PHYS 616 Multifractals and Turbulence

Lecture 2:

Introduction:

Our multifractal world part 2

. 22, 2014

The Atmosphere

4) time domain

A voyage through scales.....

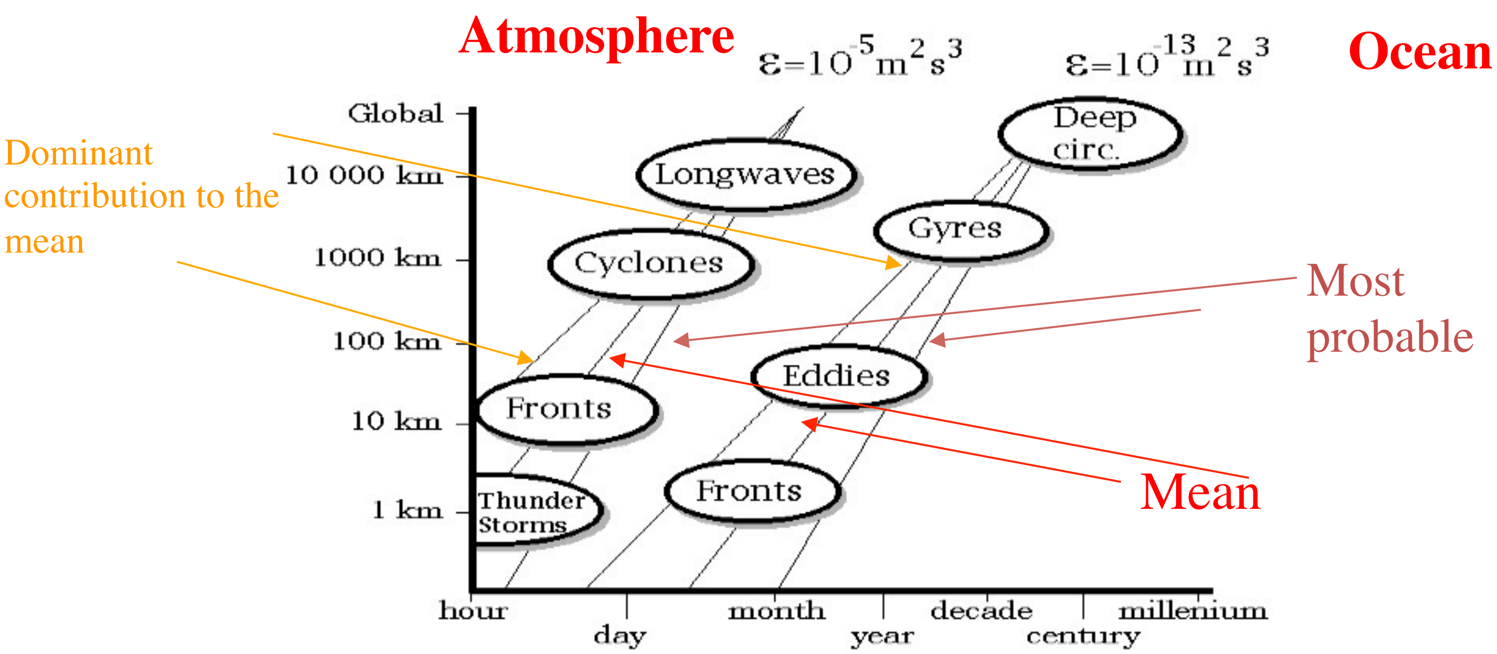
Antonie van Leeuwenhoek (1632–1723)

“A new world” in a drop of water

“Animalcules,” described in depth by Leeuwenhoek, c1695–1698. By Anton van Leeuwenhoek

…..the discovery of micro-organisms

Space-time ("Stommel") diagrammes for the ocean and atmosphere

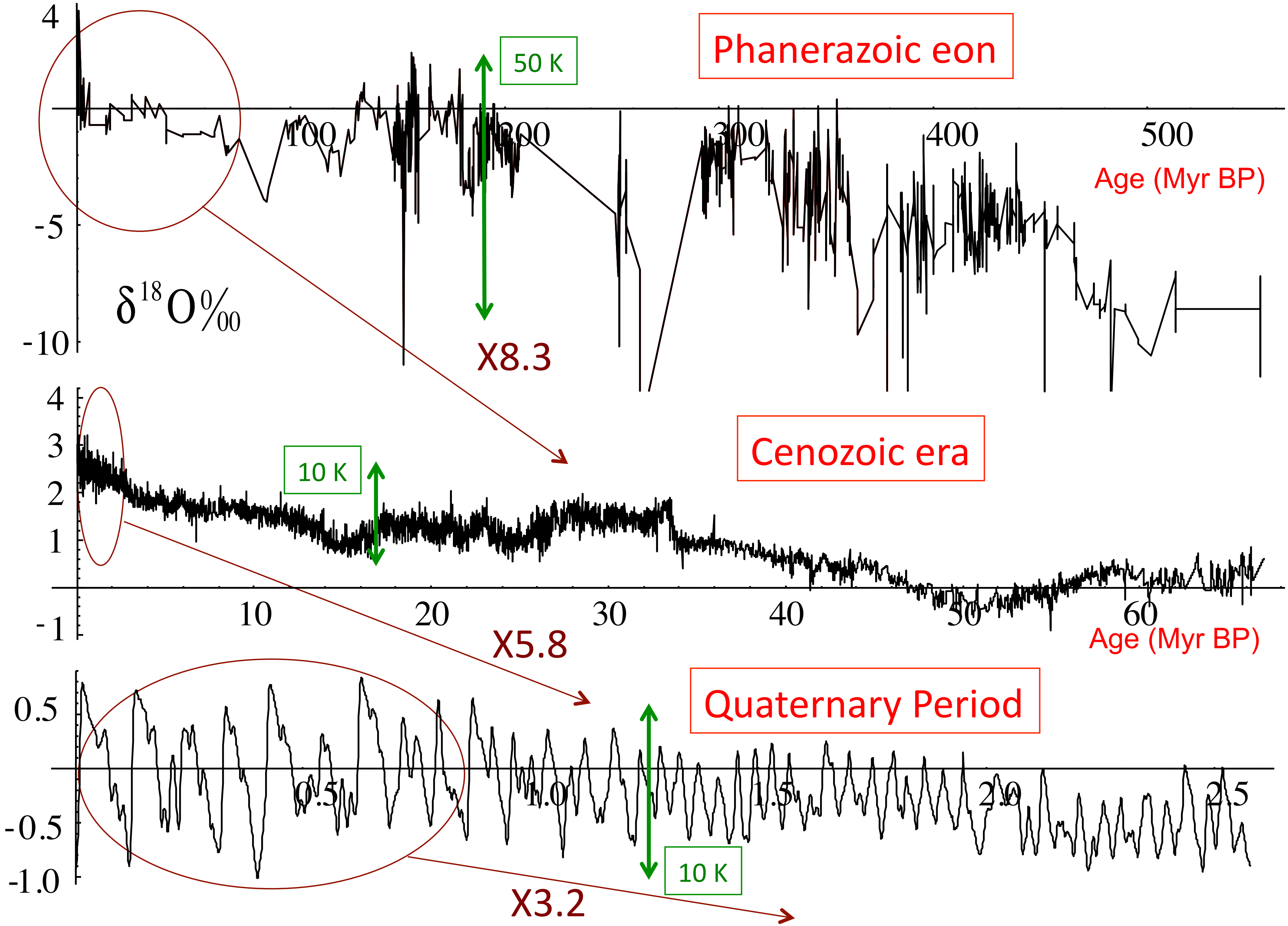


Usual scale bound interpretation: every factor of 10 (even 2!) there are new physical process requiring new models.

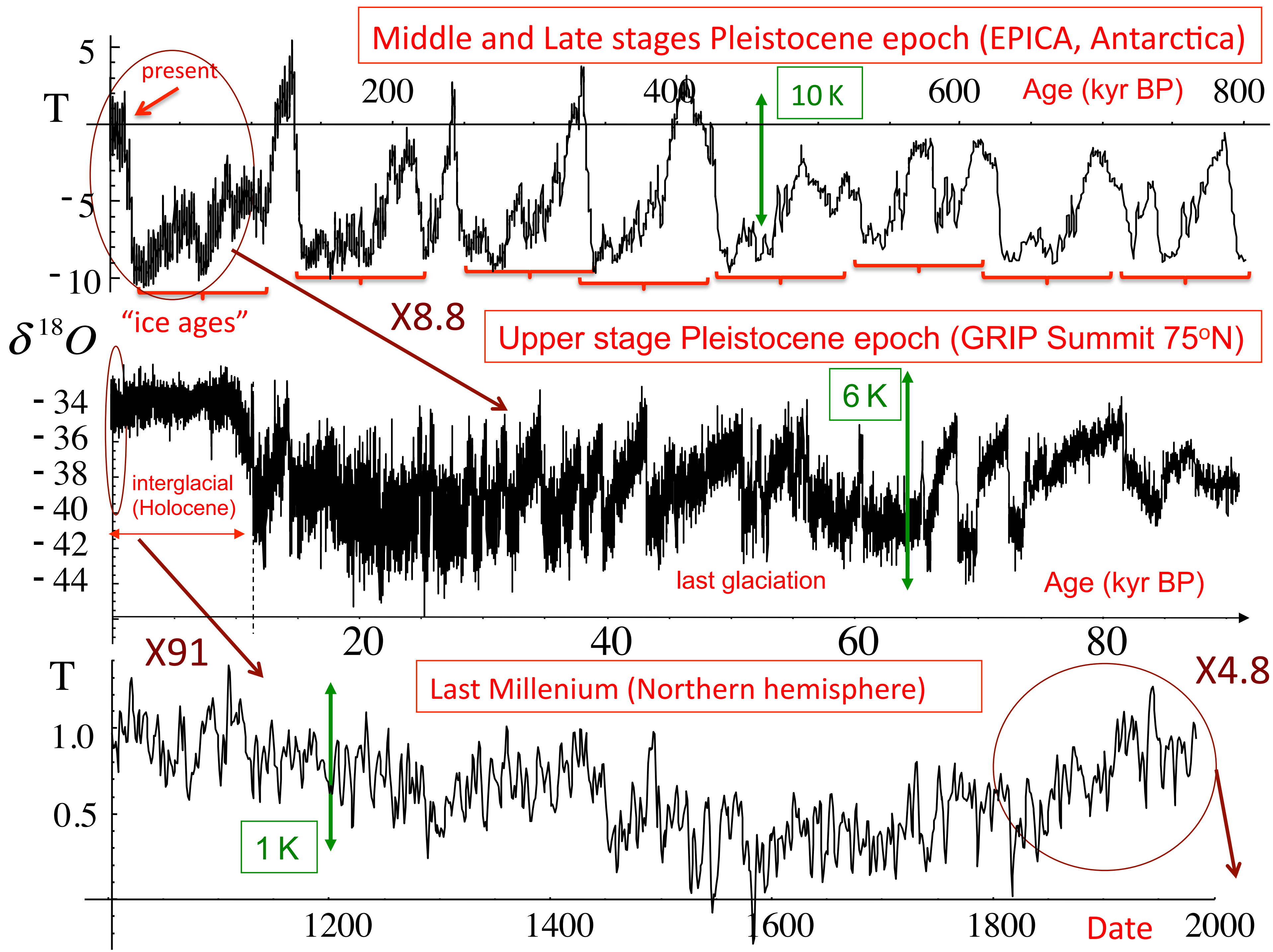
Scaling interpretation: The phenomena line up exactly as predicted by scaling, turbulence theory:

$$\tau = l^{2/3}\varepsilon^{-1/3}$$

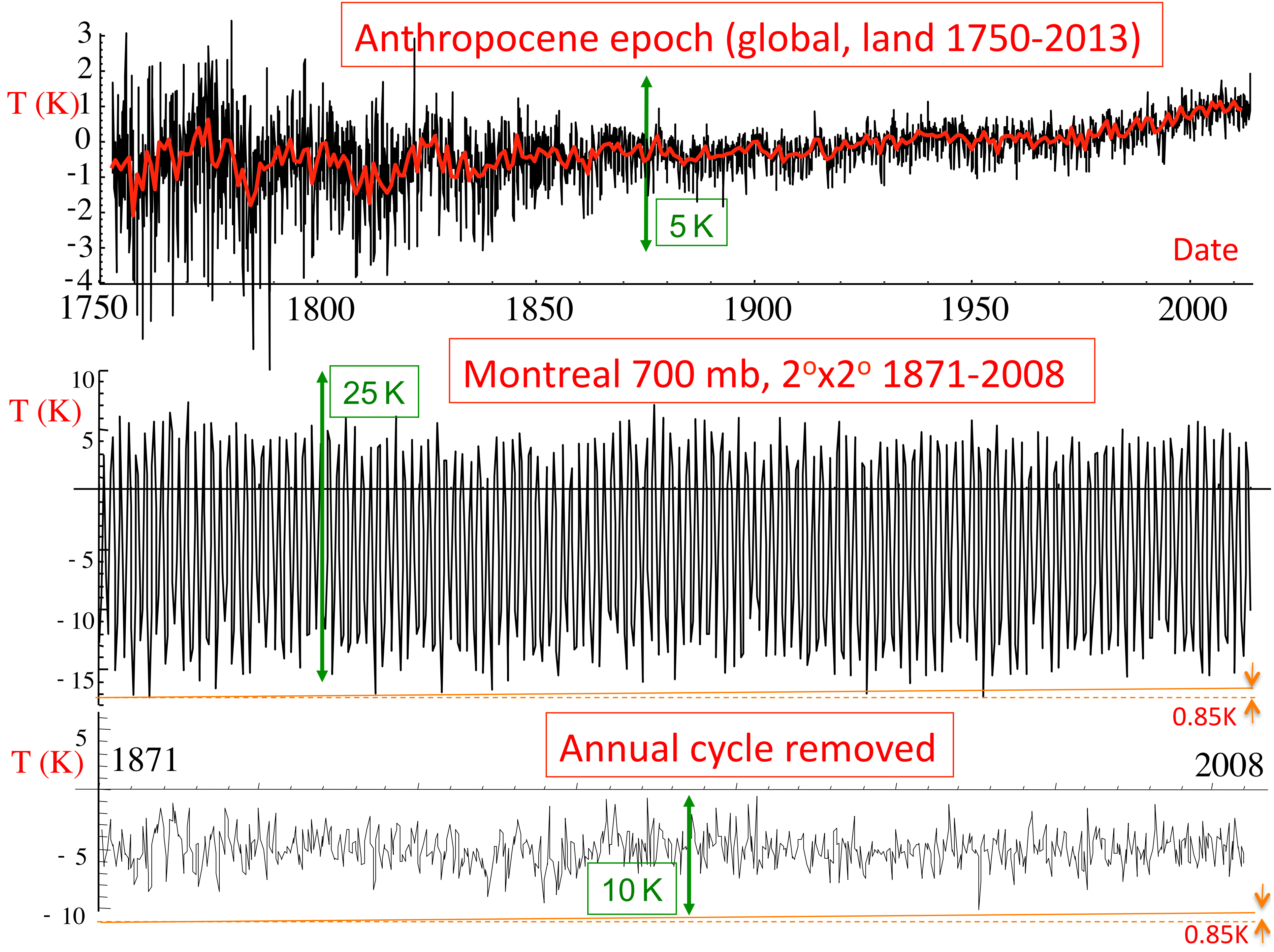
Lovejoy et al 2001



Phanerazoic eon
50 K
X8.3
δ18O‰
Age (Myr BP)
4
-5
-10
100
200
300
400
500
Cenozoic era
10 K
X5.8
4
3
2
1
-1
10
20
30
40
50
60
Age (Myr BP)
Quaternary Period
10 K
X3.2
0.5
-0.5
-1.0
0.5
1.0
1.5
2.0
2.5

