PHYS 643 Week 7: Oscillations and Instabilities

We've already seen two examples of waves in a fluid system: sound waves in a uniform gas, and fast and slow magnetosonic waves and Alfven waves in a magnetized plasma. Here I go through two examples of linear stability analysis as examples of more complex situations: first what happens when we include the energy equation explicitly for sound waves, and second how to deal with a background that has a gradient.

Sound waves with thermal conduction

Earlier, we derived the dispersion relation for sound waves by assuming a relation between the pressure and density perturbations

$$\delta P = \frac{\partial P}{\partial \rho}\delta\rho = c_s^2 \delta\rho.$$

The partial derivative can be taken at constant entropy, in which case $c_s^2 = \gamma P/\rho$ is the adiabatic sound speed, or at constant temperature, giving the isothermal sound speed $c_s^2 = P/\rho = k_B T/\mu m_p$. These two cases can be understood as limits of either very inefficient heat transfer (adiabatic) or efficient heat transfer (isothermal) on the timescale of the sound wave period ($2\pi/\omega = 2\pi/c_s k = \lambda/c_s$).

Instead of making this assumption, let's instead include the energy equation in the calculation. We will assume ideal gas, in which case the pressure, density and temperature perturbations are related by

$$\frac{\delta P}{P} = \frac{\delta\rho}{\rho} + \frac{\delta T}{T} \tag{1}$$

(since $P \propto \rho T$). The entropy equation is

$$T\frac{Ds}{Dt} = -\frac{1}{\rho}\nabla\cdot\mathbf{F} = \frac{1}{\rho}\nabla\cdot\left(K\nabla T\right),$$

where K is the thermal conductivity (we will assume this is a constant).

Perturbing the entropy equation gives

$$-i\omega T\delta s = -\frac{k^2 K\delta T}{\rho} \tag{2}$$

(the background is stationary, so only the time derivative term of D/Dt contributes at linear order). This is the extra equation that we need to eliminate δT and derive the relation between δP and $\delta\rho$. To do this, write

$$Tds = T\left.\frac{\partial s}{\partial T}\right|_P dT + T\left.\frac{\partial s}{\partial P}\right|_T dP. \tag{3}$$

We then use the fact that $T\,\partial S/\partial T|_P = c_P$ the heat capacity at constant pressure, the identity

$$\left.\frac{\partial s}{\partial P}\right|_T \left.\frac{\partial P}{\partial T}\right|_s \left.\frac{\partial T}{\partial s}\right|_P = -1,$$

and the adiabatic index
$$\frac{\gamma - 1}{\gamma} = \left.\frac{\partial \ln T}{\partial \ln P}\right|_s$$
to rewrite equation (3) as
$$T ds = c_p \left(dT - \frac{\gamma - 1}{\gamma}\frac{T}{P} dP \right). \tag{4}$$
This allows us to write down δs in terms of δT and δP. Equation (2) then gives
$$c_P T \left(\frac{\delta T}{T} - \frac{\gamma - 1}{\gamma}\frac{\delta P}{P} \right) = \frac{k^2 K T}{i\omega\rho}\left(\frac{\delta T}{T} \right).$$
Using the ideal gas relation from equation (1) to eliminate δT in favour of δP and $\delta\rho$, we find
$$\boxed{\frac{1}{\gamma}\frac{\delta P}{P} - \frac{\delta\rho}{\rho} = \frac{k^2 K}{i\omega\rho c_P}\left(\frac{\delta P}{P} - \frac{\delta\rho}{\rho} \right)} \tag{5}$$

I've put a box around this result because it is the relation between δP and $\delta\rho$ that we've been looking for. The quantity $D = K/\rho c_P$ is the thermal diffusivity (units of cm^2/s), since we can write
$$T\frac{Ds}{Dt} = c_P \frac{DT}{Dt} = \frac{K}{\rho}\nabla^2 T \Rightarrow \frac{DT}{Dt} = \frac{K}{\rho c_P}\nabla^2 T = D\nabla^2 T$$
(working at constant pressure for simplicity and again assuming constant K). The thermal timescale associated with the perturbation is therefore $1/(k^2 D)$. When the mode frequency ω is either large or small compared with $k^2 D$, we recover the adiabatic or isothermal limits discussed earlier. However, in general, we see that the relation between δP and $\delta\rho$ has a complex prefactor. The dispersion relation will be
$$\frac{\omega^2}{k^2} = c_s^2 = \frac{\gamma P}{\rho}\left(\frac{i\omega - k^2 D}{i\omega - \gamma k^2 D} \right).$$
In general we see that k^2 will be complex, so that for a given ω there will be a propagating wave (real part of k) but with a decaying amplitude (imaginary part of k).

