Week 9: Rotating systems

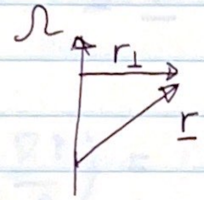
Equations of motion: need to add Coriolis force $-\rho\, 2\underline{\Omega}\times\underline{v}$ (per unit volume)

centrifugal force

$$-\rho\, \underline{\Omega}\times(\underline{\Omega}\times\underline{r}) = +\rho|\Omega|^2\, \underline{r}_\perp$$

perp. to rotation axis

$$|\underline{r}_\perp| = \frac{|\Omega\times r|}{\Omega}$$



can write this as $\underline{\Omega}\times(\underline{\Omega}\times\underline{r}) = -\underline{\nabla}\left(\frac{1}{2}|\underline{\Omega}\times\underline{r}|^2\right)$

and absorb into the gravitational potential

eg. • Earth $\frac{2\pi}{\left(\frac{GM}{R^3}\right)^{1/2}} \simeq 90$ mins

$$\Rightarrow \left(\frac{\Omega}{\Omega_K}\right)^2 \simeq \left(\frac{90 \text{ mins}}{24h}\right)^2$$

$$= 0.3\% = \frac{\Delta R}{R}$$

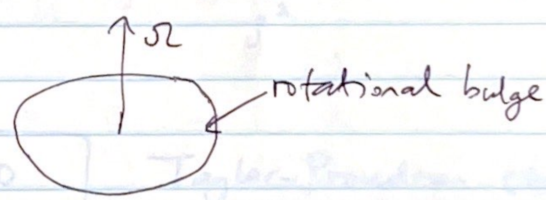
• Jupiter $\frac{2\pi}{\Omega_K} = 2.4h$

rotation is 10h

$$\Rightarrow \left(\frac{\Omega}{\Omega_K}\right)^2 = \left(\frac{2.4}{10}\right)^2 = 0.06$$

• Saturn $\frac{2\pi}{\Omega_K} = 3.6h$ $P_{rot} = 11h$

$$\left(\frac{\Omega}{\Omega_K}\right)^2 = 10\%.$$



$$g_{equator} \simeq g_{pole} - R\Omega^2$$

$$\Delta g \simeq g\left(1 - \left(\frac{\Omega}{\Omega_K}\right)^2\right)$$

where $\Omega_K^2 = \frac{GM}{R^3}$.

Coriolis force gives some new effects, we'll focus on that.

New number: ROSSBY NUMBER

$$\boxed{R_o = \frac{\cancel{\rho} v^2/L}{2v\Omega} = \frac{v}{2\Omega L}}$$

$R_o \ll 1 \Rightarrow$ rotation is important!

Momentum equation: $$\frac{D\underline{v}}{Dt} + 2\underline{\Omega}\times\underline{v} = -\frac{\underline{\nabla}P}{\rho} + \underline{\nabla}\Phi$$ (ignore friction)

$$Ro \ll 1 \Rightarrow \quad 2\underline{\Omega}\times\underline{v} = -\frac{\underline{\nabla}P}{\rho} + \underline{\nabla}\Phi$$

take the curl: $$\cancel{2}\,\underline{\nabla}\times(\underline{\Omega}\times\underline{v}) = -(\underline{\Omega}\cdot\underline{\nabla})\underline{v} + \underline{\Omega}(\underline{\nabla}\cdot\underline{v}) \quad -(*)$$

$$-\underline{\nabla}\times\left(\frac{\underline{\nabla}P}{\rho}\right) = \underline{\nabla}P\times\underline{\nabla}\left(\frac{1}{\rho}\right)$$
$$= \cancel{-}\,\frac{\underline{\nabla}\rho\times\underline{\nabla}P}{\rho^2} \quad \left(\begin{array}{c}\text{baroclinic}\\ \text{vector}\end{array}\right)$$
$$= \frac{\underline{\nabla}P\times\underline{\nabla}T}{\rho^2}$$

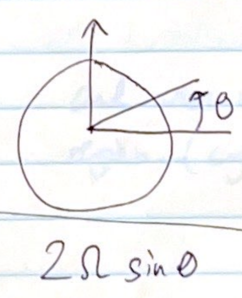
If $\underline{\nabla}\cdot\underline{v} = 0$ incompressible
$\underline{\nabla}P \parallel \underline{\nabla}\rho$ no baroclinicity

then $$\boxed{(\underline{\Omega}\cdot\underline{\nabla})\,\underline{v} = 0}$$ Taylor–Proudman theorem

Relax the assumption of baroclinicity $\Rightarrow$ thermal wind

Think about horizontal motions, vertical balance is $\frac{\partial P}{\partial z} = -\rho g$.