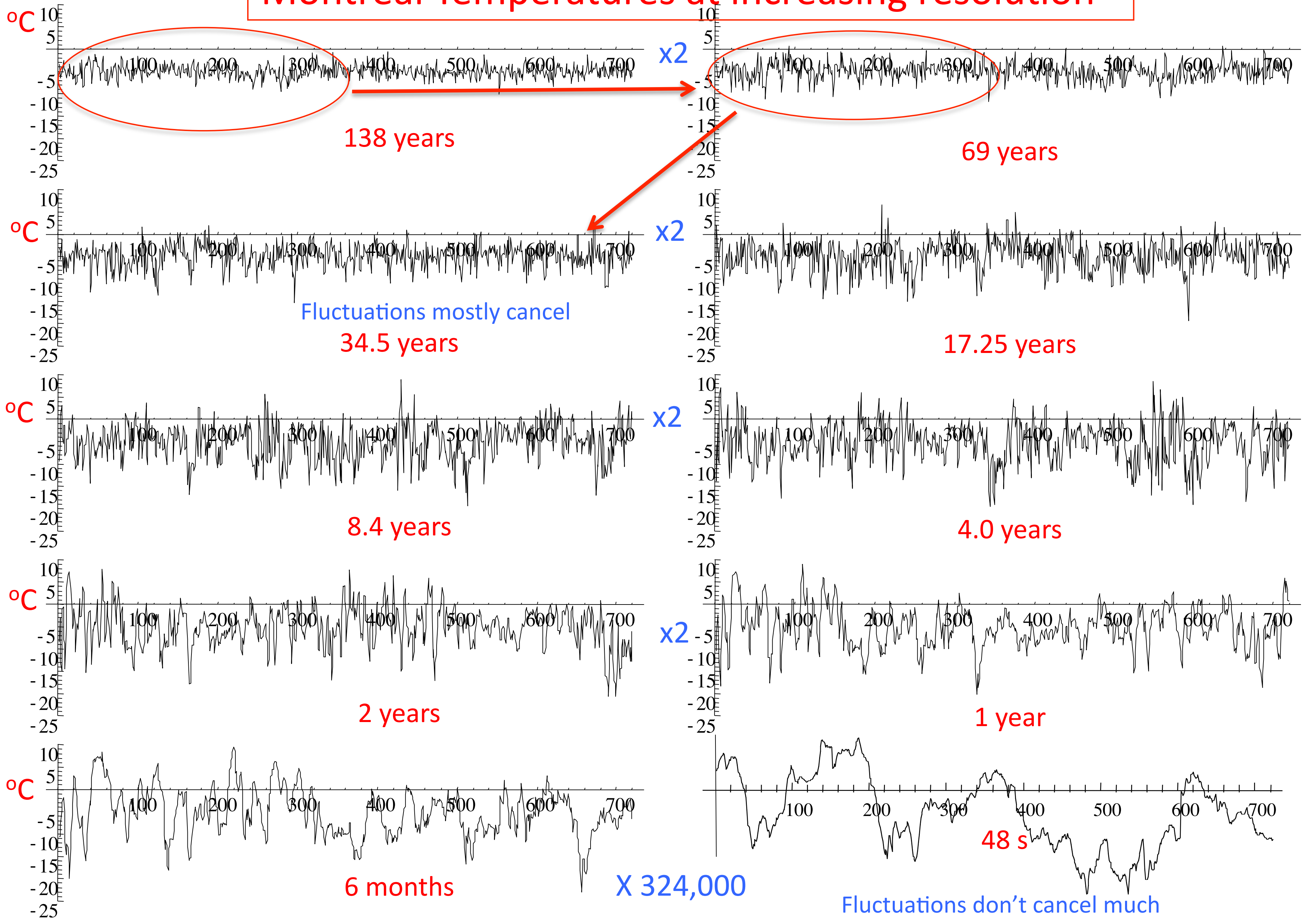


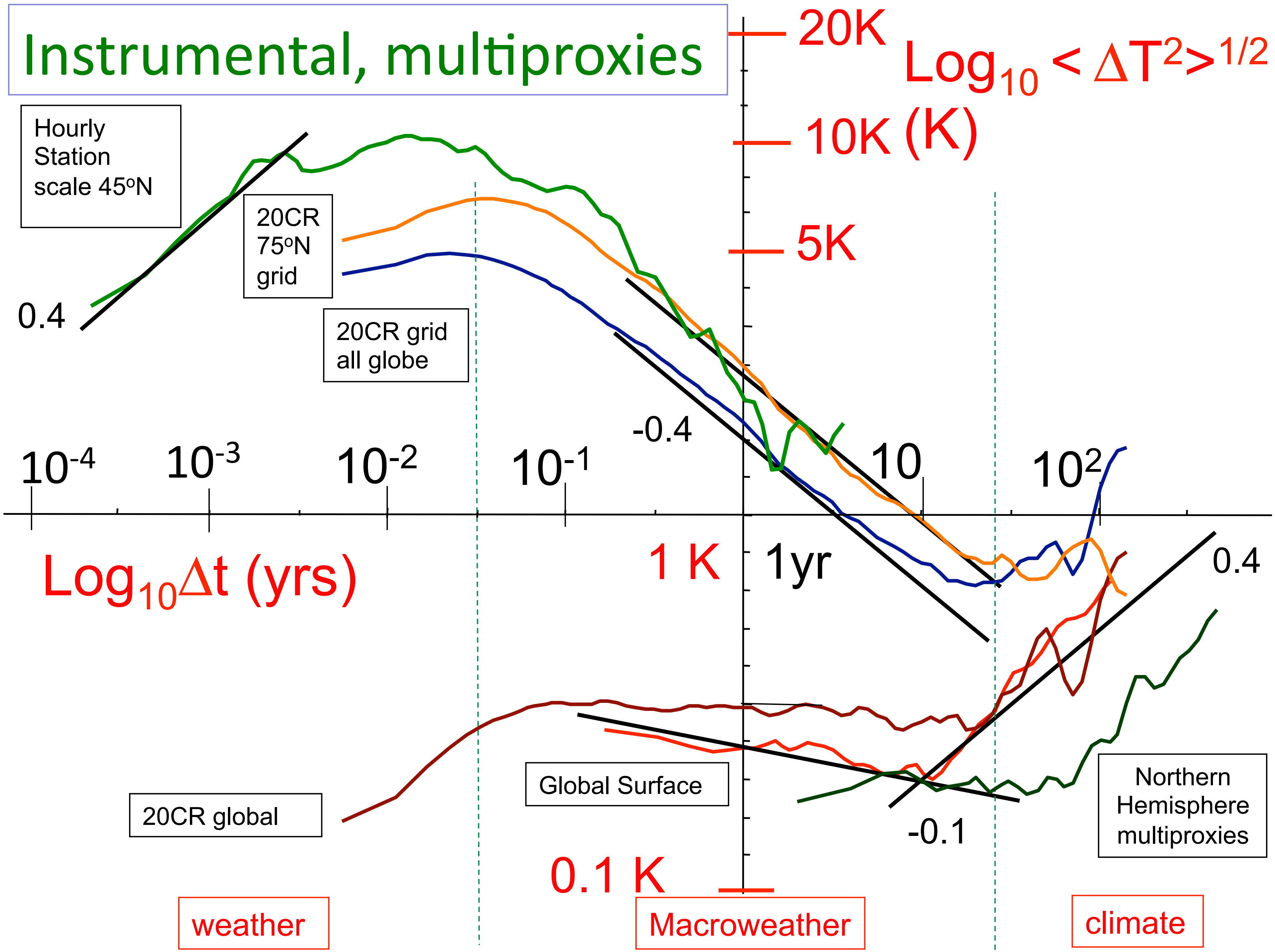
Middle and Late stages Pleistocene epoch (EPICA, Antarctica)
T
5
-5
-10
present
200
400
600
800
Age (kyr BP)
10 K
"ice ages"
X8.8
Upper stage Pleistocene epoch (GRIP Summit 75°N)
δ18O
- 34
- 36
- 38
- 40
- 42
- 44
6 K
interglacial
(Holocene)
last glaciation
Age (kyr BP)
20
40
60
80
X91
X4.8
Last Millenium (Northern hemisphere)
T
1.0
0.5
1 K
1200
1400
1600
1800
Date
2000



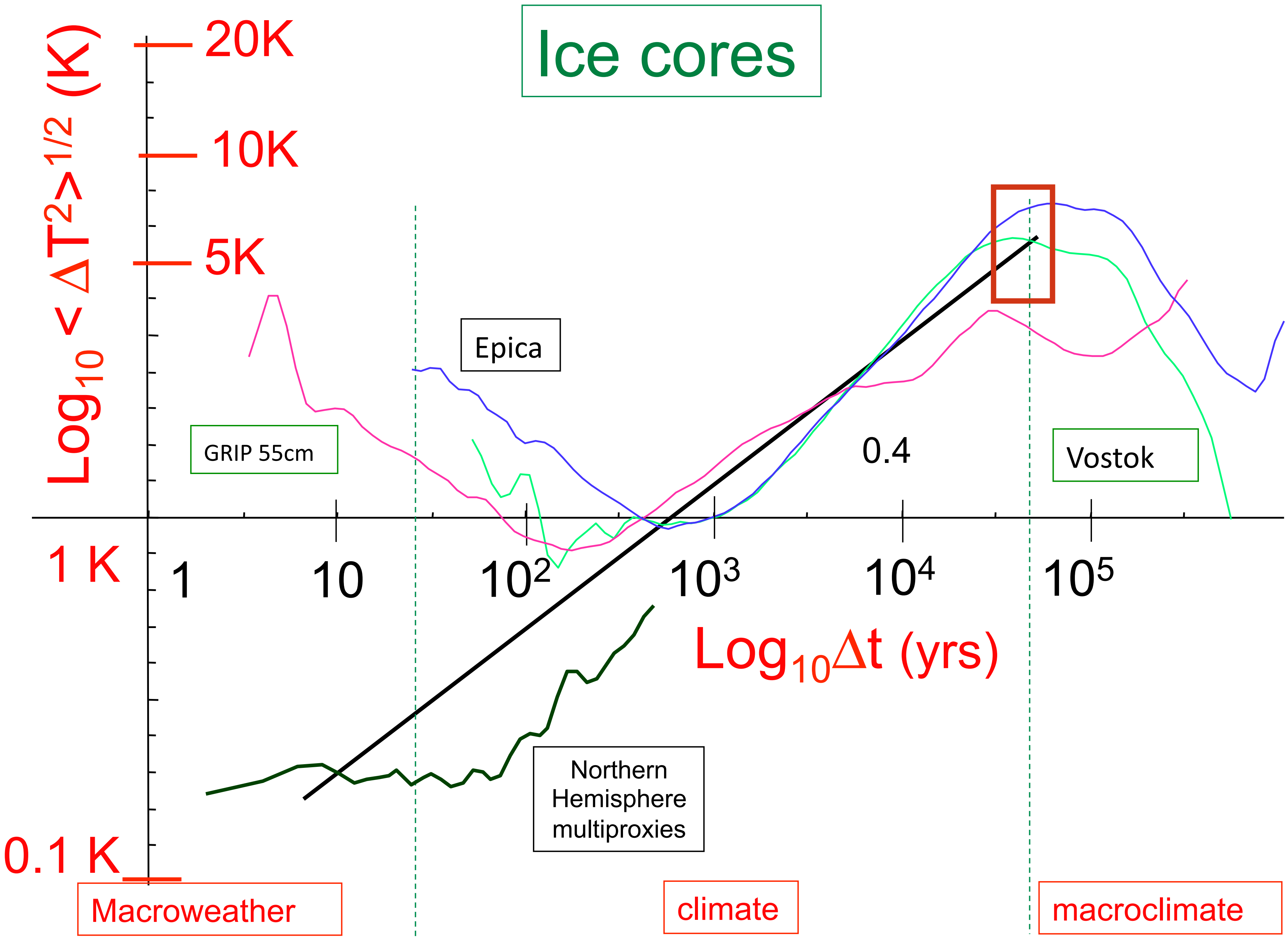
Anthropocene epoch (global, land 1750-2013)
T (K)
3
2
1
0
-1
-2
-3
-4
5 K
Date
1750
1800
1850
1900
1950
2000
Montreal 700 mb, 2°x2° 1871-2008
25 K
T (K)
10
5
- 5
- 10
- 15
0.85K
Annual cycle removed
T (K)
5
1871
2008
- 5
- 10
10 K
0.85K

