PHYS 643 Week 3: Hot Stars — Energy Transport, Nuclear Burning, and Stellar Evolution

We now move onto "hot stars" for which $k_B T \gg E_F$ and temperature matters in describing their structure. An important difference from cold stars is that hot stars can cool, so we need to understand energy sources and sinks and energy transport inside the star.

Radiative diffusion, opacity, and the luminosity of stars

The main energy transport mechanism in stars is diffusion of photons. The mean free path of a photon is $\lambda = 1/n\sigma$ where n is the number density of scatterers or absorbers and σ is the cross-section. In astrophysics, we usually write everything per gram, so that $\lambda = 1/\rho\kappa$ where κ is the cross-section per gram, or the *opacity*. For example, free electrons scatter photons with the Thomson cross-section

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 = 6.67\times 10^{-25}\ \mathrm{cm}^2.$$

For pure hydrogen, the opacity is $\kappa = \sigma_T/m_p = 0.40\ \mathrm{cm}^2\ \mathrm{g}^{-1}$. The photon mean free path in the center of the Sun is then $\lambda \approx 10^{-2}$ cm (taking $\rho = 150\ \mathrm{g\ cm}^{-3}$). This is obviously much less than the solar radius, so photons are scattered or absorbed many times on traversing the Sun, but it is also much longer than the particle mean free path ($\sim 10^{-6}$ cm; as we discussed in Week 1), so that photons carry information about the temperature at their origin to the location where they are absorbed. Other important opacity sources in stars are *free-free* and *bound-free absorption*, associated with an electron absorbing a photon in the presence of a nucleus. Unlike electron scattering, the bound-free and free-free opacities depend on density and temperature, with the *Kramer's* scaling $\kappa \propto \rho T^{-7/2}$.

The heat flux carried by the diffusing photons is

$$F = -\frac{1}{3}c\,\lambda\,\frac{d}{dr}\left(aT^4\right) = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr}.$$

The outwards luminosity at radius r is then $L = 4\pi r^2 F$. Note that in general the opacity depends on the local density, temperature and composition so we can write $\kappa(\rho, T, X_i)$ where X_i is a set of mass fractions describing the composition. The heat flux is of the form we discussed in Week 1, $F = -K\nabla T$ where $K \propto T^3/\kappa\rho$ is the thermal conductivity.

Let's use the radiative diffusion equation to estimate the luminosity of a star. We mentioned last time that hydrostatic balance is enough to estimate the central temperature of a star if we know its mass and radius,

$$k_B T_c \approx \frac{GMm_p}{R} \Rightarrow T_c \approx 2\times 10^7\ \mathrm{K}\left(\frac{M}{M_\odot}\right)\left(\frac{R}{R_\odot}\right)^{-1}.$$

The hot interior implies a luminosity

$$L \sim 4\pi R^2 \frac{4acT^4}{3\kappa R}\frac{3R^3}{4\pi M} \sim \frac{4acT^4R^4}{\kappa M},$$

where we write $r \approx R$, $\rho \approx (4\pi/3)(M/R^3)$, and $dT/dr \approx T/R$. Now putting in T_c for the temperature,

$$L \sim \frac{4ac}{\kappa}\left(\frac{Gm_p}{k_B}\right)^4 M^3 \sim 8\times 10^{36}\ \mathrm{erg\ s^{-1}}\ \left(\frac{M}{M_\odot}\right)^3\left(\frac{\kappa}{0.4\ \mathrm{cm^2\ g^{-1}}}\right)^{-1}.$$

This is about 1000 times too big for the Sun which has $L_\odot \approx 4\times 10^{33}$ erg s^{-1}; putting in an average temperature (e.g. at $r \approx 0.5R$ the temperature in the Sun is about $T_c/5$) would give a more reasonable value. The important thing is the scaling $L \propto M^3$ which is seen in models for stars with mass $M \gtrsim M_\odot$ for which the central temperature is large enough that electron scattering dominates the opacity. For $M \lesssim 1M_\odot$, free-free opacity dominates instead, introducing a temperature and density scaling into κ. These low mass main-sequence stars have a steeper dependence $L \propto M^{5.5}$.

