PHYS 643 Computational Exercise: The white dwarf mass–radius relation

You should send me a write-up of your results and the code that you used to do the calculation and make plots by email before the class on September 27.

Overview. The goal of this exercise is to calculate the mass-radius relation for $T = 0$ white dwarfs. This requires numerically integrating the equation of hydrostatic balance. You can do this using whatever method you wish, but to help you I describe a possible procedure below.

Equations and boundary conditions. The structure of the white dwarf is given by the equations of hydrostatic balance

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2} \qquad \frac{dm}{dr} = 4\pi r^2 \rho.$$

The boundary conditions are $m = 0$ at $r = 0$ and $P = \rho = 0$ at $r = R$.

Equation of state. The integration variables are m and P, so at each step you will need to compute the density from the pressure using the equation of state. Usually the equation of state is given the other way round, as a function $P(\rho)$, often as a numerical table for complex equations of state. Given $P(\rho)$, you can find the density ρ_0 corresponding to a particular pressure P_0 by solving the equation $P(\rho_0) = P_0$ numerically using a root-finding algorithm.

You can use this root-finding technique in your code, but there is also another option since we have an analytic expression for the equation of state. In this case, we can change integration variables from $P \to \rho$, ie. use the analytic equation of state to write dP/dr in terms of $d\rho/dr$. As the white dwarf mass increases, the electrons go from being non-relativistic ($P \propto \rho^{5/3}$) to relativistic ($P \propto \rho^{4/3}$). To take this into account, we can use the analytic fitting formula for the equation of state derived by Paczynksi (1983) that I mentioned in the notes:

$$P^{-2} = P_{nr}^{-2} + P_r^{-2},$$

where $P_{nr} = K_{nr}\rho^{5/3}$ is the non-relativistic degenerate electron pressure and $P_r = K_r\rho^{4/3}$ is the relativistic degenerate electron pressure. It is then straightforward to show that

$$\frac{d\ln P}{dr} = \frac{d\ln\rho}{dr}\left[\frac{5}{3}\left(\frac{P}{P_{nr}}\right)^2 + \frac{4}{3}\left(\frac{P}{P_r}\right)^2\right].$$

Use the class notes to determine the constants K_{nr} and K_r. Assume a carbon/oxygen white dwarf which has $Y_e = 0.5$.

Integration. Now the idea is to integrate outwards from the center of the star to the surface. At the center, there is a problem at $r = 0$ since the equation for dP/dr has an

r in the denominator and we can't divide by zero! To avoid this, start the integration at a small distance $r = \epsilon$ from the center, where $\rho \approx \rho_c$ and $m \approx 4\pi\rho_c\epsilon^3/3$. Here I've written the central density as ρ_c.

Integrate outwards until the density falls to zero. The radius at which $\rho = 0$ is the radius of the star $r = R$, and the value of m at this point is the mass M of the star. How you do this step in practice depends on your integrator. Most likely you will have to tell your integration routine to integrate from $r = r_1 = 0$ to $r = r_2$. In that case, try different values of r_2 until you find the one that gives $\rho = 0$ at the edge.

Repeat this integration for several different choices for central density ρ_c logarithmically spaced from about 10^6 g cm^{-3} to 10^9 g cm^{-3}. You'll have to experiment to get the correct range in central density that covers a mass range up to the Chandrasekhar mass at $\approx 1.4\ M_\odot$.

Questions

1. *Mass-radius relation*. Plot the curve of radius against mass. For low masses, when the central density is small and $\gamma = 5/3$, you should find $R \propto M^{-1/3}$, but you'll see the slope changes at large masses. How does your answer compare with the analytic approximation from the lecture notes,

$$R = 8.7 \times 10^8 \text{ cm} \left(\frac{M}{M_\odot}\right)^{-1/3} \left[1 - \left(\frac{M}{M_{Ch}}\right)^{4/3}\right]^{1/2},$$

   and what do you determine to be the Chandrasekhar mass?

2. *Investigate the density profile.* Plot the density profile as a function of radius for different mass white dwarfs. How does the density profile change as you change the white dwarf mass?

3. *The extent to which the electrons are relativistic.* Plot the value of γ and $E_F/m_e c^2$ at the center of the white dwarf as a function of mass.

A possible extension is to add the Coulomb energy of the ions (see eq. [4] of the notes) to the pressure. Are there masses where the Coulomb energy becomes important?