Montreal Temperatures at increasing resolution

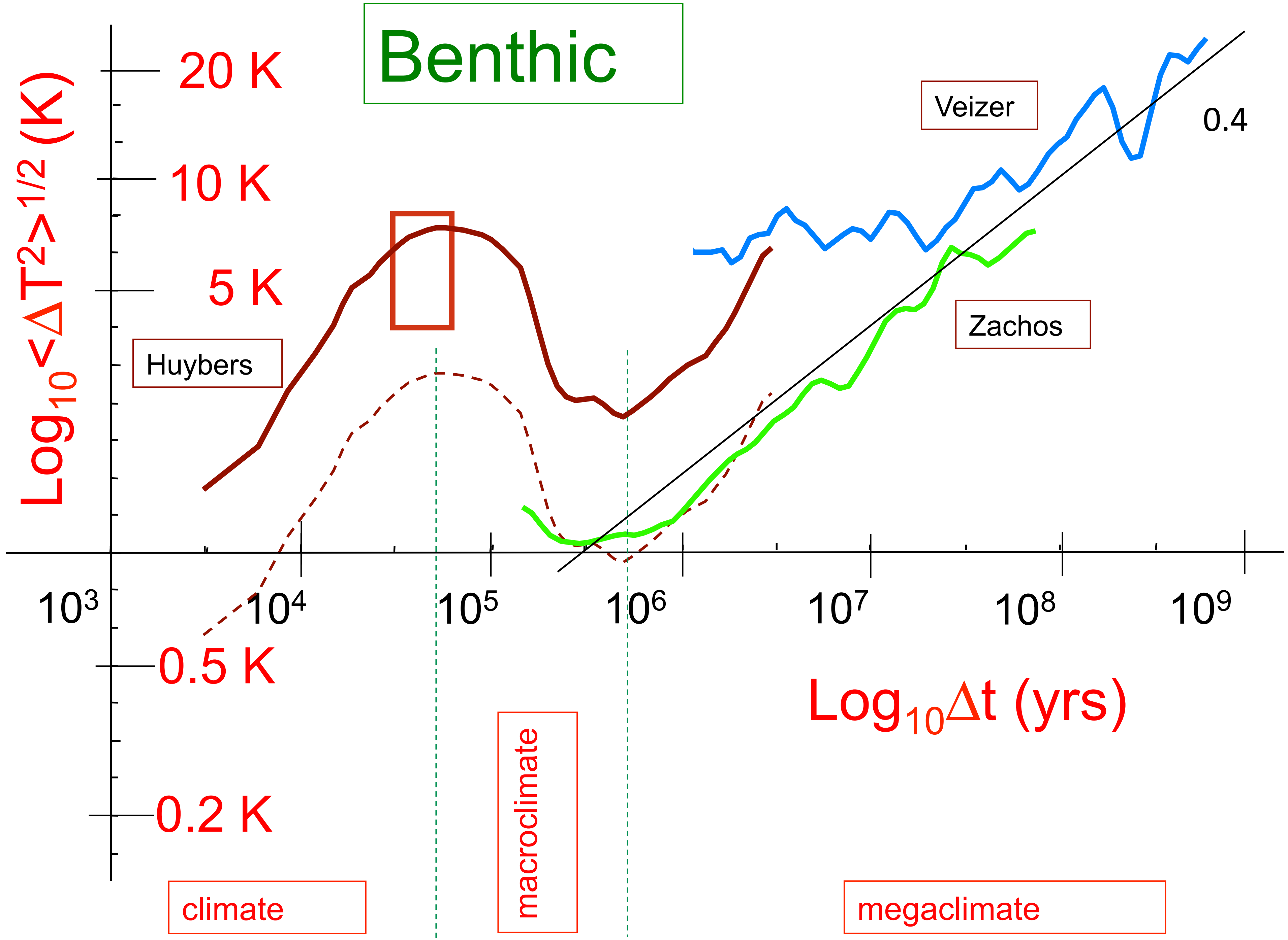




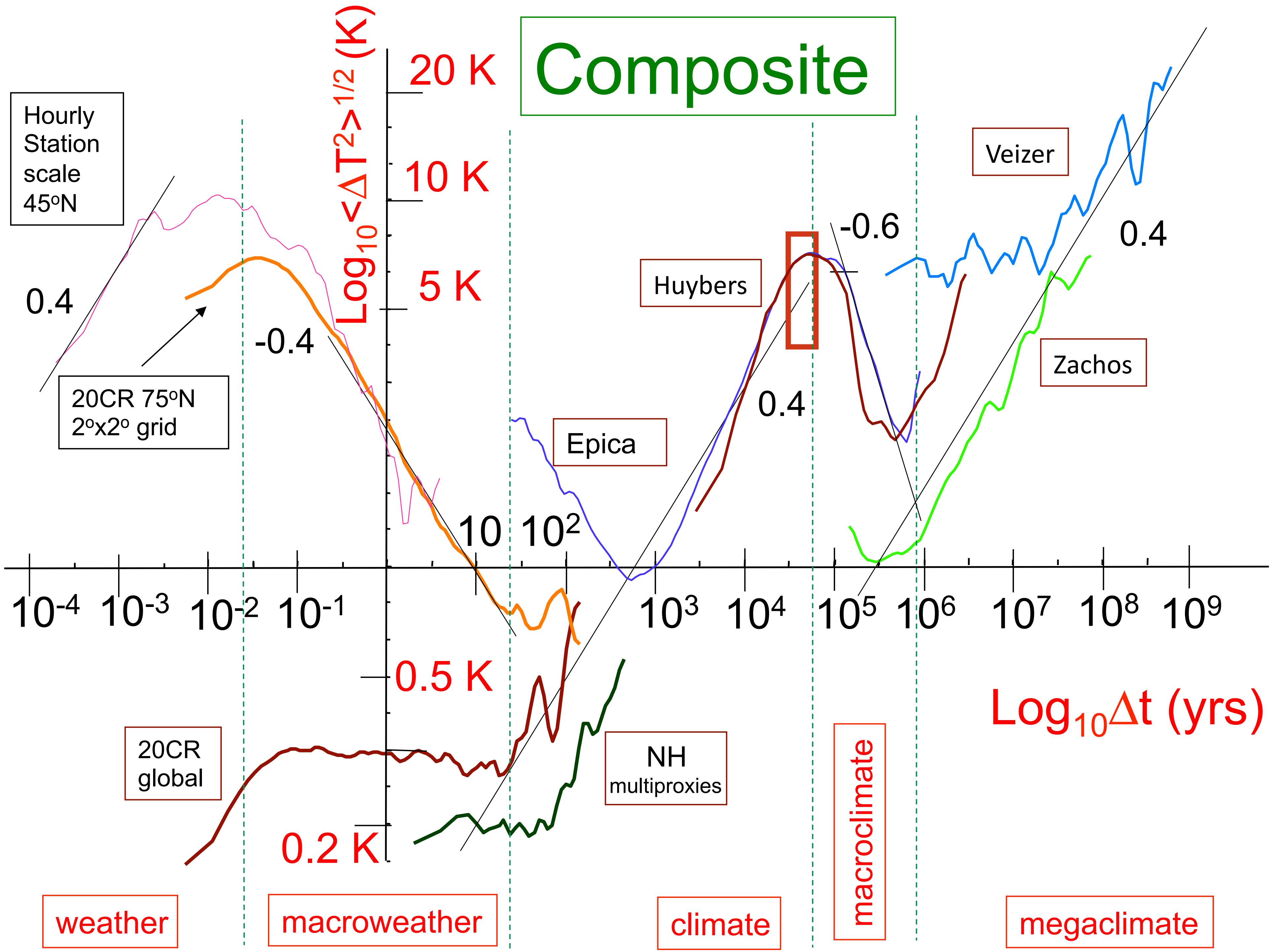
Instrumental, multiproxies
Hourly Station scale 45°N
20CR 75°N grid
20CR grid all globe
20CR global
Global Surface
Northern Hemisphere multiproxies
20K
10K
5K
1 K
0.1 K
Log10 < ΔT2>1/2 (K)
Log10Δt (yrs)
10-4
10-3
10-2
10-1
1yr
10
102
0.4
-0.4
0.4
-0.1
weather
Macroweather
climate



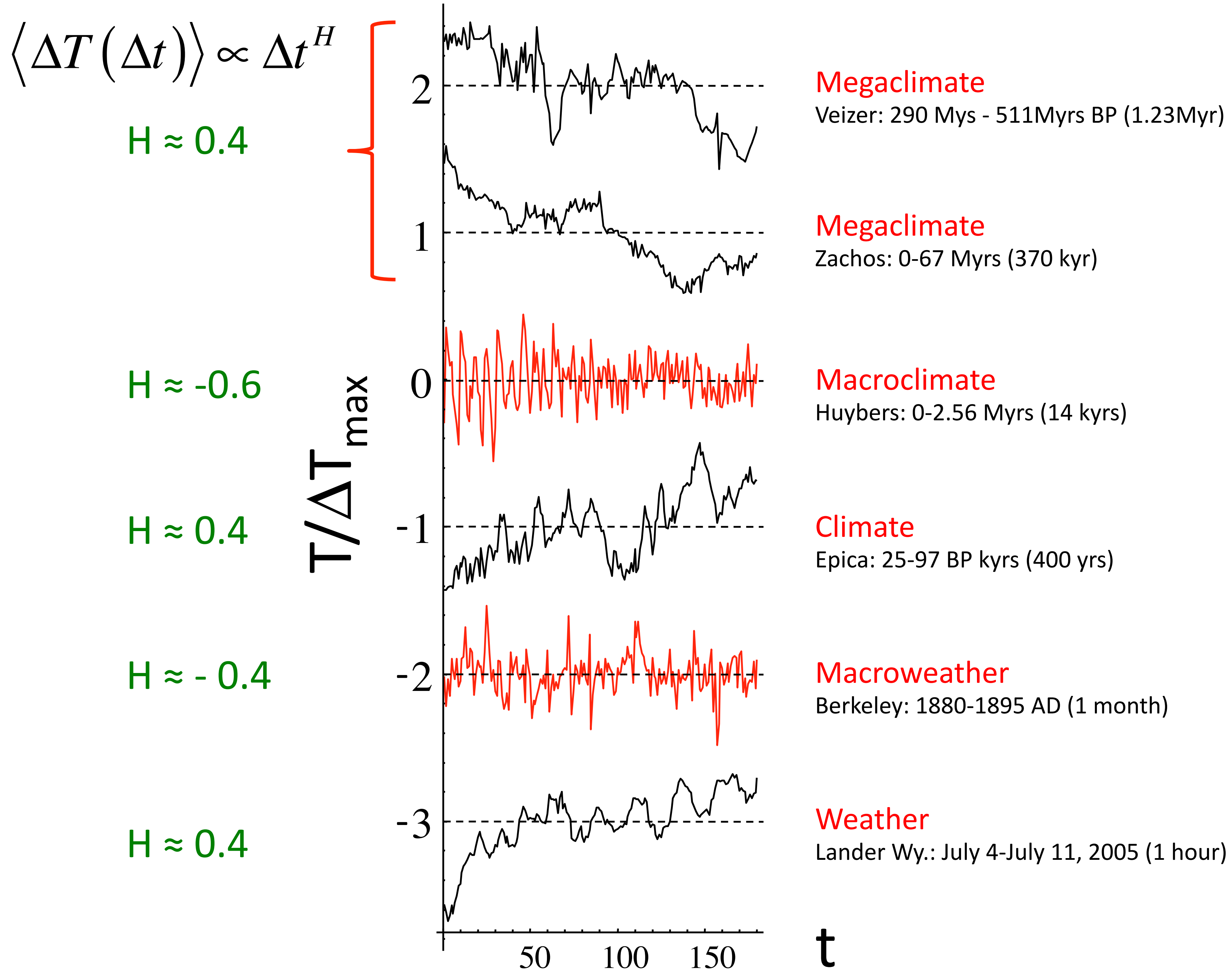
Ice cores
$Log_{10} < \Delta T^2 >^{1/2}$ (K)
20K
10K
5K
1 K
0.1 K
1
10
10^2
10^3
10^4
10^5
$Log_{10}\Delta t$ (yrs)
Epica
GRIP 55cm
Vostok
0.4
Northern Hemisphere multiproxies
Macroweather
climate
macroclimate



Benthic
20 K
10 K
5 K
0.5 K
0.2 K
Log10<ΔT2>1/2 (K)
Huybers
Veizer
Zachos
0.4
10^3
10^4
10^5
10^6
10^7
10^8
10^9
Log10Δt (yrs)
climate
macroclimate
megaclimate



Composite
Hourly Station scale 45°N
20CR 75°N 2°x2° grid
20CR global
Epica
Huybers
Veizer
Zachos
NH multiproxies
20 K
10 K
5 K
0.5 K
0.2 K
0.4
-0.4
-0.6
0.4
0.4
10
10^2
10^{-4} 10^{-3} 10^{-2} 10^{-1} 10^3 10^4 10^5 10^6 10^7 10^8 10^9
$\mathrm{Log}_{10}\langle\Delta T^2\rangle^{1/2}$ (K)
$\mathrm{Log}_{10}\Delta t$ (yrs)
weather
macroweather
climate
macroclimate
megaclimate

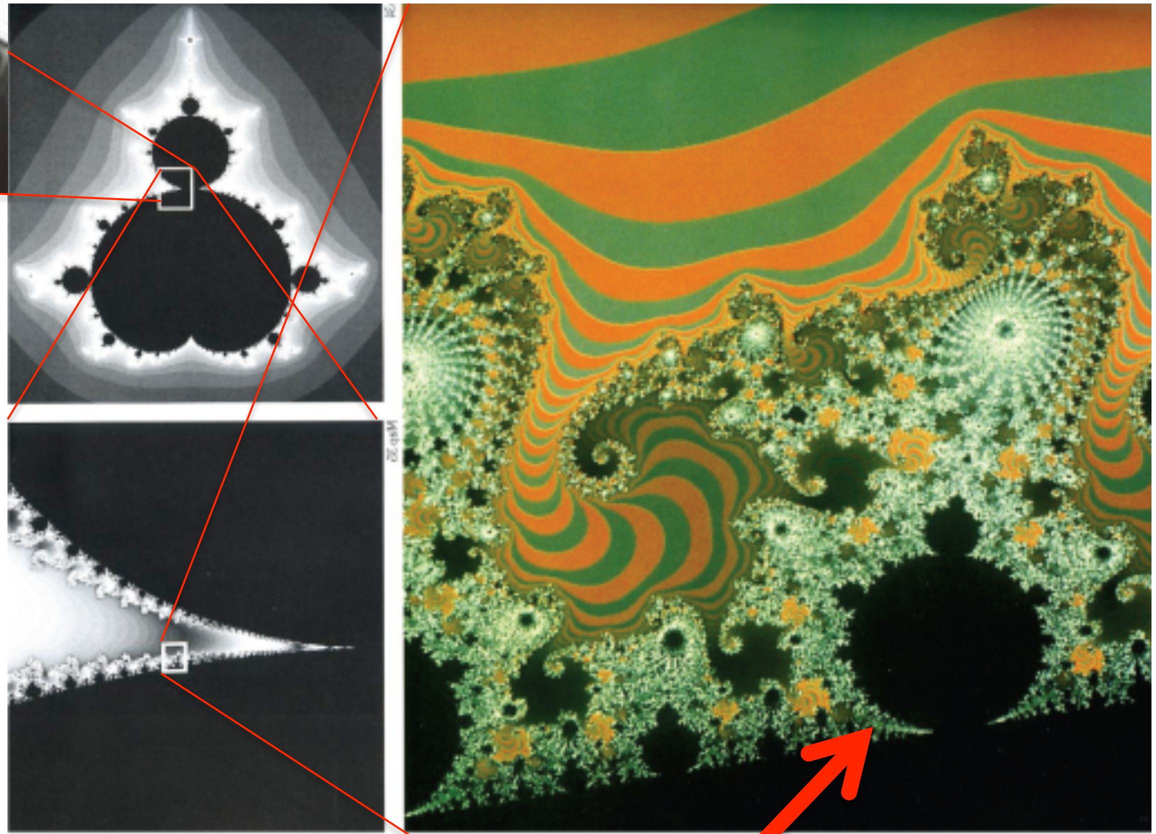


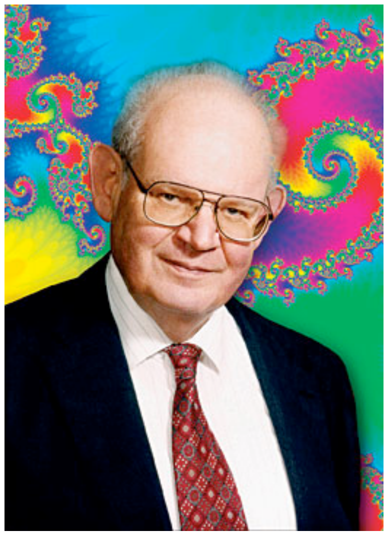
$\langle \Delta T(\Delta t) \rangle \propto \Delta t^H$
H ≈ 0.4
H ≈ -0.6
H ≈ 0.4
H ≈ - 0.4
H ≈ 0.4
T/ΔT max
2
1
0
-1
-2
-3
50
100
150
t
Megaclimate
Veizer: 290 Mys - 511Myrs BP (1.23Myr)
Megaclimate
Zachos: 0-67 Myrs (370 kyr)
Macroclimate
Huybers: 0-2.56 Myrs (14 kyrs)
Climate
Epica: 25-97 BP kyrs (400 yrs)
Macroweather
Berkeley: 1880-1895 AD (1 month)
Weather
Lander Wy.: July 4-July 11, 2005 (1 hour)

How to understand such variability?

What if....