An alternative energy transport mechanism in stars is *convection*, in which fluid motions transport heat. We'll look more into this when we talk about instabilities, but the basic idea is that if the temperature gradient is steep enough, the entropy gradient in the star can become negative (entropy decreases outwards). High entropy material underneath low entropy material is unstable to mixing and results in convection. Stars can be fully-convective (low mass stars $\lesssim 0.3M_\odot$), have a surface convection zone ($M \sim M_\odot$), or a convective core ($M \gtrsim M_\odot$).

Thermonuclear reactions

We've seen that a star must be hot to hold itself up against gravity $T_c \propto M/R$, and that implies a certain luminosity ($L \propto M^3$ for electron scattering). The luminosity is supplied by nuclear burning – at each stage of a star's life, the radius of the star adjusts to give the right central temperature at which nuclear burning can balance the luminosity.

For two nuclei to fuse, they must approach to a distance $\sim 10^{-13}$ cm (about the size of a nucleus) at which strong forces operate. In practise, this is not possible because of Coulomb repulsion between nuclei. For example, at the Sun's central temperature, the average energy of protons is ≈ 1 keV. We know that the binding energy of hydrogen e^2/a_0 is about 10 eV for $a_0 \sim 10^{-8}$ cm, so at 1 keV, the closest approach distance must be $\sim 10^{-10}$ cm. This is a factor of 1000 too large for fusion to occur. How then do nuclear reactions happen? The answer is that the protons tunnel through the Coulomb barrier.

We can estimate the probability for quantum tunnelling by saying that the wavefunction drops by a factor of e^{-kx} for a barrier of width x where $k \approx \sqrt{2mV_0}/\hbar$ is the wavevector of the evanescing wavefunction. For a closest approach r_c, the potential barrier height is $V_0 \sim e^2/r_c$ and the width of the barrier is $r_c \approx e^2/E$ where E is the center of mass energy of the two protons. Therefore

$$kx \sim \frac{e}{\hbar}\sqrt{2mr_c} \sim \sqrt{\frac{2mc^2}{E}}\frac{e^2}{\hbar c} = \sqrt{\frac{2\alpha^2 mc^2}{E}},$$

where $\alpha = e^2/\hbar c = 1/137$ is the fine-structure constant. A more detailed treatment which integrates through the barrier gives a similar result but with an extra factor of π in the prefactor. Also including the charges of the fusing nuclei Z_1 and Z_2, the final tunnelling probability is

$$\text{Prob} \propto \exp\left(-\sqrt{\frac{E_G}{E}}\right)$$

where

$$E_G = 2\pi^2\alpha^2 mc^2 (Z_1 Z_2)^2 \approx 1\text{ MeV } Z_1^2 Z_2^2 \left(\frac{m}{m_p}\right)$$

is the *Gamow energy* and m is the reduced mass $m = m_1 m_2/(m_1 + m_2)$.

The higher the energy E, the more likely tunnelling is to occur, but the probability two particles have that energy is smaller, $\propto e^{-E/k_B T}$. The tunnelling rate is therefore a convolution between the tunnelling probability and Maxwell-Boltzmann distribution of particle energies. The tunnelling is most likely for energy E_0 where $\exp(-E_0/k_B T - \sqrt{E_G/E_0})$ has a maximum, or $E_0 = (k_B T)^{2/3}(E_G/2)^{1/3}$. For $k_B T \approx 1$ keV and $E_G \approx 1$ MeV, this is $E_0 \approx 6$ keV. The energies around E_0 where the reaction is most likely to occur is called the *Gamow window*. For many reactions, the energy-dependence of the cross-section must also be taken in to account, particularly when there is a resonance which boosts the cross-section at the resonant energy.

The fact that nuclear fusion is happening only for particles in the tail of the Maxwell-Boltzmann distribution means that thermonuclear reaction rates are extremely temperature sensitive. Another property of nuclear burning is that heavier nuclei have larger Z's and so a larger Coulomb barrier, and require higher temperatures to fuse. The larger Z nuclei have a larger factor in the exponent and so have reaction rates that are more temperature sensitive than lower Z nuclei. One impact of this for main sequence stars is that massive main sequence stars $M \gtrsim M_\odot$ which burn hydrogen via the CNO cycle have T_c roughly independent of mass and so $R \propto M$. (It's actually a bit shallower because T_c increases a little bit with M.)