I will leave this as an exercise, but for example one limit to consider is when the wave is *almost* adiabatic, so that $k = k_R + ik_I$ with $k_I \ll k_R$. In this limit, $\omega^2 \approx k_R^2 c_{\rm ad}^2$, where $c_{\rm ad}$ is the adiabatic sound speed, and
$$\frac{k_I}{k_R} \approx \frac{\gamma - 1}{2}\frac{k_R^2 D}{\omega} \approx \frac{\gamma - 1}{2}\frac{\omega D}{c_{\rm ad}^2}.$$
For air, $c_s \approx 330\ \mathrm{m\ s^{-1}}$ and $D \approx 2 \times 10^{-5}\ \mathrm{m^2/s}$, giving a decay length of about 10^6 wavelengths.

Gravity waves

As a second example, let's look at the waves in a plane-parallel atmosphere. The additional ingredient here is gravity, since the atmosphere is in hydrostatic balance

$$\frac{dP}{dz} = -\rho g$$

so that there are background pressure, density and temperature gradients.

When dealing with problems with background gradients, it can be useful to write things in terms of Lagrangian perturbations. The perturbations we have been writing down so far are Eulerian, since at any given time t they give the difference between the perturbed and unperturbed flows at the same point in space, e.g.

$$\delta\rho\left(\mathbf{r},t\right) = \rho\left(\mathbf{r},t\right) - \rho_0\left(\mathbf{r},t\right),$$

where ρ is the perturbed density and ρ_0 is the unperturbed density. Instead we could define the Lagrangian perturbation

$$\Delta\rho\left(\mathbf{x_0},t\right) = \rho\left(\mathbf{r}(\mathbf{x_0},t),t\right) - \rho_0\left(\mathbf{r_0}(\mathbf{x_0},t),t\right),$$

where $\mathbf{x_0}$ is a Lagrangian label that identifies the fluid element, for example a good choice would be the initial location of the fluid element. The difference in the positions of the fluid element in the unperturbed flow $\mathbf{r_0}(\mathbf{x_0},t)$ and perturbed flows $\mathbf{r}(\mathbf{x_0},t)$ is the Lagrangian displacement

$$\xi = \mathbf{r}(\mathbf{x_0},t) - \mathbf{r_0}(\mathbf{x_0},t).$$

The Eulerian and Lagrangian perturbations at a particular spatial location $\mathbf{r}$ are related by

$$\Delta\rho\left(\mathbf{r}(\mathbf{x_0},t),t\right) - \delta\rho\left(\mathbf{r},t\right) = -\rho_0\left(\mathbf{r_0}(\mathbf{x_0},t),t\right) + \rho_0\left(\mathbf{r},t\right) \approx \xi\cdot\nabla\rho_0(\mathbf{r}),$$

or

$$\boxed{\Delta\rho = \delta\rho + \xi\cdot\nabla\rho}$$

As an application, consider the perturbed continuity equation

$$-i\omega\delta\rho = -\nabla\cdot\left(\rho\delta\mathbf{v}\right).$$

We assume $\mathbf{v} = 0$ in the background, in which case we can also write

$$\delta\mathbf{v} = \frac{\partial\xi}{\partial t} = -i\omega\xi$$

and so

$$\delta\rho = -\nabla\cdot(\rho\xi) = -\rho\nabla\cdot\xi - \xi\cdot\nabla\rho$$

$$\Rightarrow \boxed{\frac{\Delta\rho}{\rho} = -\nabla\cdot\xi} \tag{6}$$

If the Lagrangrian displacements have a non-zero divergence, it implies a Lagrangian density change. Note that if the background is moving then δv and ξ have a more complex relation.

Going back to the plane-parallel atmosphere, the perturbed continuity equation is therefore equation (6). We consider adiabatic perturbations which we can immediately write down taking advantage of the Lagrangian formalism as

$$\frac{\Delta P}{P} = \gamma \frac{\Delta \rho}{\rho}.$$

Therefore

$$\frac{\delta P}{P} = \gamma \frac{\delta \rho}{\rho} - \xi_z \left[\frac{d \ln P}{dz} - \gamma \frac{d \ln \rho}{dz} \right]$$

or

$$\frac{\delta P}{\rho c_s^2} = \frac{\delta \rho}{\rho} - \frac{N^2 \xi_z}{g}, \tag{7}$$

where we define the Brunt-Väisälä frequency N and *convective discriminant* $\mathcal{A}$ according to

$$N^2 = -g\mathcal{A} = -g \left[\frac{d \ln \rho}{dz} - \frac{1}{\gamma} \frac{d \ln P}{dz} \right].$$

The momentum equations are

$$-\rho \omega^2 \xi_z = -\frac{d\delta P}{dz} + g\delta\rho \tag{8}$$

$$-\rho \omega^2 \xi_x = -ik_x \delta P. \tag{9}$$

Note that whereas we have assumed an x-dependence for the perturbations of $e^{ik_x x}$, we do not specify a functional form for the z-dependence; it will be determined by how the background changes with height. To solve the equations in a realistic atmosphere or star requires integration of the equations over height z. The equations form an eigenvalue problem: in general, only a certain set of frequencies ω_n give solutions that satisfy the boundary conditions at $z = 0$ or $z = \infty$ ($r = 0$ and $r = R$ in the case of a star).