Vertical component of Ω is $\Omega\sin\theta$ where θ = latitude



Coriolis parameter $\boxed{f = 2\Omega\sin\theta}$

(*) becomes
$$-f\frac{\partial \underline{v}_\perp}{\partial z} = \frac{1}{\rho^2}\nabla\rho \times \left(-\rho g\hat{\underline{z}}\right)$$

$$\boxed{\frac{\partial \underline{v}_\perp}{\partial z} = \frac{g}{f\rho}\hat{z}\times\underline{\nabla}\rho}$$

thermal wind equation → horizontal density / temperature variations ⇒ vertical shear.

The dynamical balance when $Ro \ll 1$ is <u>geostrophic balance</u>

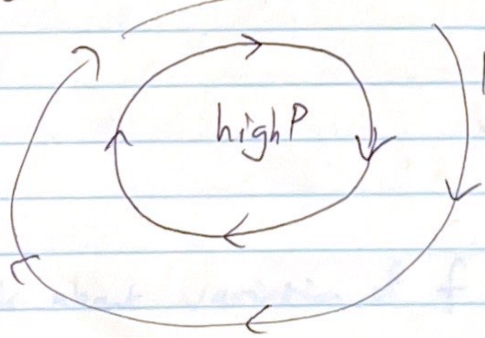
$$2\underline{\Omega}\times\underline{v} = -\frac{\underline{\nabla}P}{\rho}$$

horizontal winds:

$\underline{v}_\perp = (u, v)$

$$fu = -\frac{1}{\rho}\frac{\partial P}{\partial y}$$
$$-fv = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

Velocity flow is along constant P surfaces



"anticyclone"
(northern hemisphere $\hat{\underline{z}}\cdot\underline{\Omega} > 0$)

and vice versa for low P region (cyclone)

The Coriolis force can also support new oscillation modes.

eg. from last time g-modes with rotation

$$-\rho\omega^2\xi_x - \rho 2\Omega(-i\omega\xi_y) = -ik_x\delta P$$

$$-\rho\omega^2\xi_y + \rho 2\Omega(-i\omega\xi_x) = -ik_y\delta P$$

add these terms

if you take $\underline{k}\cdot\underline{\xi} = 0$ (incompressible)

$$\rightarrow \quad \omega^2 = N^2\left(\frac{k_\perp}{k}\right)^2 + (2\Omega)^2\left(\frac{k_z}{k}\right)^2$$

inertial modes
(present when $N^2 \to 0$)

Also rotational splitting

eg. in rotating frame with $\Omega \ll \omega$ the mode is not affected

in inertial frame $\quad e^{\pm im\phi}\,e^{-i\omega t} \longrightarrow e^{\pm im(\phi+\Omega t) - i\omega t}$

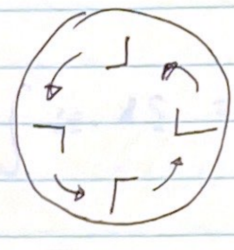
or $\quad e^{im\phi}\,e^{-i(\omega - m\Omega)t}$

$$\Rightarrow \omega \rightarrow \omega \pm m\Omega$$

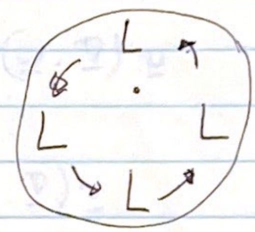
Now think about variation of f with latitude: β-effect.

Vorticity $\underline{\omega} = \underline{\nabla} \times \underline{v}$ (2× local ang. velocity of the fluid)

eg. rigid body rotation $\underline{\omega} = 2\underline{\Omega}$



line vortex $\underline{v}_\phi \propto \frac{1}{r}$ $\underline{\omega} = 0$



Vorticity equation

$$\frac{\partial \underline{v}}{\partial t} + \underbrace{\underline{v}.\underline{\nabla}\,\underline{v}}_{-\underline{v}\times(\underline{\nabla}\times\underline{v}) + \underline{\nabla}\left(\frac{1}{2}v^2\right)}$$

goes into gradient terms

$$\Rightarrow \quad \frac{\partial \underline{v}}{\partial t} - \underline{v}\times\underline{\omega} = +\underline{\nabla}\phi - \frac{\underline{\nabla}P}{\rho}$$

$\cancel{\phi}$ $\frac{1}{2}v^2 - \phi_g$

take curl $\Rightarrow$

$$\boxed{\frac{\partial \underline{\omega}}{\partial t} - \underline{\nabla}\times(\underline{v}\times\underline{\omega}) = \cancel{0}}$$

etc.