Stellar evolution

The full set of equations that are needed to follow the evolution of a hot star are

$$\frac{\partial m}{\partial r} = 4\pi r^2 \rho$$

$$\frac{\partial P}{\partial r} = -\frac{\rho G m}{r^2}$$

$$T\frac{\partial S}{\partial t} = \epsilon_{\rm nuc} - \epsilon_\nu - \frac{1}{4\pi r^2 \rho}\frac{\partial L}{\partial r}$$

$$\frac{\partial T}{\partial r} = \frac{\partial P}{\partial r}\frac{T}{P}\nabla$$

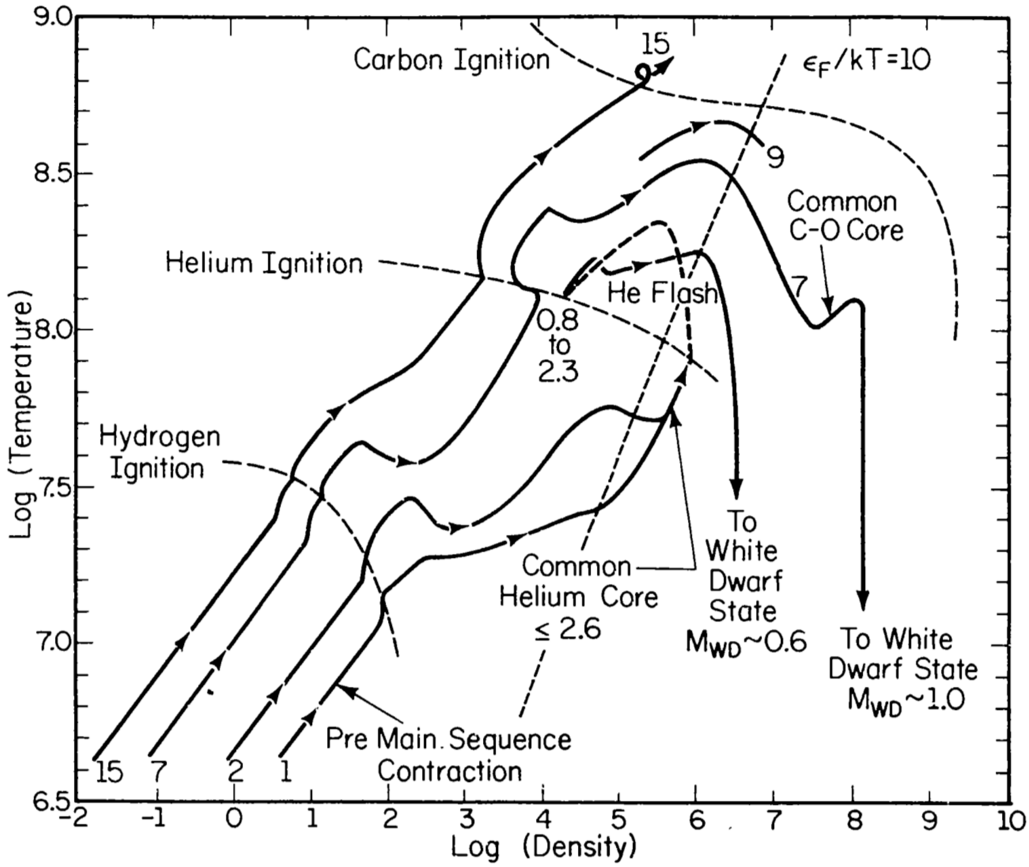
$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\rho}\sum_j \left(r_{ji} - r_{ij}\right) + D\nabla^2 X_i$$

where the temperature gradient ∇ is determined by the energy transport process. If radiation is transporting energy,

$$\nabla = \nabla_{\rm rad} = \frac{3\kappa P L}{16\pi a c G m T^4},$$

from the radiative diffusion equation. When convection operates, the temperature gradient is usually close to the adiabatic gradient $\nabla = \nabla_{\rm ad}$ (because this is the entropy-neutral gradient that marks the onset of convection). The nuclear energy generation rate per gram is written as $\epsilon_{\rm nuc}$ (units are erg g^{-1} s^{-1}). In massive stars in late burning stages the temperature and density can be large enough that neutrinos become an effective cooling source. The local neutrino cooling rate is written as ϵ_ν. The last equation is actually a set of equations, one for each species, which follow the change in composition as nuclear reactions occur and as diffusion, convection or other processes mix composition in the star (for simplicity, I just put a diffusion term here).