A useful limit to consider however is when the vertical wavelength of the waves is much smaller than the pressure or density scale heights. Then the coefficients in the equations remain constant on the scale of a wavelength, and we can write a local WKB solution $e^{ik_z z}$. Substituting this into the continuity equation, momentum equations and adiabatic condition (eqs. [6], [7], [8], and [9]) gives the dispersion relation

$$c_s^2 k_z^2 = \left(\omega^2 - N^2\right) \left(1 - \frac{k_x^2 c_s^2}{\omega^2}\right).$$

A vertically-propagating wave requires $k_z^2 > 0$ so that k_z is real. This can happen in two ways. The first is $\omega^2 \gg N^2$, when $\omega^2 = c_s^2(k_z^2 + k_x^2) = c_s^2 k^2$. These are the sound waves or acoustic waves we have encountered before. We see them again in the plane-parallel atmosphere.

The second solution for propagating waves is when $\omega^2 < N^2$ and $\omega^2 < c_s^2 k^2$. Then both terms on the right hand side of the dispersion relation are negative and so $k_z^2 > 0$. The dispersion relation when $\omega^2 \ll N^2$ is

$$\omega^2 = N^2 \left(\frac{k_x}{k}\right)^2.$$

These waves are *internal gravity waves*. They are incompressible waves, ie. they satisfy $\nabla \cdot \xi \approx 0$ (if you repeat the calculation setting $\nabla \cdot \xi = 0$ exactly, you'll find that the sound waves go away but the gravity waves survive). The restoring force for the wave is from horizontal pressure gradients that arise from horizontal variations in the hydrostatic column that arise as the fluid moves. One interesting fact about gravity waves is that the phase and group velocities of a wavepacket are orthogonal (try proving this using the dispersion relation). In the context of stars, standing gravity waves or acoustic waves can exist that occupy the entire stellar volume in some cases or may propagate in only a limited region of the stellar interior (where $k^2 > 0$). In this context, gravity waves are referred to as *g-modes* and the acoustic waves as *p-modes* (g for gravity and p for pressure).

The convective discriminant $\mathcal{A}$ is so-named because it indicates whether the atmosphere is unstable to convection. In a situation in which $\mathcal{A} > 0$, $N^2 < 0$ and $\omega^2 < 0$ indicating instability. The way to understand this is to consider moving a fluid element upwards slowly enough that it stays in pressure equilibrium with its surroundings, but quickly enough that the motion is adiabatic. The density contrast between the fluid element and its surroundings after moving a vertical distance Δz is

$$\Delta z \left.\frac{\partial \rho}{\partial P}\right|_S \frac{dP}{dz} - \Delta z \frac{d\rho}{dz} = -\rho \Delta z \left[\frac{d\ln\rho}{dz} - \frac{1}{\gamma}\frac{d\ln P}{dz}\right] = -\rho \Delta z \mathcal{A}.$$

If $\mathcal{A} > 0$, we see that the fluid element will be less dense than its surroundings and so will buoyantly rise further: the atmosphere is unstable to vertical perturbations. The criterion $\mathcal{A} > 0$ is the *Schwarzschild criterion* for convection.

Papers

There are many different types of waves and instabilities relevant for astrophysical objects. Here is a selection of recent papers that give some nice examples:

- Fuller (2014) "Saturn ring seismology: Evidence for stable stratification in the deep interior of Saturn"
  http://adsabs.harvard.edu/abs/2014Icar..242..283F
- Showman & Polvani (2011) "Equatorial Superrotation on Tidally Locked Exoplanets"
  http://adsabs.harvard.edu/abs/2011ApJ...738...71S
- Philippov et al. (2016) "Spreading Layers in Accreting Objects: Role of Acoustic Waves for Angular Momentum Transport, Mixing, and Thermodynamics"
  https://ui.adsabs.harvard.edu/#abs/2016ApJ...817...62P/abstract
- Levin (2007) "On the theory of magnetar QPOs"
  https://ui.adsabs.harvard.edu/#abs/2007MNRAS.377..159L/abstract