Peitgen et al



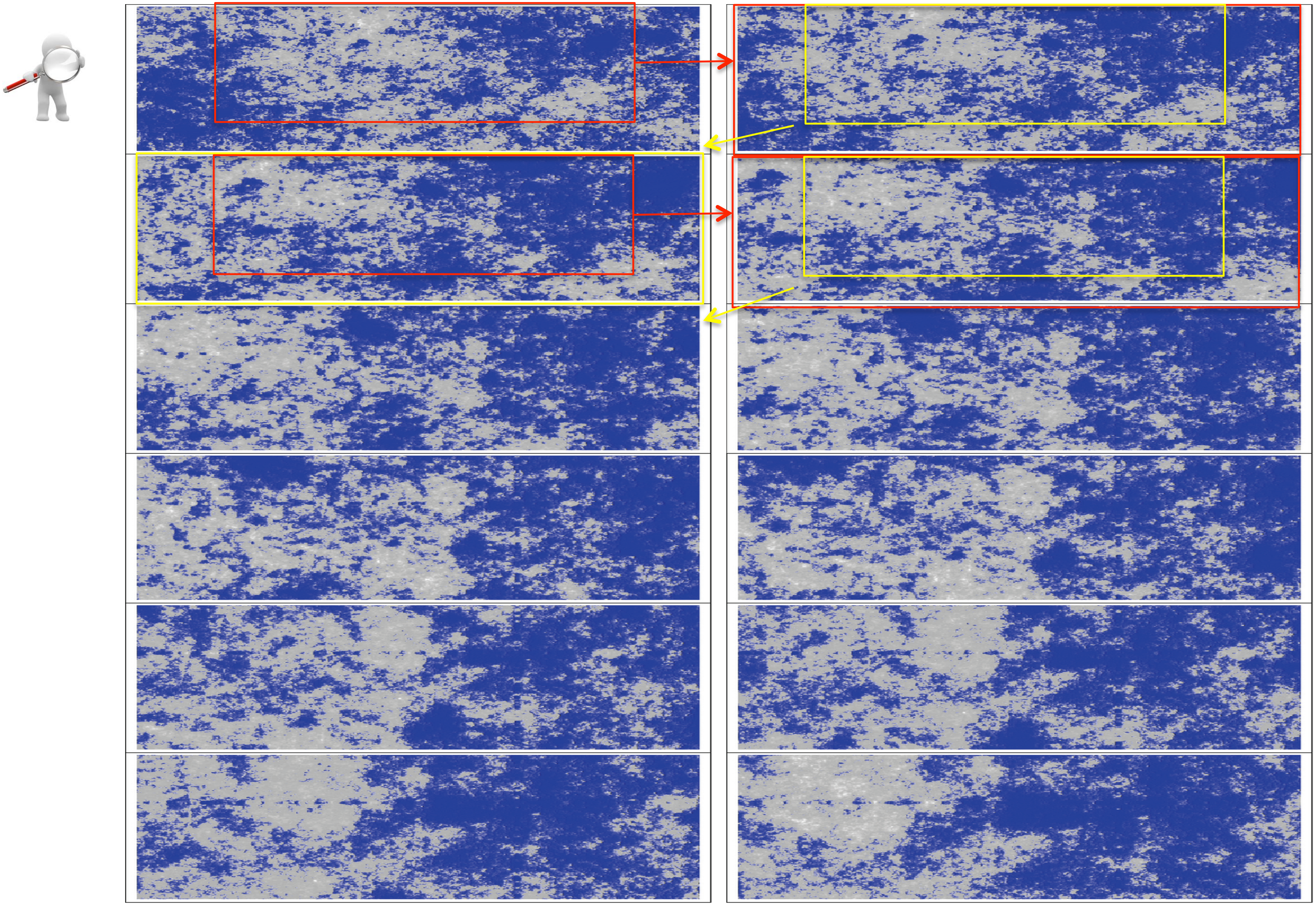


Mandelbrot 1924-2010

We found the same!!!
"Scaling"

(the Mandelbrot set)

Scale invariant Clouds..... Zooming in by factors of 1.7

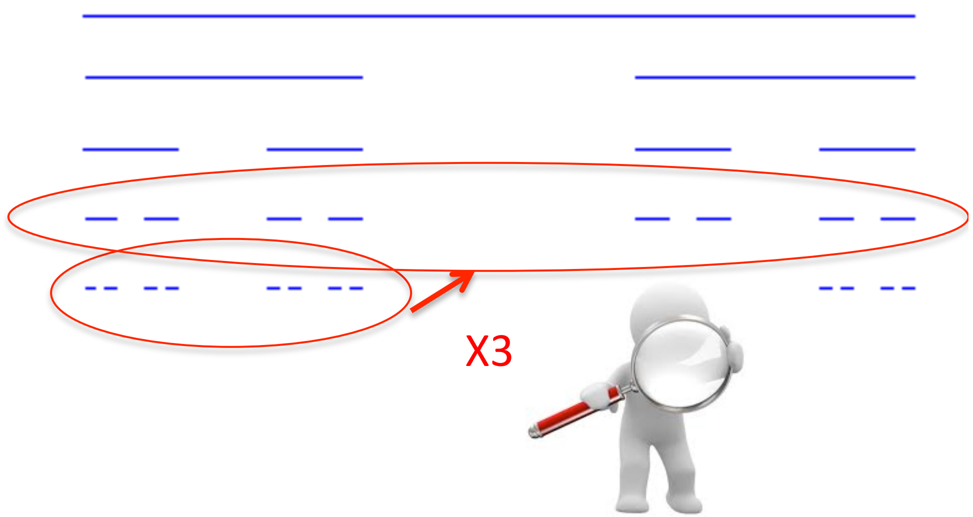


Scale invariant geometric sets: Fractals

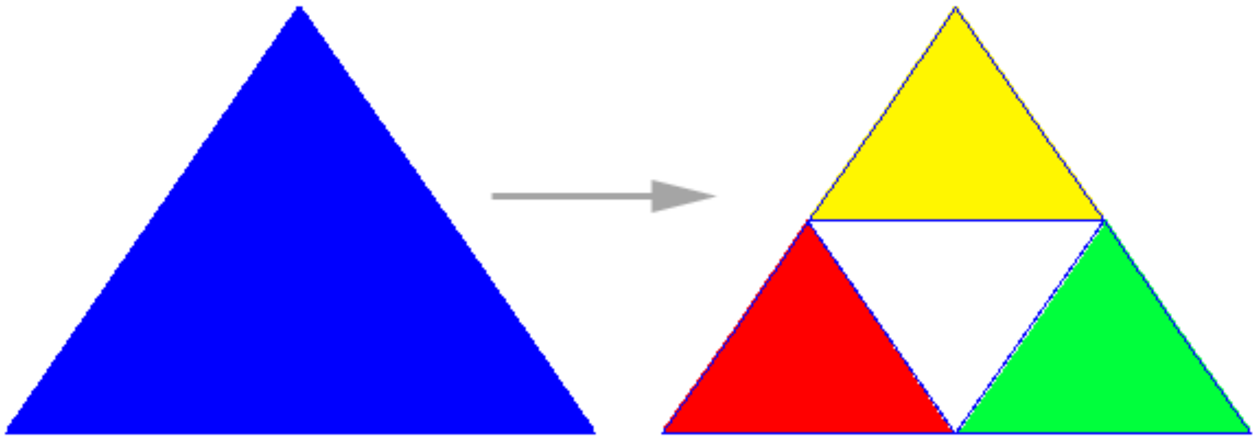
The simplest fractal, the Cantor set (1871)

- Start with:

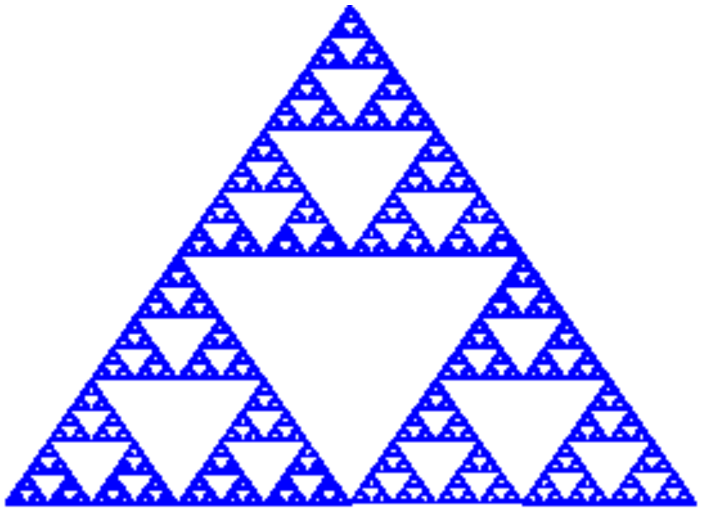
iterate:



Sierpinski Triangle



1 2

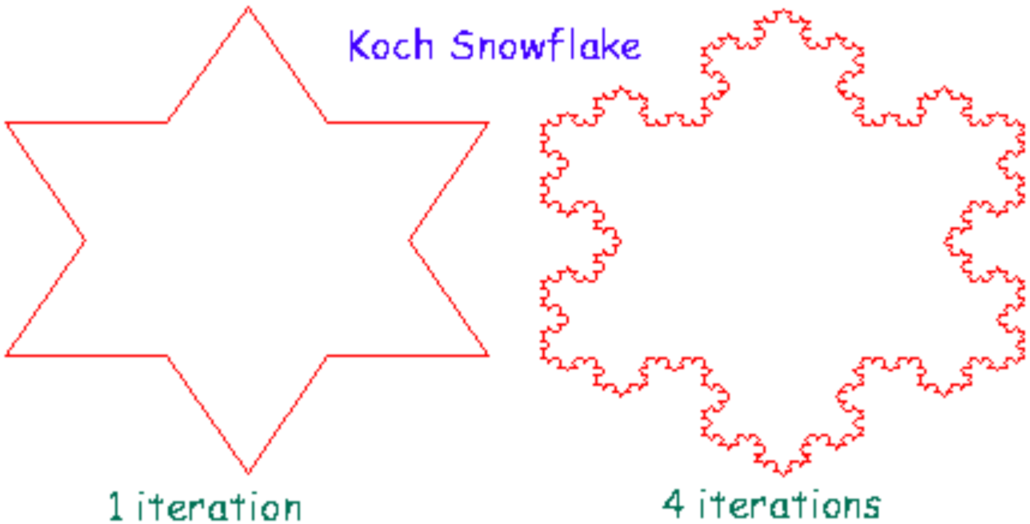


10 iterations

The Koch snowflake

- Start with:

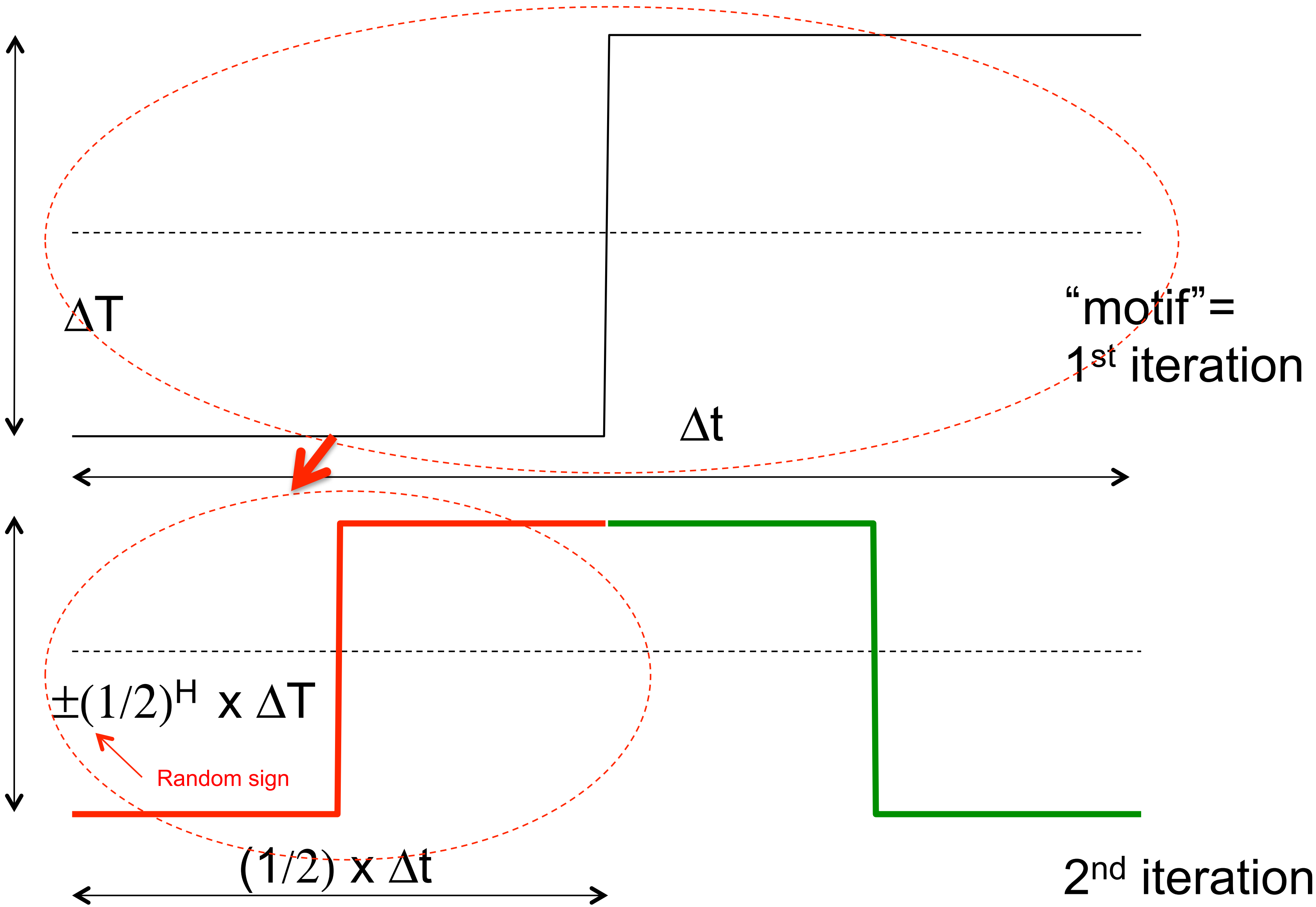
iterate:

