PHYS 643 Week 1: Introduction

These notes are for the first week of PHYS 643 Astrophysical Fluids. The idea is to introduce the fluid equations, laying the groundwork for the specific topics in future weeks.

What is a fluid?

The fluid equations apply when the mean free path of particles λ is much smaller than the distances over which bulk properties, such as temperature and density, are varying.

For example, let's estimate λ in the Sun. At the center of the Sun, the temperature is $T \approx 10^7$ K and density $\rho \approx 150$ g cm^{-3}, so the matter is a completely-ionized plasma of (mostly) protons and electrons. The mean free path is given by $n\sigma\lambda = 1$ where n is the number density of scatterers and σ is the scattering cross-section. We can get n from the density, $n \approx \rho/m_p \sim 10^{26}$ cm^{-3}. The Coulomb cross-section is given roughly by writing down the scalings

$$\frac{e^2}{r} \sim k_B T \qquad \sigma \sim \pi r^2 \sim \frac{e^4}{(k_B T)^2}.$$

Plugging in numbers gives

$$\lambda \sim 10^6 \text{ cm } \frac{T^2}{n} \sim 10^{-6} \text{ cm}.$$

Much smaller than the radius of the sun $R_\odot \approx 7 \times 10^{10}$ cm, which is the scale on which the temperature varies. This large difference in scales means that the particles are in *local thermodynamic equilibrium* (LTE). For example, they have a Maxwell-Boltzmann distribution at the local temperature.

Under these conditions, we can treat the matter as a continuum and describe the matter with a set of conservation equations for mass, momentum and energy — the fluid equations. We don't have to worry about following the trajectories and interactions of individual particles (like in an N-body simulation of a star system for example), although there is a systematic derivation of the fluid equations from such a starting point (the Boltzmann equation), expanding in the small parameter λ/L. This is covered in the early chapters of Choudhuri.

Just to give a couple of situations in astrophysics where the fluid approximation is not so good, consider gas in a galaxy cluster with temperature $\sim 10^8$ K and $n \sim 10^{-3}$ cm^{-3}, then $\lambda \sim 10^{24}$ cm ≈ 0.3 Mpc, which is a large fraction of a typical cluster size. In the solar wind near Earth, $T \sim 10^5$ K and $n \sim 10$ cm^{-3} gives $\lambda \sim 10^{14}$ cm which is several AU.

The continuity equation, advective derivative, Eulerian and Lagrangian approaches, incompressible fluid

The continuity equation describes mass conservation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u}) .$$

This has a flux-conservative form: rate of change of a density on one side and the divergence of the flux of that quantity on the other side. The mass flux is $\rho\mathbf{u}$ (units: g cm^{-2} s^{-1}). Using the advective derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

this can be rewritten

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u}.$$

Make sure you are comfortable with the physical interpretation of this: e.g., if the flow converges, mass is flowing to a point and so the local density has to increase.

The advective derivative is also known as the Lagrangian derivative. It represents the rate of change of a quantity following along with the fluid element (Lagrangian approach) rather than asking what is the rate of change of the quantity at a fixed point in space (Eulerian approach).

An incompressible fluid (e.g. water) has a constant density, so that $D\rho/Dt = 0$ and $\nabla \cdot \mathbf{u} = 0$. (Think about how when a river widens, the water slows down so that the mass flow rate is the same). Incompressibility is a good approximation when the flow is subsonic $|u| \ll c_s$ because then any density variations will be rapidly smoothed out by sound waves much faster than the fluid motion.

Momentum equation; body and surface forces, viscosity, equation of state

The momentum equation in flux conservative form is

$$\frac{\partial}{\partial t} (\rho u_i) + \frac{\partial}{\partial x_j} \left(\rho u_i u_j\right) = f_i + \frac{\partial}{\partial x_j} T_{ij}.$$

This describes conservation of the ith component of momentum density ρu_i (momentum per unit volume). The flux of the ith component of momentum in the j-direction is $\rho u_i u_j$.

On the right hand side, forces act to change the momentum. They are of two types. *Body forces* act on each particle in the fluid element. The body force per unit volume in the i-direction is f_i. Examples are gravity, $\mathbf{f} = \rho\mathbf{g} = -\rho\nabla\Phi$, and magnetic force, $\mathbf{f} = \mathbf{J} \times \mathbf{B}/c$.