$$\frac{\underline{\nabla}\rho \times \underline{\nabla}P}{\rho^2}$$

$$\Rightarrow \quad \frac{\partial \underline{\omega}}{\partial t} = \underline{\nabla}\times(\underline{v}\times\underline{\omega})$$

Same eqn as for $\underline{B}$! (induction)

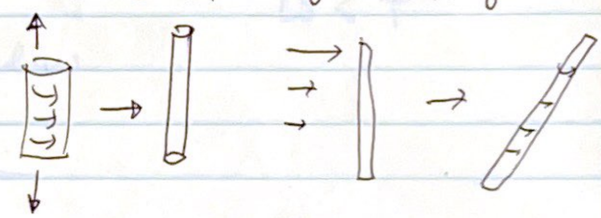
$\Rightarrow$ Vortex lines move with fluid.

Kelvin's theorem $\quad \dfrac{D}{Dt}\oint \underline{u}\cdot d\underline{l} = \dfrac{D}{Dt}\int \underline{\omega}\cdot d\underline{S} = 0.$

$$\underline{\nabla}\times(\underline{v}\times\underline{\omega}) \rightarrow -(\underline{v}\cdot\underline{\nabla})\underline{\omega} + (\underline{\omega}\cdot\underline{\nabla})\underline{v}$$

$$\Rightarrow \frac{D\underline{\omega}}{Dt} = (\underline{\omega}\cdot\underline{\nabla})\underline{v}$$

vortex stretching + tilting.

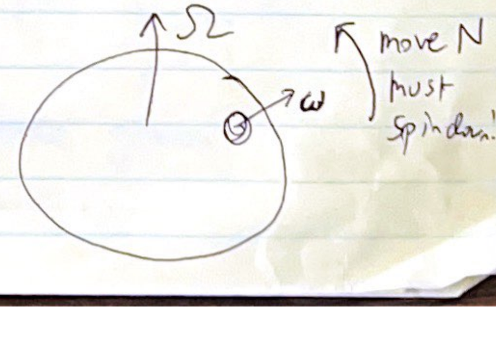


<u>With rotation</u>:

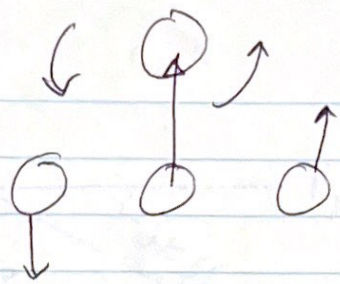
$$\frac{\partial \underline{v}}{\partial t} \underbrace{- \underline{v}\times\underline{\omega} + 2\underline{\Omega}\times\underline{v}}_{+\,(\underline{\omega}+2\underline{\Omega})\times\underline{v}} = (\quad)$$

absolute vorticity $\underline{\omega}_a = 2\underline{\Omega} + \underline{\omega}$

As before but replace $\underline{\omega} \rightarrow \underline{\omega}_a$



Rossby wave



$$\beta = \frac{\partial f}{\partial y}$$

"f-plane" "β-plane" $f(y) = f_0 + \beta y$

$$\omega = \frac{-\beta k}{k^2 + \ell^2 + 1/R^2}$$

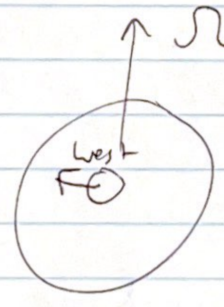
perturbations $\sim e^{i(kx + \ell y - \omega t)}$

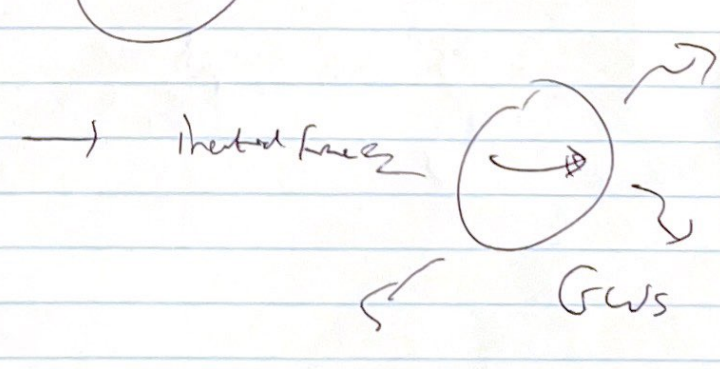
$$R = \frac{\sqrt{gH}}{f}$$

Rossby radius

$\omega < f$

CFS instability





lose J $\Rightarrow$ wave amplitude grows!

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baroclinicity