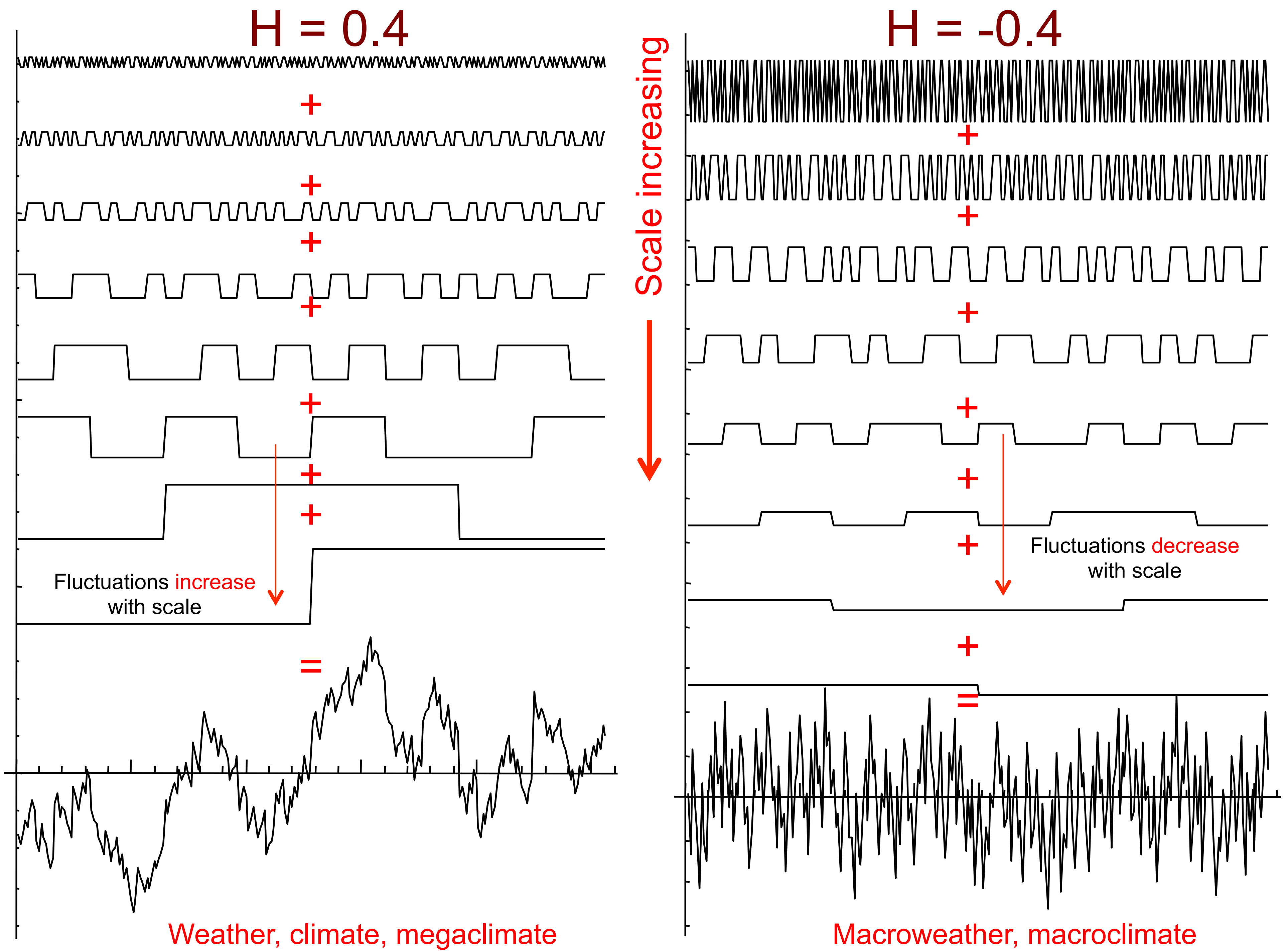


A simple fractal model to help understand the different regimes

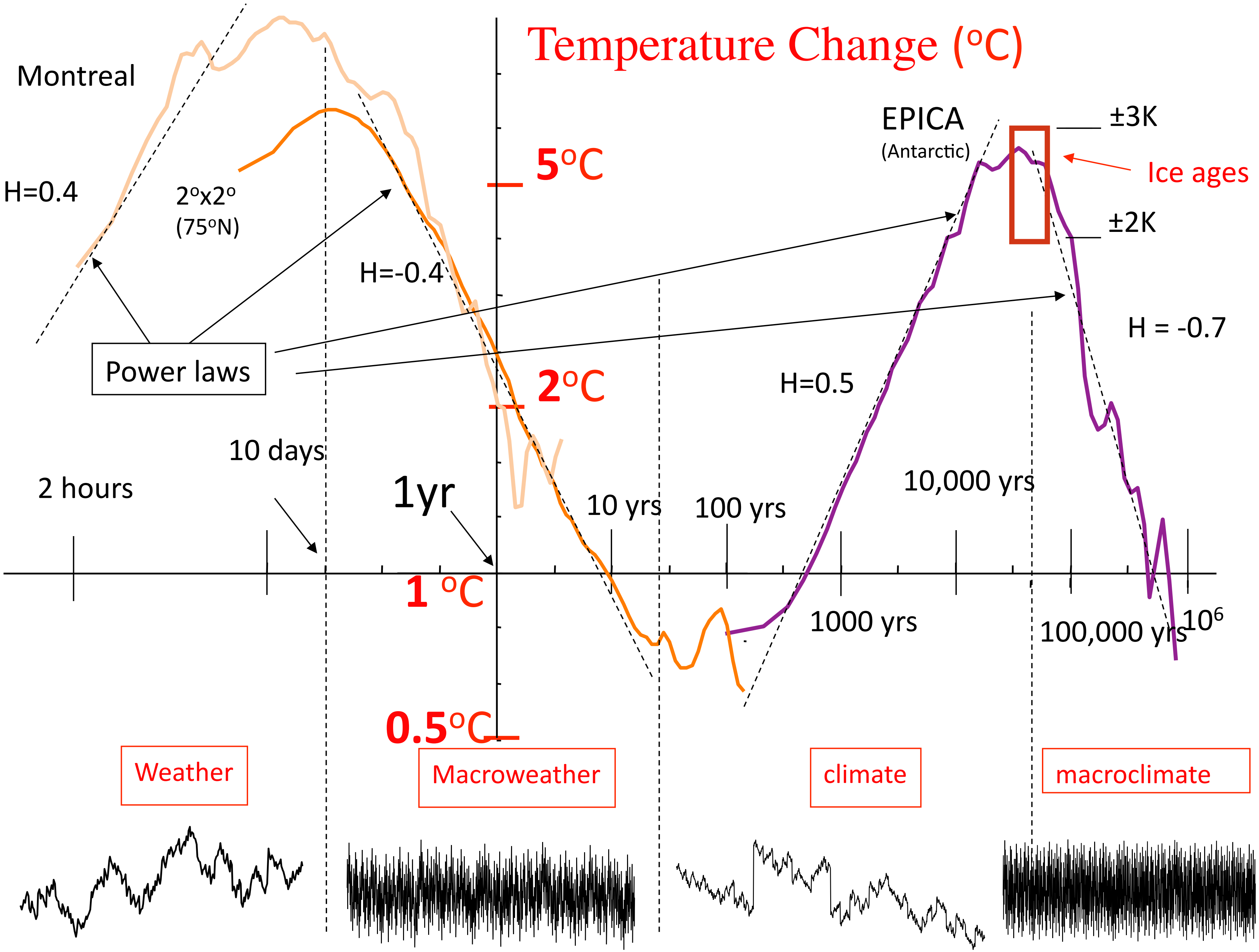
Understanding the fluctuation exponent:

The fractal H model





H = 0.4
H = -0.4
Scale increasing
Fluctuations increase with scale
Fluctuations decrease with scale
Weather, climate, megaclimate
Macroweather, macroclimate


Temperature Change (°C)
Montreal
H=0.4
2°x2°
(75°N)
H=-0.4
Power laws
EPICA
(Antarctic)
±3K
Ice ages
±2K
H = -0.7
H=0.5
5°C
2°C
1 °C
0.5°C
2 hours
10 days
1yr
10 yrs
100 yrs
1000 yrs
10,000 yrs
100,000 yrs
10^6
Weather
Macroweather
climate
macroclimate

Conclusion:

“Macroweather is what you expect
The climate is what you get!”

Weather, macroweather and the climate are distinguished by the way the change under a zoom!


The unity of clouds
and rocks:
Scaling
Multifractal simulation

What is the tangent of the coast of Brittany?

Perrin 1913:
"Consider the difficulty in finding the tangent to a point of the coast of Brittany... depending on the resolution of the map the tangent would change. The point is that a map is simply a conventional drawing in which each line has a tangent. On the contrary, an essential feature of the coast is that ... without making them out, at each scale we *guess* the details which prohibit us from drawing a tangent..."

How Long is the Vistula… the coast of Britain?

Steinhaus 1954: "... The left bank of the Vistula when measured with increased precision would furnish lengths ten, hundred, and even a thousand times as great as the length read off a school map. A statement nearly adequate to reality would be to call most arcs encountered in nature as not rectifiable. This statement is contrary to the belief that not rectifiable arcs are an invention of mathematicians and that natural arcs are rectifiable: it is the opposite which is true..."

Richardson 1961: Empirical scaling of coast of Britain and of several frontiers using "Richardson dividers" method.

Mandelbrot 1967: paper "How long is the coast of Britain?" interprets Richardson's scaling exponent in terms of a fractal dimension.

By the 1980's: The fractality of coastlines became "obvious"!

But… the coastline is a level set of the topography.

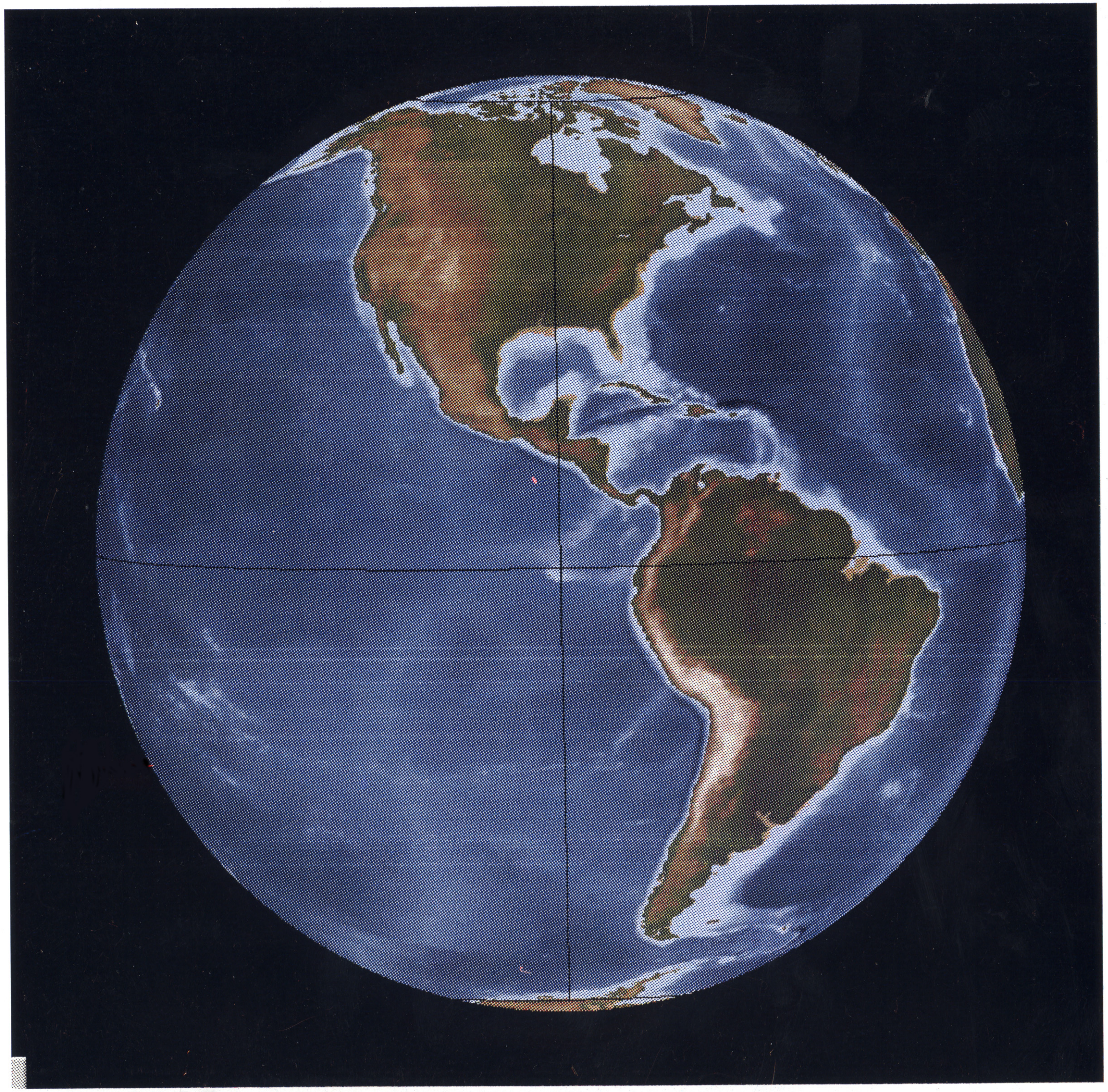
So what are the statistics of the topography field h(x,y)?

Some early scaling results the isotropic spectrum E(k) of h(x,y):

Vening-Meinesz 1951: $E(k)=k^{-\beta}$; $\beta=2$

Balmino et al 1973, Bell 1975 : $E(k)=k^{-\beta}$ ωιτη β νεαρ 2

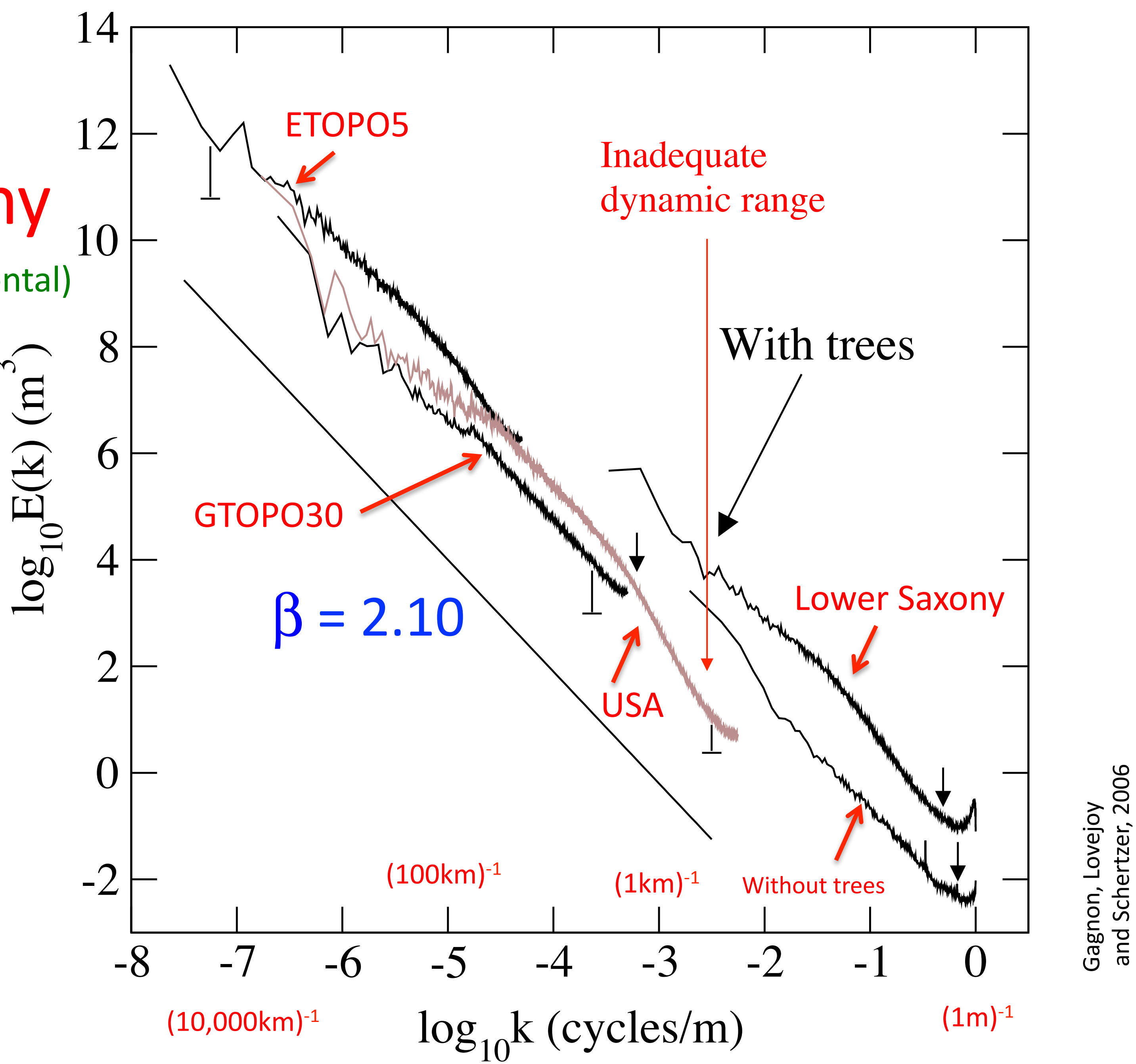
Topography



ETOPO5
altitude data
(5 ’ arc, roughly 10km
Resolution)

Topography

(scaling in the horizontal)



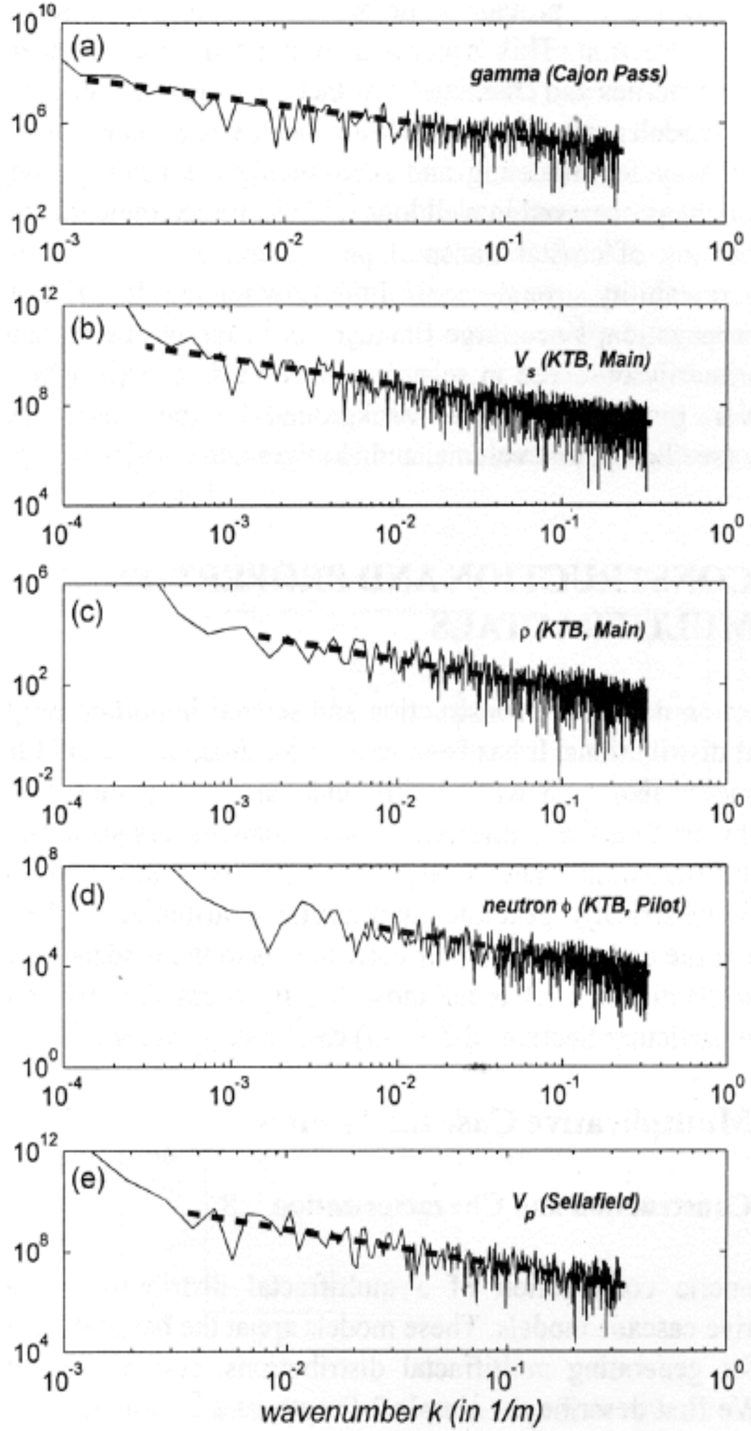
Energy spectra over a scale range of 10^8 Global (ETOPO5, 10km), continental US (GTOPO30: 1km and 90m), Lower Saxony, 20cm).

