from composition differences in this paragraph. Earth is about 4 times denser than Jupiter, and the Y_e in a white dwarf is ≈ 0.5 whereas Jupiter is mostly hydrogen so will have Y_e closer to 1.

This overpredicts the density of terrestrial metals: for example, copper has $A \approx 64$ and $Z = 29$, giving $\rho_0 \sim 300$ g cm^{-3} (actual density is 9 g cm^{-3}), but the electronic configuration is much more complex than we have assumed in our simple model. The important point is that we have found a self-bound state which exists without any confining pressure. So in the low mass limit we might expect to see $M \sim \rho_0 R^3$ as expected.

The second thing is then to add in gravity:

$$\frac{GM^2}{R^4} \approx K_e \left(\frac{M}{R^3}\right)^{5/3} - K_C \left(\frac{M}{R^3}\right)^{4/3}.$$

Solving for R, we get

$$R = \frac{K_e}{GM^{1/3} + K_C M^{-1/3}}.$$

The two limits are $R = (K_e/G)M^{-1/3}$ "white dwarf" and $R = (K_e/K_C)M^{1/3}$ "rock". The maximum radius is where $M = (K_C/G)^{3/2} \approx 0.4\ M_J$.

The interplay between degeneracy pressure and Coulomb pressure, leading to the turnover of the $R(M)$ relation, is the reason why the radii of brown dwarfs are about the same as Jupiter, despite being 30–100 times more massive!

Papers

- Chandrasekhar 1931 "Maximum mass of white dwarfs" http://adsabs.harvard.edu/abs/1931ApJ....74...81C and "The density of white dwarf stars" http://www.tandfonline.com/doi/abs/10.1080/14786443109461710
- Holberg et al. 2012 WD "Observational constraints on the degenerate mass-radius relation" http://adsabs.harvard.edu/abs/2012AJ....143...68H, or more recent Parsons et al. 2017 "Testing the white dwarf mass-radius relationship with eclipsing binaries" https://ui.adsabs.harvard.edu/#abs/2017MNRAS.470.4473P/abstract
- Salpeter 1961 "Energy and pressure of a zero-temperature plasma" http://adsabs.harvard.edu/abs/1961ApJ...134..669S
- Seager et al. 2007 "Mass-radius relationships for solid exoplanets" http://adsabs.harvard.edu/abs/2007ApJ...669.1279S
- Lattimer and Prakash 2001 "Neutron star structure and the equation of state" http://adsabs.harvard.edu/abs/2001ApJ...550..426L

Exercises

1. *Gravitational energy of a star.* Even without any knowledge of the equation of state, there are certain integral relations that can be derived using only the fact that a star is in hydrostatic balance. Here is an example. The gravitational binding energy of a star is

$$\Omega = -\int \frac{Gm}{r} dm.$$

Using equations (1) and (2) and an integration by parts, show that

$$\Omega = -3 \int P dV,$$

where $dV = 4\pi r^2 dr$ is the volume element.

2. *Gravitational energy of a polytrope.* We can use the result from exercise 1 to derive an expression for the gravitational energy of a polytrope.

(a) First show by integrating by parts that

$$\int P dV = \int m \, d\left(\frac{P}{\rho}\right) = \left(\frac{\gamma - 1}{\gamma}\right) \int m \frac{dP}{\rho}.$$

(b) Next, use equation (1) to change integration variables to r and integrate by parts to find

$$\int P dV = \left(\frac{\gamma - 1}{\gamma}\right) \left[\frac{GM^2}{R} + 2\Omega\right].$$

(c) Now apply the result from exercise 1 to show that

$$-\Omega = \frac{3(\gamma - 1)}{5\gamma - 6} \frac{GM^2}{R} = \frac{3}{5 - n} \frac{GM^2}{R}.$$

As a check, what is the answer for an incompressible equation of state? Does it look familiar?

Appendix: TOV equations

When calculating the structure of a neutron star, general relativistic corrections are important, since

$$\frac{GM}{Rc^2} = 0.15 \left(\frac{M}{M_\odot}\right) \left(\frac{R}{10\ \mathrm{km}}\right)^{-1}.$$

The GR version of the stellar structure equations are known as the Tolman-Oppenheimer-Volkoff (TOV) equations. They are

$$\begin{aligned}
\frac{dm}{dr} &= 4\pi r^2 \rho \\
\frac{dP}{dr} &= -\rho \frac{Gm}{r^2} \left(1 + \frac{P}{\rho c^2}\right) \left(1 + \frac{4\pi r^3 P}{Gm}\right) \left(1 - \frac{2Gm}{rc^2}\right)^{-1} \\
\frac{d\Phi}{dr} &= -\frac{1}{\rho c^2} \frac{dP}{dr} \left(1 + \frac{P}{\rho c^2}\right)^{-1}.
\end{aligned}$$

As well as the continuity and momentum equations, there is an additional equation for the metric function Φ, which is defined such that the metric is

$$ds^2 = -e^{2\Phi} dt^2 + e^{2\lambda} dr^2 + r^2 d\Omega,$$

with

$$e^{2\lambda(r)} = \left(1 - \frac{2Gm}{r}\right)^{-1}.$$

To match onto the exterior Schwarzschild metric, $\Phi(R) = (1/2)\ln(1 - 2GM/Rc^2)$ at the surface of the star. Note that r is defined such that it corresponds to the sphere with surface area $4\pi r^2$ (or circumference $2\pi r$). The proper distance between two shells is $dr(1 - 2Gm/rc^2)^{-1/2}$, giving a volume element

$$\left(1 - \frac{2Gm}{rc^2}\right)^{-1/2} 4\pi r^2 dr.$$

The quantity $m(r)$ is the gravitational mass interior to coordinate r, equal to M at the surface.