The second term represents surface forces, and T_{ij} is a stress tensor. The diagonal elements of the stress tensor are forces that push inwards or outwards on the surface of a fluid element (direction along the normal to the surface). An example is pressure, described by

$$T_{ij} = -P\delta_{ij}$$

which gives

$$\frac{\partial}{\partial x_j} T_{ij} = -\frac{\partial}{\partial x_i} P = -(\nabla P)_i$$

the ith component of the pressure gradient. Fluid elements feel a force down the pressure gradient. Physically, the pressure force on one side of the fluid element outbalances the pressure force on the other, giving a net acceleration.

With pressure and gravity forces only, and using the continuity equation to simplify the left hand side, a common form of the momentum equation is

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \rho \mathbf{g}.$$

(Think of this as $F = ma$ for a fluid element).

Viscosity in a fluid resists shear (as the random motions of particles transfer momentum between parts of the fluid moving with different velocities). It gives off-diagonal contributions to T_{ij}, i.e. the viscous force acts in a direction parallel to the surface of a fluid element rather than normal to it. In general, the viscous stress can be written

$$\sigma_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij} \nabla \cdot \mathbf{u} \right) + \xi \delta_{ij} \nabla \cdot \mathbf{u},$$

where μ is the *shear viscosity* and ξ is the *bulk viscosity* (units of viscosity: g cm^{-1} s^{-1}). The velocity derivatives in the first term describe shearing motions of the fluid (as opposed to rotation of a fluid element which would have a minus sign — see vorticity below), and is the usual viscosity that we worry about. The bulk viscosity is not usually important, it describes irreversible processes that occur when a fluid element is compressed. The quantity $\nu = \mu/\rho$ is the *kinematic viscosity* (units: cm^2 s^{-1}). For an ideal gas this is roughly $\nu \sim \lambda^2/t_c \sim \lambda v_{th}$, where t_c is the collision time and v_{th} is the thermal velocity of the particles in the gas. If the fluid motions have $\nabla \cdot \mathbf{u} \approx 0$, the viscous term in the momentum equation simplifies to

$$\frac{\partial}{\partial x_j} T_{ij} \approx \frac{\partial}{\partial x_j} \left(\mu \frac{\partial u_i}{\partial x_j} \right)$$

or for constant μ,

$$\approx \mu \nabla^2 \mathbf{u}.$$

This last form shows that viscosity leads to diffusion of momentum.

The momentum and continuity equations describe the fluid motion completely if we know how to relate P and ρ, which depends on the equation of state of the fluid. For example, if the fluid flow is rapid enough that there is no time for heat flow between fluid elements, the motion is adiabatic and we can write $P \propto \rho^\gamma$, where γ is the adiabatic index. The opposite limit is extremely rapid heat transport so that the gas remains isothermal, $P \propto \rho$. Intermediate cases require that we also follow the temperature of the gas which requires a third equation, the energy equation. We'll come to that soon.

Magnetic fields: the MHD equations

For a magnetized plasma, we already mentioned the $\mathbf{J} \times \mathbf{B}$ force in the momentum equation[1] We also need to discuss how the magnetic field evolves. The electric field by Ohm's law[2] is

$$\mathbf{E} = -\frac{\mathbf{u} \times \mathbf{B}}{c} + \frac{\mathbf{J}}{\sigma}.$$

The first term on the right hand side comes from the relativistic transformation from the fluid rest frame to the frame in which the fluid is moving with velocity $\mathbf{u}$. Faraday's law then gives the time-dependence of $\mathbf{B}$,

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E} = \nabla \times (\mathbf{u} \times \mathbf{B}) - c\nabla \times \left(\frac{\mathbf{J}}{\sigma}\right).$$

This is known as the *induction equation*.

The first term on the right hand side describes *flux freezing*. In the absence of ohmic dissipation (*ideal MHD*), magnetic field lines move with the fluid. A good way to see this is to derive an equation for the separation $d\ell$ between two fluid elements in a fluid. It turns out to be of the same form as the induction equation (without the ohmic term) but with $\mathbf{B} \rightarrow d\ell$. So if you take two fluid elements and follow them as they move through the flow, their separation vector and the local magnetic field vector evolve in the same way. That tells you that magnetic field lines are tied into the fluid.