The scaling of the KTB borehole

(scaling in the vertical)

(1987-1995) 9.1km deep
Russian Kola: 12.2 km

Marsan and Bean (2003)

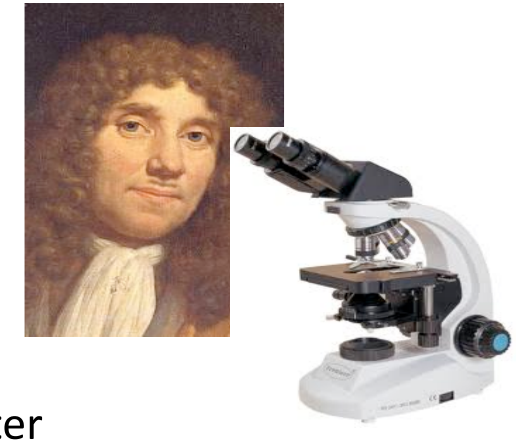


Scale bound
versus
fractal thinking

Scale bound thinking

Antonie van Leeuwenhoek

(1632–1723)

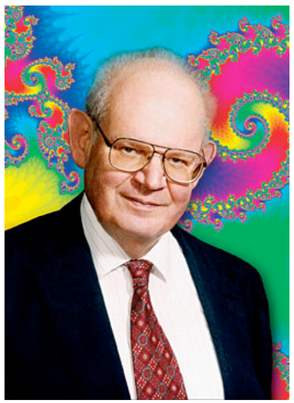


A new world in a drop of water

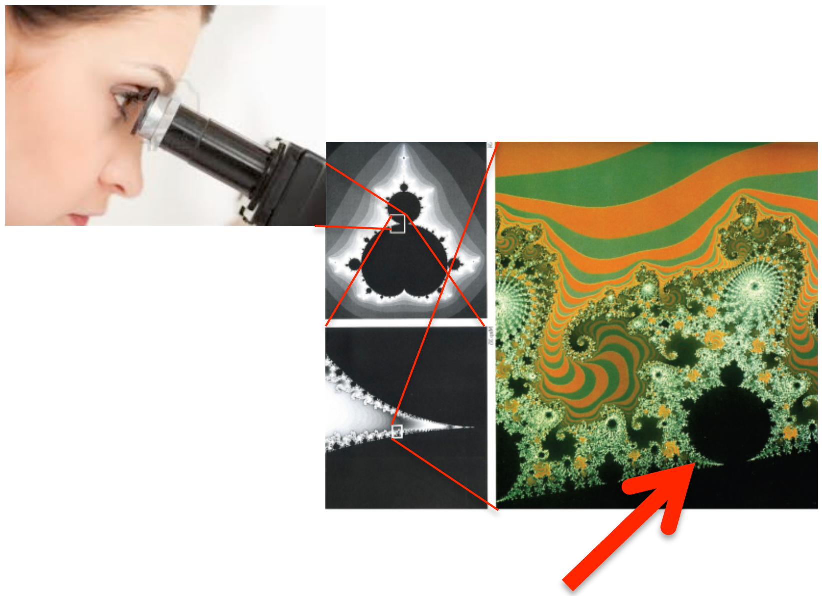
…..the discovery of micro-organisms

"Animalcules," described in depth by Leeuwenhoek, c1695–1698. By Anton van Leeuwenhoek

Pure (self-similar) Fractal thinking



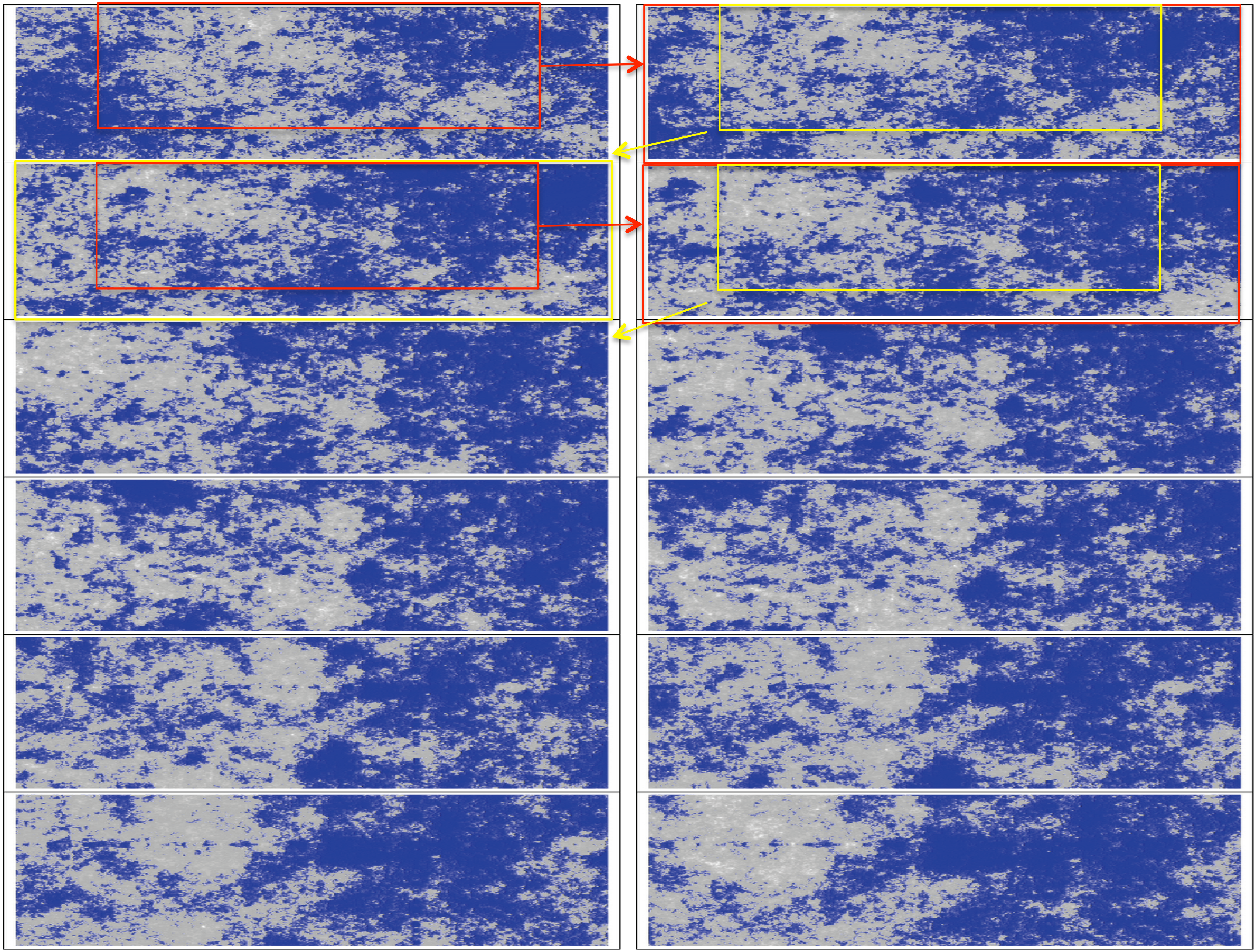
Mandelbrot 1924-2010



The same!!!

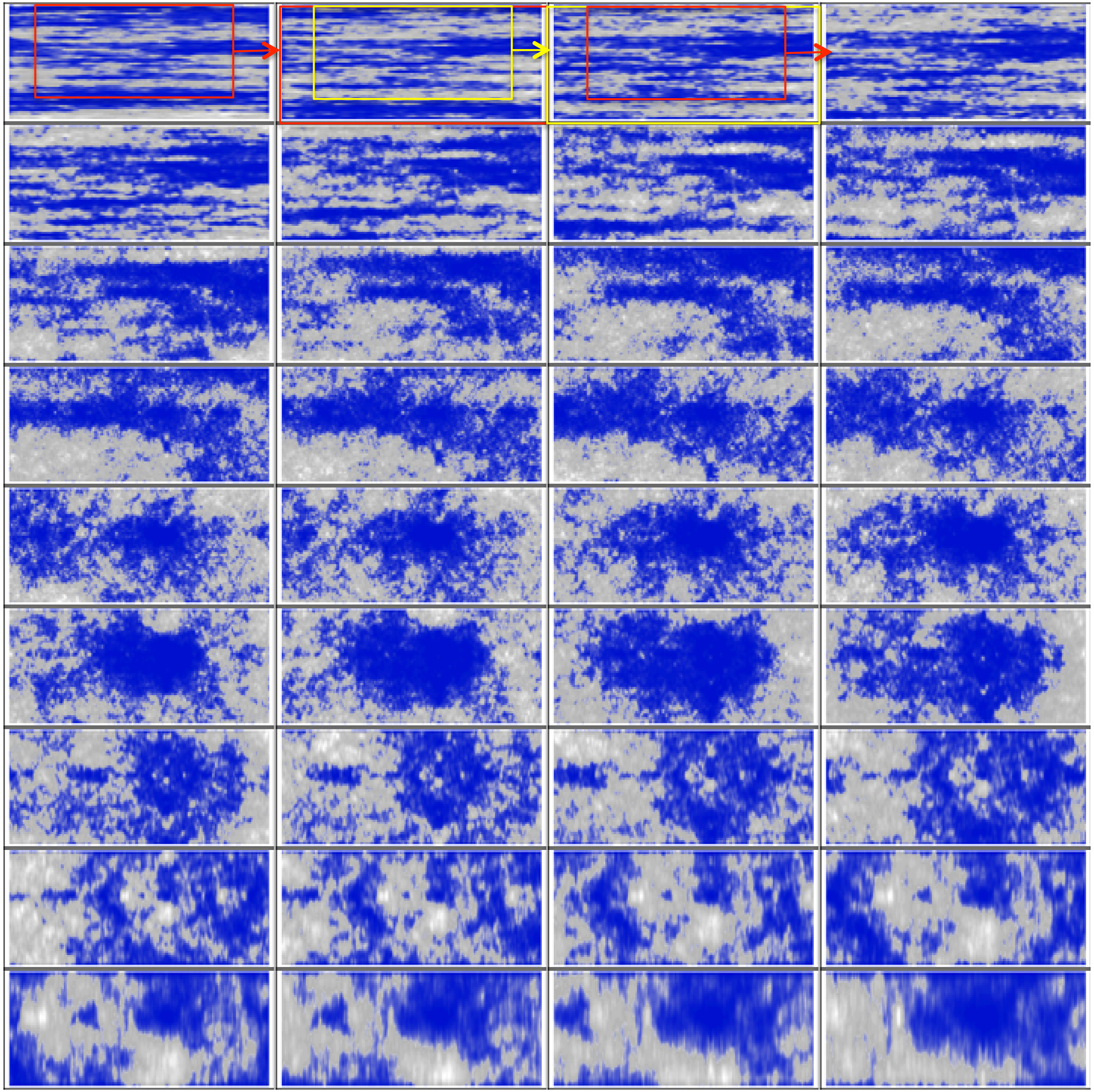
(the Mandelbrot set)

Self-similar Fractal thinking is OK here (Zooming in by factors of 1.7)



What about fractal thinking here?

(Zoom factor 1000)