Overall, the life of a star involves moving to higher central temperatures and densities, stopping at various nuclear burning stages, until the core becomes degenerate. This is illustrated in the figure below, taken from Iben (1985) http://adsabs.harvard.edu/abs/1985QJRAS..26....1I.



Several codes to follow stellar evolution are available. An interesting one to try is MESA (Modules for Experiments in Stellar Astrophysics). See their website at http://mesa.sourceforge.net for references and more details. Here are two movies showing the evolution of main sequence stars:
1 solar mass – https://www.youtube.com/watch?v=oZY3TtA63sE
3 solar masses – https://www.youtube.com/watch?v=C4tucmhAaSk
Watching these movies you'll see that the nuclear burning is often unstable, leading to a rapid local rise in temperature within the star. This happens either when the nuclear burning is in a degenerate region (e.g. when helium ignites in the core of the solar mass star) or when the burning is in a thin shell (He burning or H burning shells in giants). In either of these situations, the star is not able to lower the presure by expansion in response to nuclear energy release. The temperature rises and the nuclear burning runs away.

Cores and envelopes

In stellar evolution, there is an interesting interplay between cores and envelopes. In a main sequence star like the Sun, the star is relatively compact, with a smooth change in density, temperature and composition from the center to the surface. However, once hydrogen runs out in the core and the main sequence lifetime ends, the star adopts a very different structure. The star contracts and heats up until the hydrogen at the edge of the helium core is hot enough to ignite. The ignition of a shell source has a dramatic effect on the hydrostatic structure of the star, which becomes a red giant, with a large, low density, extended hydrogen envelope sitting on top of a compact helium core in the center. This is a general feature: *if the nuclear burning is central, the star will be compact; if burning is in a shell source, the star adopts a giant structure.*

In a red giant, the core is isothermal at a temperature that is regulated by the shell H burning. An interesting aspect of an isothermal non-degenerate core is that there is a maximum mass envelope that it can support. The way to see this is to write an equation for the pressure at the surface of the core P_s. Integrating the hydrostatic balance equation from the center to the surface of the core gives

$$P_s = A\frac{T_c M_c}{R_c^3} - B\frac{GM_c^2}{R_c^4} \tag{1}$$

for constants A and B that depend on the internal density profile (the core has mass M_c and radius R_c). Think of this as saying that the surface pressure is the mean pressure in the core reduced by the weight of the core. For zero pressure at the surface, the radius is $R_0 = BGM_c/AT_c$ (which shows the $T \propto M/R$ scaling we've seen before).

Equation (1) has the interesting feature that there is a maximum pressure. At large core radius, both terms go to zero, so the surface pressure becomes small. At small core radius, the gravitational term increases faster than the mean pressure term, also reducing the pressure. The maximum surface pressure is

$$P_{s,\max} = \frac{27}{256}B\frac{GM_c^2}{R_0^4} \propto \frac{T_c^4}{M_c^2}$$

at a radius $R = (4/3)R_0 \propto M_c/T_c$.

The maximum surface pressure means that there is a maximum mass envelope that the core can support hydrostatically. This is known as the *Schönberg-Chandrasekhar limit*, and can be written as a ratio of core mass to total mass. This is because most of the mass of the star is contained in the envelope, so the pressure at the base of the envelope is $P_b \approx GM^2/R^4 \propto T_c^4/M^2$ since the (base of the) envelope is at the same temperature as the core and $T \propto M/R$. This means that $P_b/P_{s,\max} \propto (M_c/M)^2$. Typically the limit is found to be $M_c/M \lesssim 0.1$ for stability.

For red giants, this can lead to collapse of the helium core: as the hydrogen shell adds more and more helium to the core, it grows in mass. Once it reaches the Schönberg-Chandrasekhar mass, it collapses, initiating helium burning in the core. In practise, this happens only in a limited range of masses, because massive stars leave the main sequence with a helium core that already exceeds the Schönberg-Chandrasekhar limit.