To see the effect of the ohmic term, use Ampere's law[3], which gives the current density

$$\mathbf{J} = \frac{c}{4\pi}\nabla \times \mathbf{B}.$$

The induction equation is then

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B}),$$

where the magnetic diffusivity $\eta = c^2/(4\pi\sigma)$. Since $\nabla \cdot \mathbf{B} = 0$, $\nabla \times \nabla \times \mathbf{B} = -\nabla^2\mathbf{B}$ and so for constant η the ohmic term in the induction equation is

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B},$$

a diffusion equation for $\mathbf{B}$. We see that ohmic dissipation gives rise to *ohmic diffusion* of the magnetic field. It breaks flux freezing, and leads to motion of the field lines within the fluid.

The fluid equations with the $\mathbf{J} \times \mathbf{B}$ force, the induction equation, and Ampere's law together form the equations of magnetohydrodynamics (MHD).

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[1] If there is a non-zero charge density ρ_e, there will also be an electric force $\rho_e\mathbf{E}$, but usually in astrophysical situations the plasma is electrically neutral $\rho_e = 0$.

[2] In fact, there are other terms that can appear in Ohm's law. See the Appendix for a more general derivation of Ohm's law using the two fluid equations.

[3] The term $\partial\mathbf{E}/\partial t$ in Ampere's law can be dropped as long as the timescale on which $\mathbf{B}$ is evolving is much longer than a light crossing time. Note that we then have $\nabla \cdot \mathbf{J} = 0$, consistent with charge conservation.

One more thing we can do is to look in more detail at the $\mathbf{J} \times \mathbf{B}$ force using Ampere's law to write $\mathbf{J}$ in terms of $\mathbf{B}$. Then

$$\frac{\mathbf{J} \times \mathbf{B}}{c} = \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} = -\nabla \left(\frac{B^2}{8\pi} \right) + \frac{(\mathbf{B} \cdot \nabla)\mathbf{B}}{4\pi}.$$

The first term is the gradient of the magnetic pressure $B^2/8\pi$. The second term has two pieces. One piece has a direction along the field, and cancels the gradient along the field from the first term. The net effect is that the magnetic pressure acts only perpendicular to the field (as it must since the force is $\mathbf{J} \times \mathbf{B}$). So if you grab a flux tube and squeeze it, you will feel the magnetic pressure pushing back. The other piece of the $\mathbf{B} \cdot \nabla \mathbf{B}$ term is *magnetic tension*, which tries to make the fields lines straighten (like an elastic string). The magnitude of the tension force per unit volume is $B^2/4\pi R_c$, where R_c is the radius of curvature of the field line. We'll see later this force supports Alfven waves.

Energy equation

It helps to consider the bulk kinetic energy, internal energy, and magentic energy separately.

An equation for the kinetic energy density $(1/2)\rho u^2$ comes from carrying out the dot product

$$\mathbf{u} \cdot (\text{momentum equation}) \Rightarrow$$

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho u^2 \right) + \frac{\partial}{\partial x_j} \left(\frac{1}{2} \rho u^2 u_j \right) = \mathbf{u} \cdot \mathbf{f} + u_i \frac{\partial T_{ij}}{\partial x_j}.$$

Again this is flux-conservative form and says that the kinetic energy density changes if there is mechanical work $\mathbf{u} \cdot \mathbf{f}$ on the fluid element, either from body or surface forces.

For internal energy, we can start with the 1st law of thermodynamics $dE = TdS - PdV$ which we write per unit mass as

$$de = Tds + \frac{Pd\rho}{\rho^2}. \tag{1}$$

For a given fluid element, the rate of change of entropy,

$$T\frac{Ds}{Dt} = \frac{De}{Dt} - \frac{P}{\rho^2}\frac{D\rho}{Dt}, \tag{2}$$

is the rate of change of heat content of the fluid element. It can come from internal heating or cooling (e.g. nuclear reactions that deposit energy in the gas or neutrinos that leave the volume and act as a volumetric cooling source), or from a heat flux at the surface of the fluid element. The heat flux $\mathbf{F}$ can often be written $\mathbf{F} = -K\nabla T$ where K is the thermal conductivity (heat flows down the temperature gradient). Including both contributions, we write

$$T\frac{Ds}{Dt} = \epsilon - \frac{1}{\rho} \nabla \cdot \mathbf{F},$$

the entropy equation.