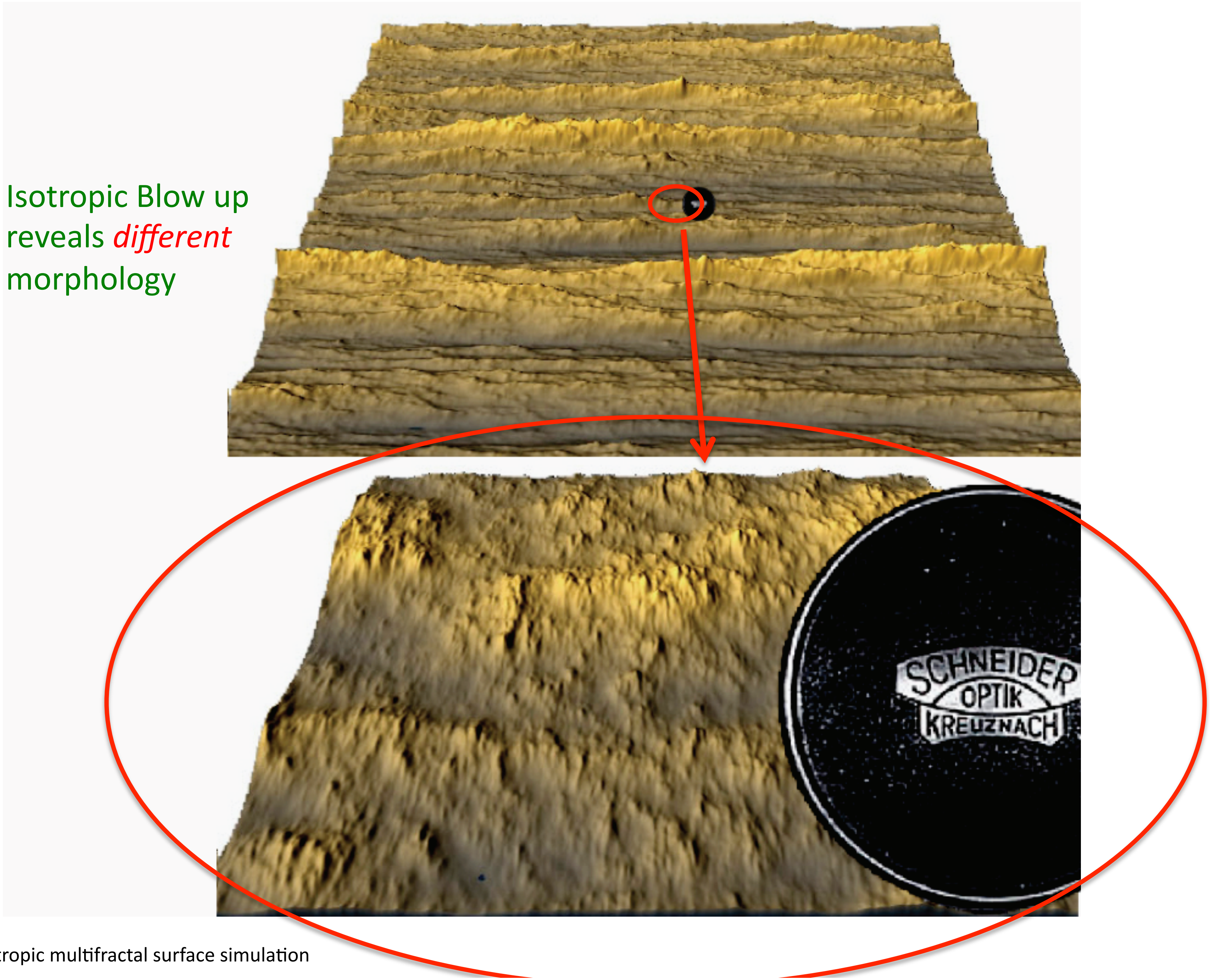


Vertical cross-section

Phenomenological Fallacy

1) Morphology not dynamics is taken as fundamental

2) Scaling is reduced to the isotropic (self-similar) special case



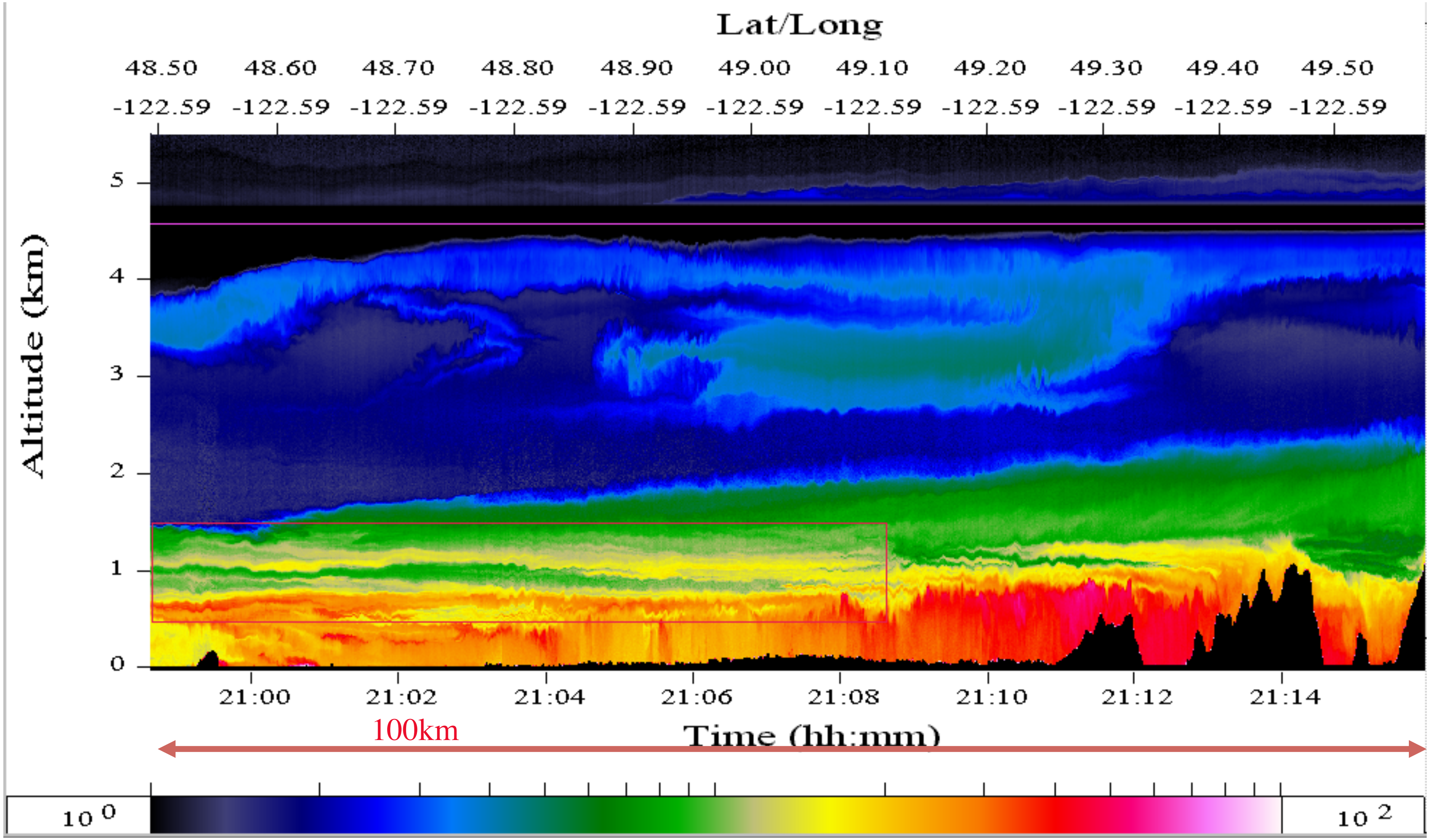
Anisotropic multifractal surface simulation

Anisotropic Scaling

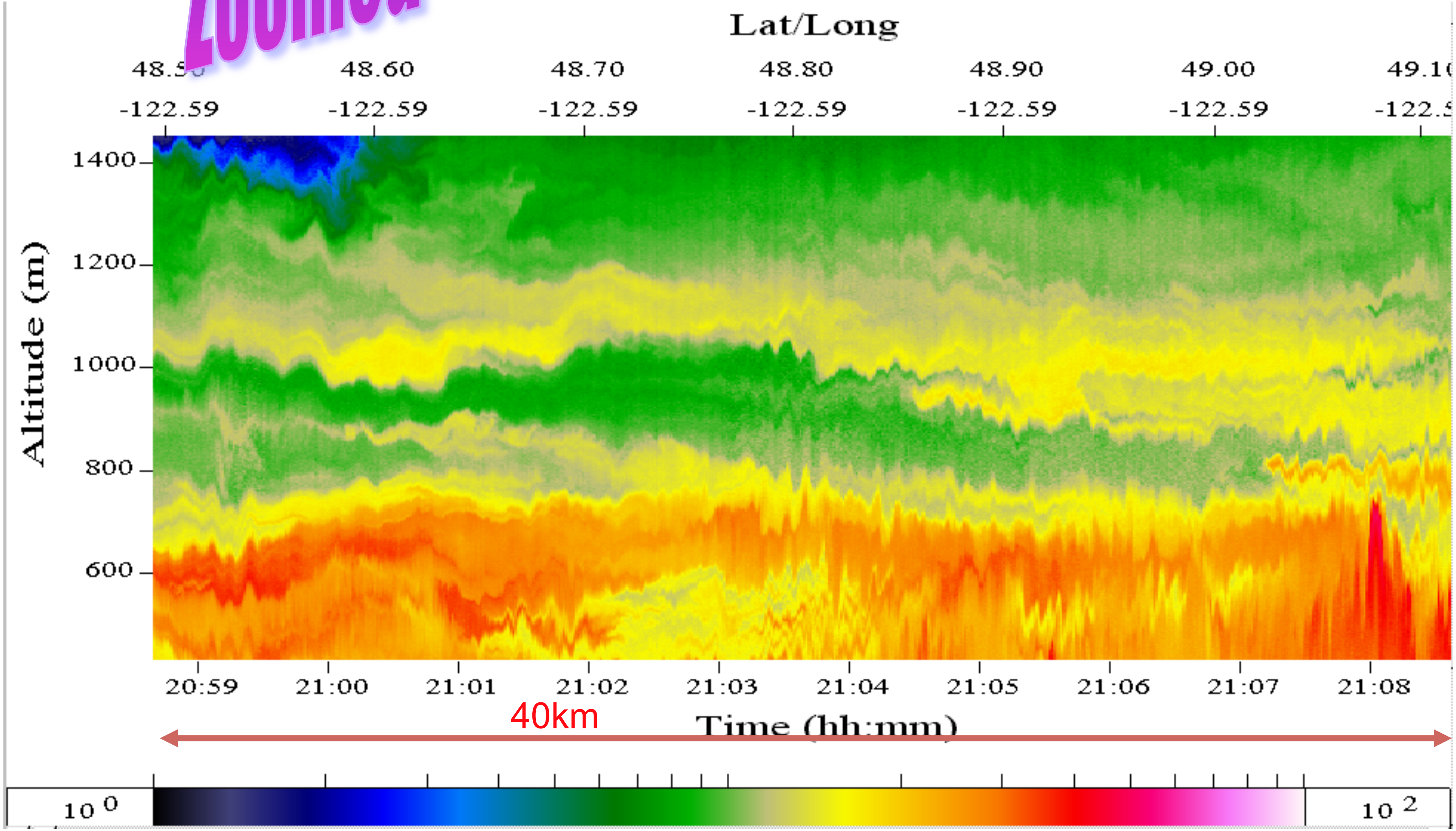
Horizontal versus vertical
(stratification)

AERIAL Lidar Data

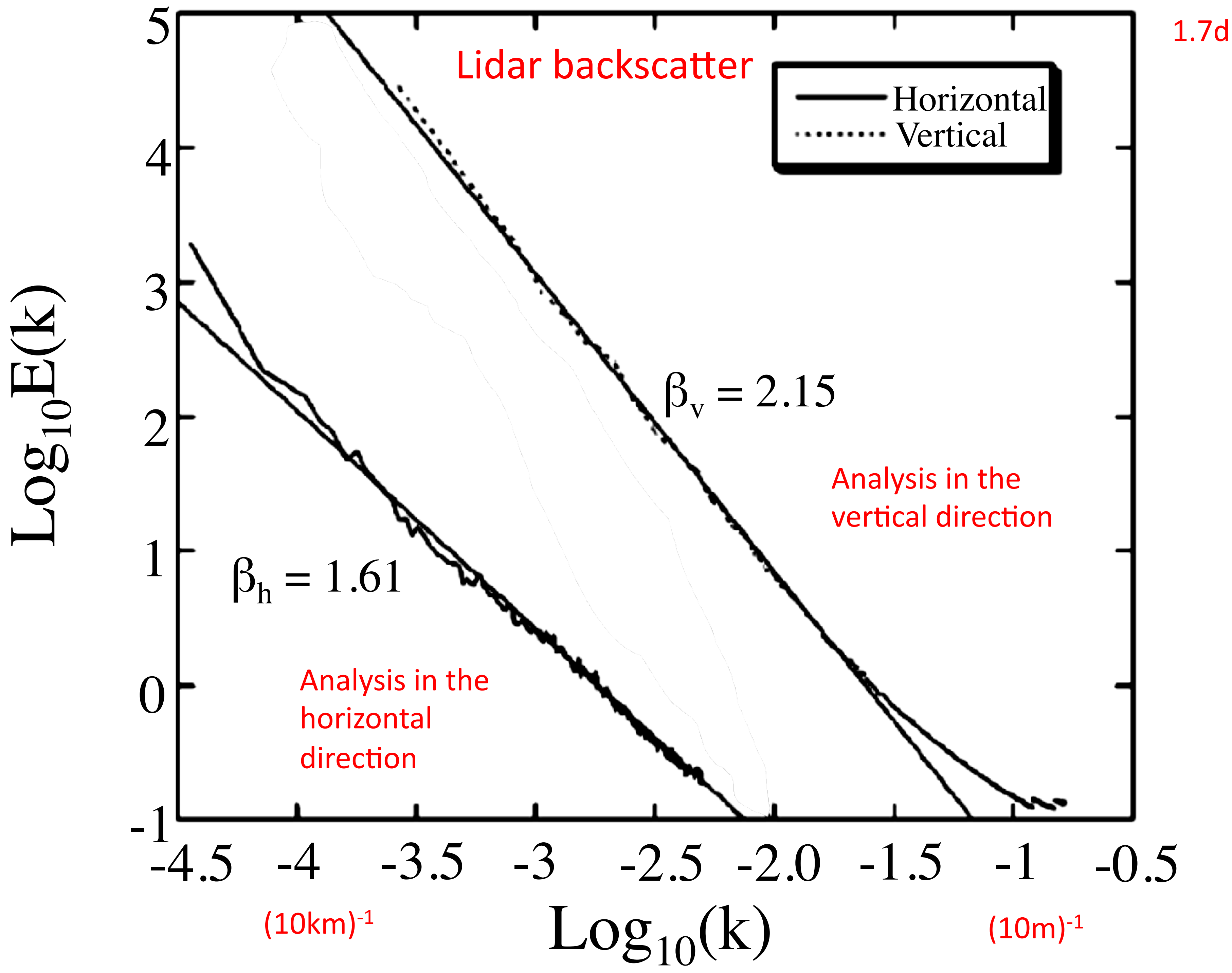
(courtesy of K. Strawbridge)



Zoomed



Lat/Long
48.60
48.70
48.80
48.90
49.00
-122.59
-122.59
-122.59
-122.59
-122.59
-122.59
Altitude (m)
1400
1200
1000
800
600
20:59
21:00
21:01
21:02
21:03
21:04
21:05
21:06
21:07
21:08
Time (hh:mm)
40km
10 0
10 2


1.7d
Lidar backscatter
Horizontal
Vertical
$\beta_v = 2.15$
$\beta_h = 1.61$
Analysis in the vertical direction
Analysis in the horizontal direction
$\text{Log}_{10}E(k)$
$\text{Log}_{10}(k)$
(10km)$^{-1}$
(10m)$^{-1}$
5
4
3
2
1
0
-1
-4.5
-4
-3.5
-3
-2.5
-2.0
-1.5
-1
-0.5

The physical scale function and differential scaling

$$|\underline{\Delta r}| \rightarrow \|\underline{\Delta r}\|$$

Usual distance (=vector norm)

Scale function (scale notion)

Scale symmetry $\left\|\lambda^{-G}\underline{r}\right\| = \lambda^{-1}\left\|\underline{r}\right\|$

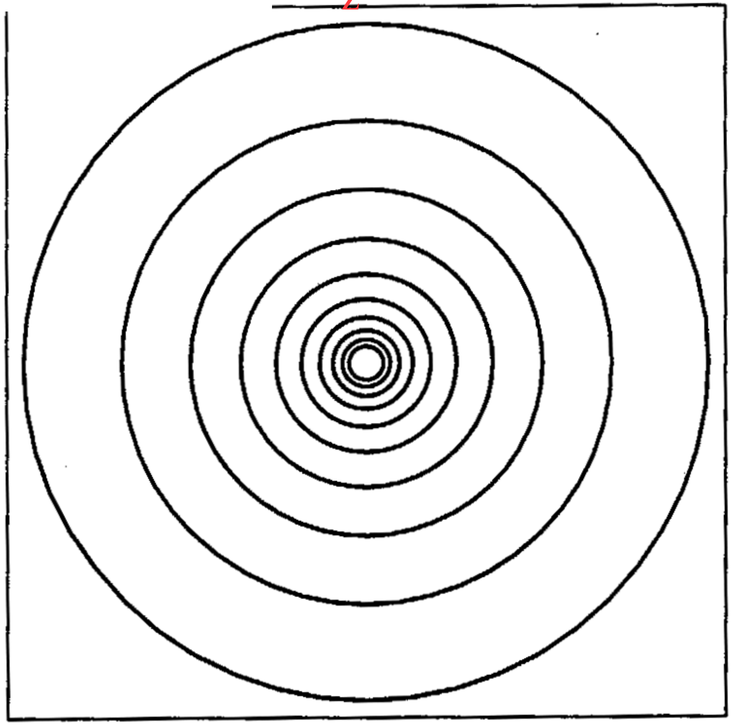
"canonical" scale function:

$$\|(\Delta x, \Delta z)\| = l_s\left(\left(\frac{\Delta x}{l_s}\right)^2 + \left(\frac{\Delta z}{l_s}\right)^{2/H_z}\right)^{1/2}$$

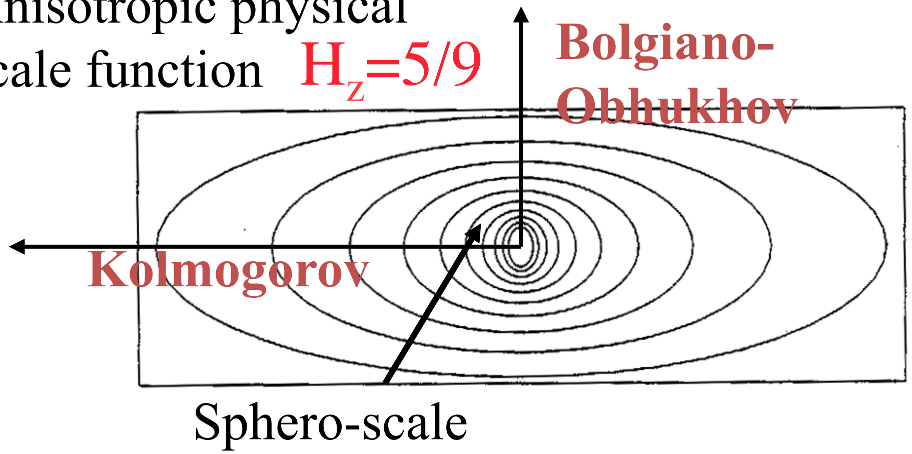
$$G = \begin{pmatrix} 1 & 0 \\ 0 & H_z \end{pmatrix}$$