Note that the surface pressure does not have this behaviour for a degenerate core: then the pressure $\propto 1/R^5$ rather than $1/R^3$ and the radius can always adjust to supply any surface pressure needed. In that case, the helium burning starts in an unstable way once the core temperature reaches a critical value, giving a core helium flash. This means that there is a separation in stellar evolution between stars that develop a degenerate helium core and undergo a helium core flash ($\lesssim 2\ M_\odot$) and those that have a non-degenerate helium core and do not undergo a core flash ($\gtrsim 2\ M_\odot$).

Papers

- Stevenson 1982 "Formation of the Giant Planets" http://adsabs.harvard.edu/abs/1982P%26SS...30..755S
- Ushomirsky et al. 1998 "Light element depletion in contracting brown dwarfs and pre-main-sequence stars" http://adsabs.harvard.edu/abs/1998ApJ...497..253U
- Deloye & Bildsten 2003 "The stellar structure of finite-entropy objects" http://adsabs.harvard.edu/abs/2003ApJ...598.1217D
- Woosley & Heger 2015 "The remarkable deaths of 9-11 solar mass stars" http://adsabs.harvard.edu/abs/2015ApJ...810...34W

Appendix: Gravothermal heat capacity

In class, we discussed the fact that the heat capacity of a star is negative: the temperature decreases in response to energy input. Here's how this works, following a similar argument to the one in Kippenhahn and Wiegert's book on stellar structure. We can ask: what is the response of the gas to entropy changes?

First, write the entropy change in terms of temperature and pressure:

$$dS = \left.\frac{\partial S}{\partial T}\right|_P dT + \left.\frac{\partial S}{\partial P}\right|_T dP.$$

Using the fact that the heat capacity at constant pressure is $c_P = T\,\partial S/\partial T|_P$, and the identity

$$\left.\frac{\partial S}{\partial T}\right|_P \left.\frac{\partial T}{\partial P}\right|_S \left.\frac{\partial P}{\partial S}\right|_T = -1,$$

we can write this as

$$TdS = c_P\left(dT - \frac{T}{P}\nabla_{\rm ad}dP\right), \tag{2}$$

where $\nabla_{\rm ad} = \partial \ln T/\partial \ln P|_S$.

So far, this is just thermodynamics, but now we put in the fact that the star is in hydrostatic balance, so that $P \propto 1/R^4$ and $\rho \propto 1/R^3$. This means that we must have

$$\frac{dP}{P} = \frac{4}{3}\frac{d\rho}{\rho}. \tag{3}$$

But the equation of state relates density to pressure and temperature changes through

$$d\ln P = \chi_T d\ln T + \chi_\rho d\ln \rho$$

where $\chi_X \equiv (\partial \ln P/\partial \ln X)$ with other variables held constant. Equation (3) becomes

$$\frac{\delta P}{P} = \frac{4\chi_T}{4-3\chi_\rho}\frac{\delta T}{T}. \tag{4}$$

Combining equations (2) and (3) gives

$$T\frac{dS}{dT} = c_P\left(1 - \frac{4\chi_T\nabla_{\rm ad}}{4-3\chi_\rho}\right) = c_\star,$$

where $c_\star$ is the effective heat capacity.

Now look at different limits:

- For an ideal gas, $\chi_T = 1$, $\chi_\rho = 1$, and for a monatomic gas $\nabla_{\rm ad} = 2/5$, so that $c_\star = -(3/5)c_P < 0$. (The Sun is stable).

- For a degenerate gas, $\chi_T \sim k_BT/E_F \to 0$ so that the correction term becomes small and $c_\star \to c_P > 0$. (Helium core flash).

- If the burning is in a thin shell, equation (3) is no longer correct. To see this, consider a shell that has mass ΔM, thickness H and is located at radius r. If the shell changes its thickness by δH, the pressure change is of order $\delta H/r$, since pressure is $\sim GM\Delta M/4\pi r^4$. On the other hand the change in density is of order $\delta H/H$ (since mass conservation $\Rightarrow r^2\rho H =$constant). Therefore for a thin shell,

  $$\frac{\delta P}{P} \sim \frac{H}{R}\frac{\delta\rho}{\rho},$$

  which means that $c_\star \approx c_P$ to first order in H/r. Burning in a thin shell is therefore unstable. This is the origin of the term *thin shell flash*.