We already mentioned the adiabatic approximation that $P \propto \rho^\gamma$ if there is no time for heat flow. Here, $\gamma = c_P/c_V$ is the ratio of specific heats. We can see this directly by demanding that fluid elements conserve entropy

$$\frac{Ds}{Dt} = \frac{D}{Dt}\left(\frac{P}{\rho^\gamma}\right) = 0.$$

In the second step, we used eq. (1) and $P = (\gamma - 1)\rho e$ to rewrite equation (2).

Adding the kinetic energy to the internal energy we get an equation for the total energy (neglecting magnetic energy)

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \frac{\partial}{\partial x_j}\left(u_j\left[\frac{1}{2}\rho u^2 + \rho e + P\right]\right) = \left(\epsilon - \frac{\nabla \cdot \mathbf{F}}{\rho}\right) + \mathbf{u} \cdot \mathbf{f}.$$

(We write only the body force piece of the mechanical work for simplicity). Note that the *enthalpy* $h = e + P/\rho$ appears in the flux term. The enthalpy is often a more useful quantity than internal energy in flows at constant pressure, since it takes into account the PdV work done as the fluid moves around. It often comes up in chemistry, for example.

To include magnetic energy, we can dot $\mathbf{B}$ into the induction equation. This gives an equation for the magnetic energy density $B^2/8\pi$,

$$\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right) = -\nabla \cdot \left(\frac{c\mathbf{E} \times \mathbf{B}}{4\pi}\right) - \mathbf{E} \cdot \mathbf{J}$$

which you may have seen before in electromagnetism. The first term on the right hand side is the divergence of the Poynting flux; the second is Ohmic dissipation.

Using Ohm's law, the $\mathbf{J} \cdot \mathbf{E}$ term can be written as two terms

$$-\mathbf{E} \cdot \mathbf{J} = -\frac{J^2}{\sigma} - \mathbf{u} \cdot \frac{\mathbf{J} \times \mathbf{B}}{c}.$$

The first is the energy dissipation rate from ohmic heating. This converts magnetic energy into internal energy: J^2/σ is the heating rate per unit volume. The second term has the same form as the mechanical work term $\mathbf{u} \cdot \mathbf{f}$ in the kinetic energy equation, but with opposite sign. This shows that the work done on the fluid by the $\mathbf{J} \times \mathbf{B}$ force takes energy from (or puts energy into) the magnetic field.

Examples

Here are two example problems to work on that will give you a chance to play around with the fluid and MHD equations:

1. *Magnetic field winding*. Consider a spherical star which is differentially rotating such that the fluid velocity is $\mathbf{u} = \hat{\phi}\, R\Omega(R)$, where we use cylindrical coordinates (R, ϕ, z) with z along the rotation axis. A poloidal magnetic field $(B_R(R, z), 0, B_z(R, z))$ threads the star initially.

(a) First assume that the velocity does not change over time. What does the induction equation imply for the subsequent evolution of the field? Explain your result physically.

(b) Now write down the momentum equation for the fluid and include the back reaction of the field on the fluid. What is the evolution in time?

2. *Electric field in an atmosphere*. Consider a plane-parallel atmosphere of fully ionized hydrogen gas. By writing down the momentum equations for the protons and electrons separately, show that (1) the structure of the atmosphere is given by $dP/dz = -\rho g$, where P is the sum of the electron and proton pressures, and (2) there is an electric field in the atmosphere. What is the value of the electric field, and what is its role?

Appendix: Two fluid equations

Another way to approach the MHD equations is to consider the electron and ions separately. Coupled by a collisional term, the momentum equations for each species are

$$n_e m_e \frac{D\mathbf{v_e}}{Dt} = -\frac{n_e m_e(\mathbf{v_e}-\mathbf{v_i})}{\tau_e} - n_e e\left(\mathbf{E}+\frac{\mathbf{v_e}\times\mathbf{B}}{c}\right) - \nabla P_e \tag{3}$$

$$n_i m_i \frac{D\mathbf{v_i}}{Dt} = -\frac{n_i m_i(\mathbf{v_i}-\mathbf{v_e})}{\tau_i} + n_i Ze\left(\mathbf{E}+\frac{\mathbf{v_i}\times\mathbf{B}}{c}\right) - \nabla P_i \tag{4}$$

where n_e, n_i, P_e and P_i are the electron and ion densities and pressures, m_e and m_i are the electron and ion masses, and τ_e and τ_i are the timescales on which the electron or ion velocity $\mathbf{v_e}$ or $\mathbf{v_i}$ relaxes due to collisions with the other species.