Appendix: Perturbation equations in spherical geometry

For stellar or planetary oscillations, we need the perturbation equations in spherical geometry. We assume the background is spherically-symmetric, so pressure or density depend only on r. We take density perturbations of the form

$$\delta\rho = \delta\rho(r) e^{im\phi} P_\ell^m(\cos\theta) e^{-i\omega t}$$

and similarly for the pressure perturbation δP and radial displacement ξ_r. The non-radial displacements have a different angular dependence:

$$\xi_\theta = \xi_\theta(r) e^{im\phi} \frac{dP_\ell^m(\cos\theta)}{d\theta} e^{-i\omega t}$$

$$\xi_\phi = \xi_\phi(r) e^{im\phi} \frac{imP_\ell^m(\cos\theta)}{\sin\theta} e^{-i\omega t}$$

With these choices for the angular dependences, the perturbation equations then depend only on r, as follows:

Adiabatic perturbations

$$\frac{\Delta\rho}{\rho} = \frac{1}{\gamma}\frac{\Delta P}{P}$$

$$\Rightarrow \frac{\delta\rho}{\rho} = \frac{1}{\gamma}\frac{\delta P}{P} + \frac{N^2\xi_r}{g}$$

(using the definition of N^2 from the text; $g(r) = Gm(r)/r^2$).

Continuity

$$\frac{\Delta\rho}{\rho} = -\nabla\cdot\xi$$

$$\frac{1}{\gamma}\frac{\delta P}{P} + \frac{\xi_r}{\gamma}\frac{d\ln P}{dr} = -\frac{1}{r^2}\frac{d(r^2\xi_r)}{dr} - \frac{\xi_\theta}{r\sin\theta P_\ell^m}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial P_\ell^m}{\partial\theta}) + \frac{m^2\xi_\phi}{r\sin^2\theta}$$

$$\Rightarrow \frac{1}{\gamma}\frac{\delta P}{P} + \frac{\xi_r}{\gamma}\frac{d\ln P}{dr} = -\frac{1}{r^2}\frac{d(r^2\xi_r)}{dr} + \ell(\ell+1)\frac{\xi_\theta}{r}$$

Momentum

$$-\rho\omega^2\xi_r = -\frac{d\delta P}{dr} - g\delta\rho$$

$$-\rho\omega^2\xi_\perp = -\frac{\delta P}{r}$$

$$\xi_\phi = \xi_\theta = \xi_\perp$$

We have made the approximation that the perturbations do not change the gravitational potential, so $\delta g = 0$; this is known as the Cowling approximation.

These simplify to give two ODEs to integrate:

$$\frac{1}{r^2}\frac{d(r^2\xi_r)}{dr} = \frac{g}{c_s^2}\xi_r - \frac{\delta P}{\rho}\left[\frac{1}{c_s^2} - \frac{\ell(\ell+1)}{\omega^2 r^2}\right] \tag{10}$$

$$\frac{d\delta P}{dr} = -\frac{g}{c_s^2}\delta P + \rho(\omega^2 - N^2)\xi_r. \tag{11}$$

We have used the fact that $\gamma H = \gamma P/\rho g = c_s^2/g$, where c_s^2 is the adiabatic sound speed and $H = -dz/d\ln P$ is the pressure scale height.

The boundary condition at the stellar surface is that the Lagrangian pressure perturbation should vanish there

$$\frac{\Delta P}{P} = 0 \Rightarrow \frac{\delta P}{P} = \frac{\xi_r}{H} \qquad \text{at } r = R. \tag{12}$$

At the center $r = 0$, we see that there are terms that diverge, so we need to step away from the origin and begin our integration at a small non-zero value of r. The boundary conditions for $\ell > 0$ (non-radial oscillations) are

$$\frac{d\delta P}{dr} = \frac{\ell}{r}\delta P \qquad \frac{d\xi_r}{dr} = \frac{\ell - 1}{r}\xi_r,$$

or for $\ell = 0$ (radial oscillations)

$$\frac{d\delta P}{dr} = 0 \qquad \frac{d\xi_r}{dr} = \frac{\xi_r}{r}.$$

An equivalent way to write the first $\ell > 0$ boundary condition is

$$\xi_r = \ell\xi_\perp, \tag{13}$$

where $\xi_\perp$ can be expressed in terms of δP using the horizontal momentum equation.

Equations (10) and (11) and the boundary conditions of equations (12) and (13) define an eigenvalue problem for the mode frequency ω.

There are three "quantum numbers" that label the modes: ℓ, m, and the number of radial nodes n. However note that the azimuthal wavenumber m doesn't enter into the equations, so the frequency of the mode depends on the number of radial nodes n and the angular quantum number ℓ, but not m. This changes if spherically symmetry is broken, eg. a rotating or magnetized star.