Charge neutrality implies that $n_e = Zn_i$. Momentum conservation also tells us that the collisional terms must cancel, i.e. $n_e m_e/\tau_e = n_i m_i/\tau_i$ or $\tau_e = (Zm_e/m_i)\tau_i$. This means that the electron velocity changes on a much faster timescale due to collisions with protons than vice versa. This makes sense if we consider two body collisions between particles with very different masses: the heavy particle undergoes a smaller velocity change by roughly the ratio of the particle masses.

The electrons and ions satisfy the continuity equations

$$\frac{\partial n_e}{\partial t} + \nabla\cdot(n_e\mathbf{v_e}) = 0$$

$$\frac{\partial n_i}{\partial t} + \nabla\cdot(n_i\mathbf{v_i}) = 0.$$

Multiplying by the particle masses and adding, we find

$$\frac{\partial}{\partial t}(n_e m_e + n_i m_i) + \nabla\cdot(n_e m_e\mathbf{v_e} + n_i m_i\mathbf{v_i}) = 0$$

or

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0,$$

where $\rho = n_e m_e + n_i m_i$ is the mass density and we define the fluid velocity $\mathbf{u}$ such that

$$\rho\mathbf{u} = n_e m_e\mathbf{v_e} + n_i m_i\mathbf{v_i}.$$

Note that since $m_e \ll m_i$, the fluid velocity is close to the ion velocity $\mathbf{u} \approx \mathbf{v_i}$. Subtracting the continuity equations and assuming charge neutrality gives $\nabla\cdot\mathbf{J} = 0$ as required for charge conservation.

Now add the two momentum equations (3) and (4). On the left hand side this gives

$$n_e m_e\frac{D\mathbf{v_e}}{Dt} + n_i m_i\frac{D\mathbf{v_i}}{Dt} = \rho\frac{D\mathbf{u}}{Dt}.$$

On the right hand side, the pressure gradient terms add $\nabla P_i + \nabla P_e = \nabla P$, where P is the total pressure, and the Lorentz force terms are

$$-n_e e\left(\mathbf{E}+\frac{\mathbf{v_e}\times\mathbf{B}}{c}\right) + n_i Ze\left(\mathbf{E}+\frac{\mathbf{v_i}\times\mathbf{B}}{c}\right) = n_e e\frac{(\mathbf{v_i}-\mathbf{v_e})\times\mathbf{B}}{c} = \frac{\mathbf{J}\times\mathbf{B}}{c}.$$

The final result is

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \frac{\mathbf{J} \times \mathbf{B}}{c},$$

which is the familiar momentum equation for the fluid.

Ohm's law can be obtained from the electron equation of motion. We neglect the acceleration term on the left hand side, assuming that the electron velocity quickly adjusts to changes in Lorentz forces since the electrons are much less massive than the ions. Therefore

$$0 = -\frac{n_e m_e(\mathbf{v_e} - \mathbf{v_i})}{\tau_e} - n_e e \left(\mathbf{E} + \frac{\mathbf{v_e} \times \mathbf{B}}{c} \right) - \nabla P_e$$

or

$$\mathbf{E} = \frac{m_e \mathbf{J}}{n_e e^2 \tau_e} - \frac{(\mathbf{v_e} - \mathbf{v_i}) \times \mathbf{B}}{c} - \frac{\mathbf{v_i} \times \mathbf{B}}{c} - \frac{\nabla P_e}{n_e e}.$$

The electrical conductivity is $\sigma = n_e e^2 \tau / m_e$, and since $\mathbf{u} \approx \mathbf{v_i}$, we have

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} - \frac{\mathbf{u} \times \mathbf{B}}{c} + \frac{\mathbf{J} \times \mathbf{B}}{n_e e c} - \frac{\nabla P_e}{n_e e}. \tag{5}$$

Equation (5) is the Ohm's law we wrote down in the text, except for the last two terms which are the Hall term and battery term. The first of these is the Hall electric field that you may have come across before that arises when a current flows perpendicular to a magnetic field. The Lorentz force deflects the current-carrying charges until the Hall electric field grows to balance it. The battery term enters the induction equation as the cross product of the electron pressure and density gradients $\nabla P_e \times \nabla n_e$ (after taking the curl of $\mathbf{E}$) so that misalignment of the surfaces of constant electron density and constant electron pressure leads to magnetic field growth (the "battery effect").