Vertical sections

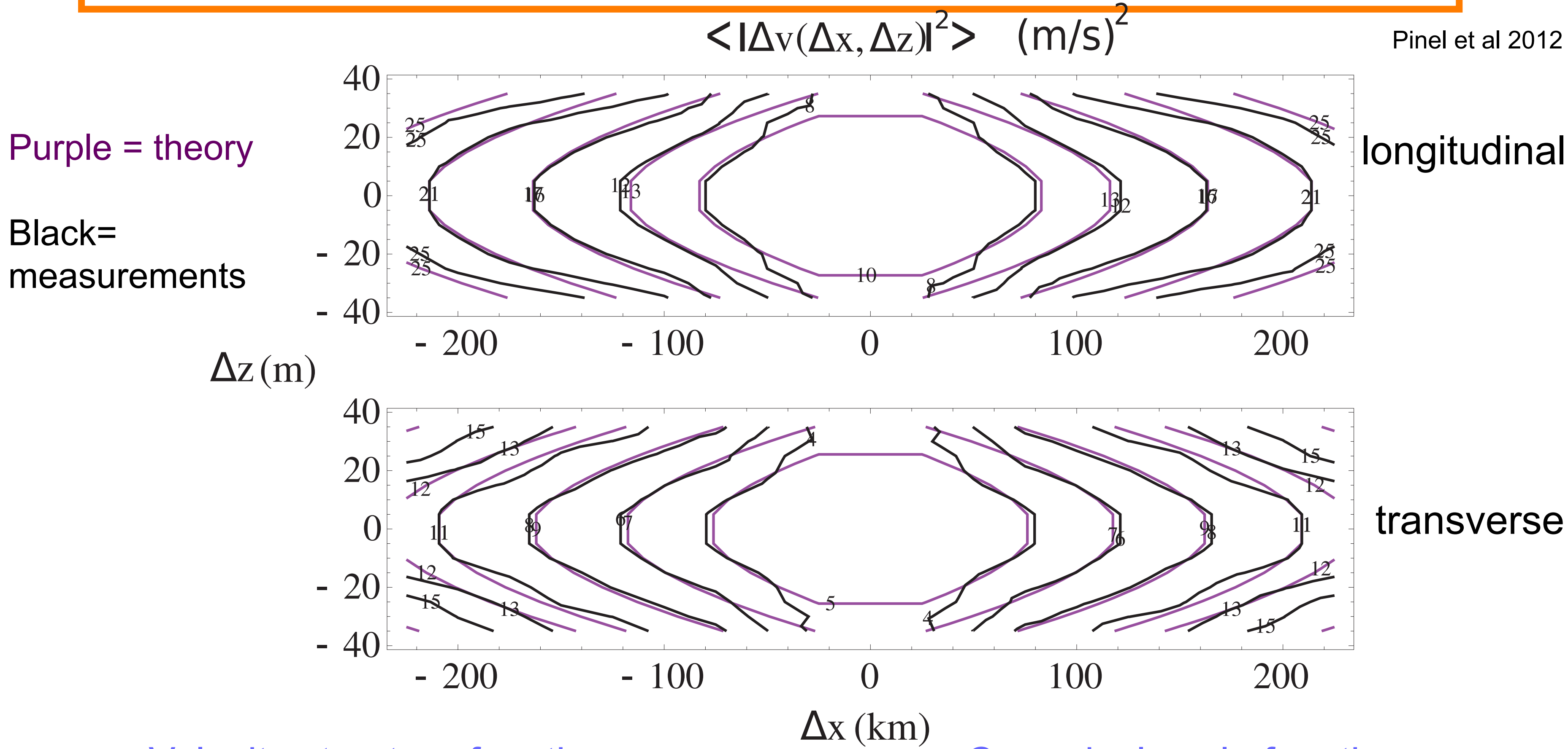
Isotropic function $H_z=1$



Anisotropic physical scale function $H_z=5/9$



14500 aircraft flights: 5-5.5km altitude, 2009, US (TAMDAR data)



Velocity structure function

$$\left\langle \Delta v^2 (\Delta x, \Delta z) \right\rangle = C \left\| (\Delta x, \Delta z) \right\|^{\xi(2)}$$

$$\xi(2) \approx 0.80$$

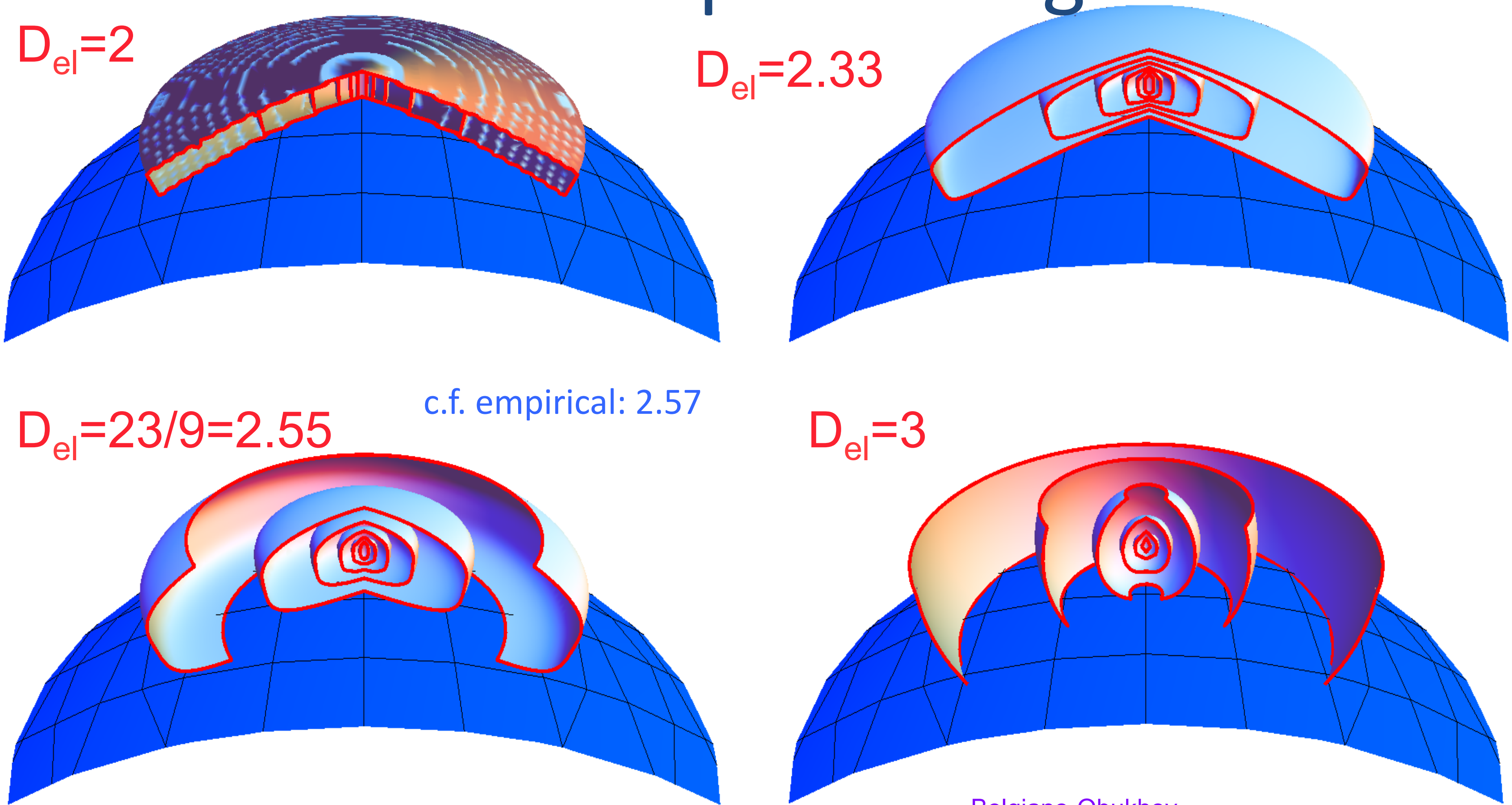
Canonical scale function

$$\left\| (\Delta x, \Delta z) \right\| = \left(\left(\frac{\Delta x}{l_s} \right)^2 + \left(\frac{\Delta z}{l_s} \right)^{2/H_z} \right)^{1/2}$$

$$H_z \approx 0.57 \pm 0.01$$

(Theory: 5/9=0.555…)

Anisotropic Scaling



The 23/9D model:

Bolgiano-Obukhov

$$\underbrace{\Delta v(\Delta x) = \varepsilon^{1/3}\Delta x^{1/3}}_{\text{Kolmogorov}}; \quad \overbrace{\Delta v(\Delta z) = \phi^{1/5}\Delta z^{3/5}}^{\text{Bolgiano-Obukhov}}$$

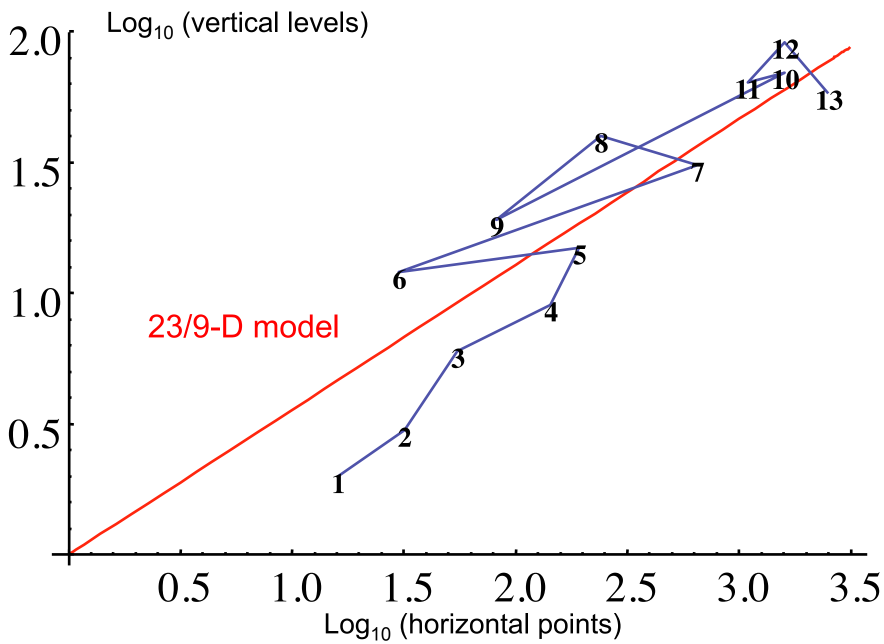
Kolmogorov

$H_z=(1/3)/(3/5)=5/9$

Volume $\approx L x L x L^{Hz} \approx L^{Del}$

$D_{el}=2+H_z=23/9$

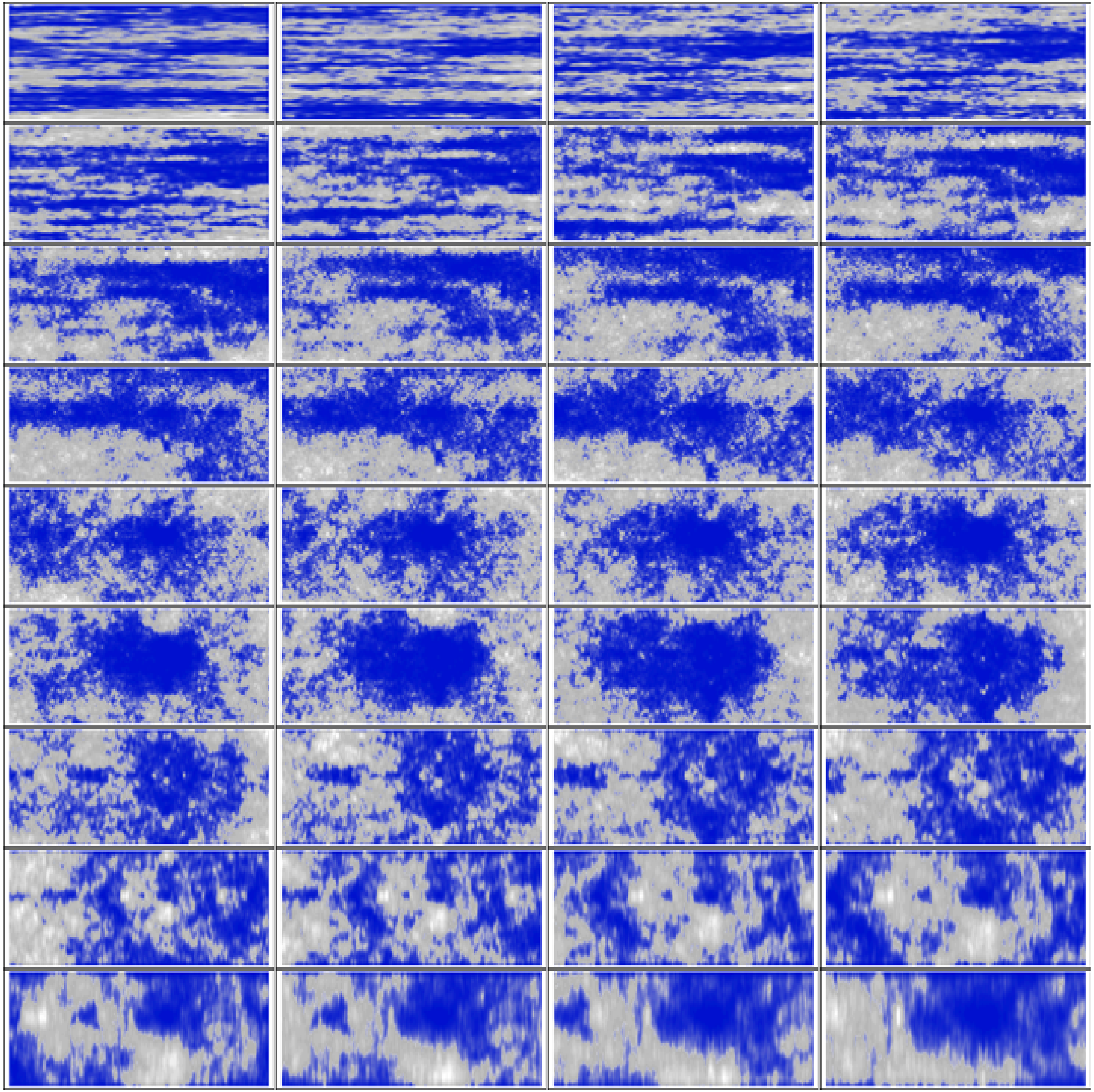
The Historical development of numerical atmospheric models and 23/9D dynamics



	Date (approx.)	name	Horizontal (points, rough)	Vertical (levels)	Comment
1	1956	Phillips	16x17	2	First simulation of long range general circulation
2	1958	Barotropic	30x34	3	Early Operational model
3	1966	Joint Numerical Weather Prediction Unit	53x57	6	First operational primitive equation model
4	1974	Global model	144	9	First global model
5	1979	ECMWF	192	15	First ECMWF model
6	1980	Global spectral model	30	12	First spectral model
7	1991	ECMWF	650	31	high res spectral model
8	2006-present	Weather Research and Forecasting	240 (typical high resolution)	40	Public domain
9	≈2000	HadCM3,	96x73	19	Met Office coupled Atmospehre-Ocean model
10	2010-present	Unified Model	1600	70	Met Office operational model
11	current	Global Forecasting System	1100 (high res mode)	64	NWS, operational model: 1988-present
12	current	Integrated Forecast System	1600	91	ECMWF operational model
13	current	Global Environmental Multiscale model	≈2500	58	Canadian Meteorological service operational model 1997-present

Fly by of anisotropic (multifractal, cascade) cloud

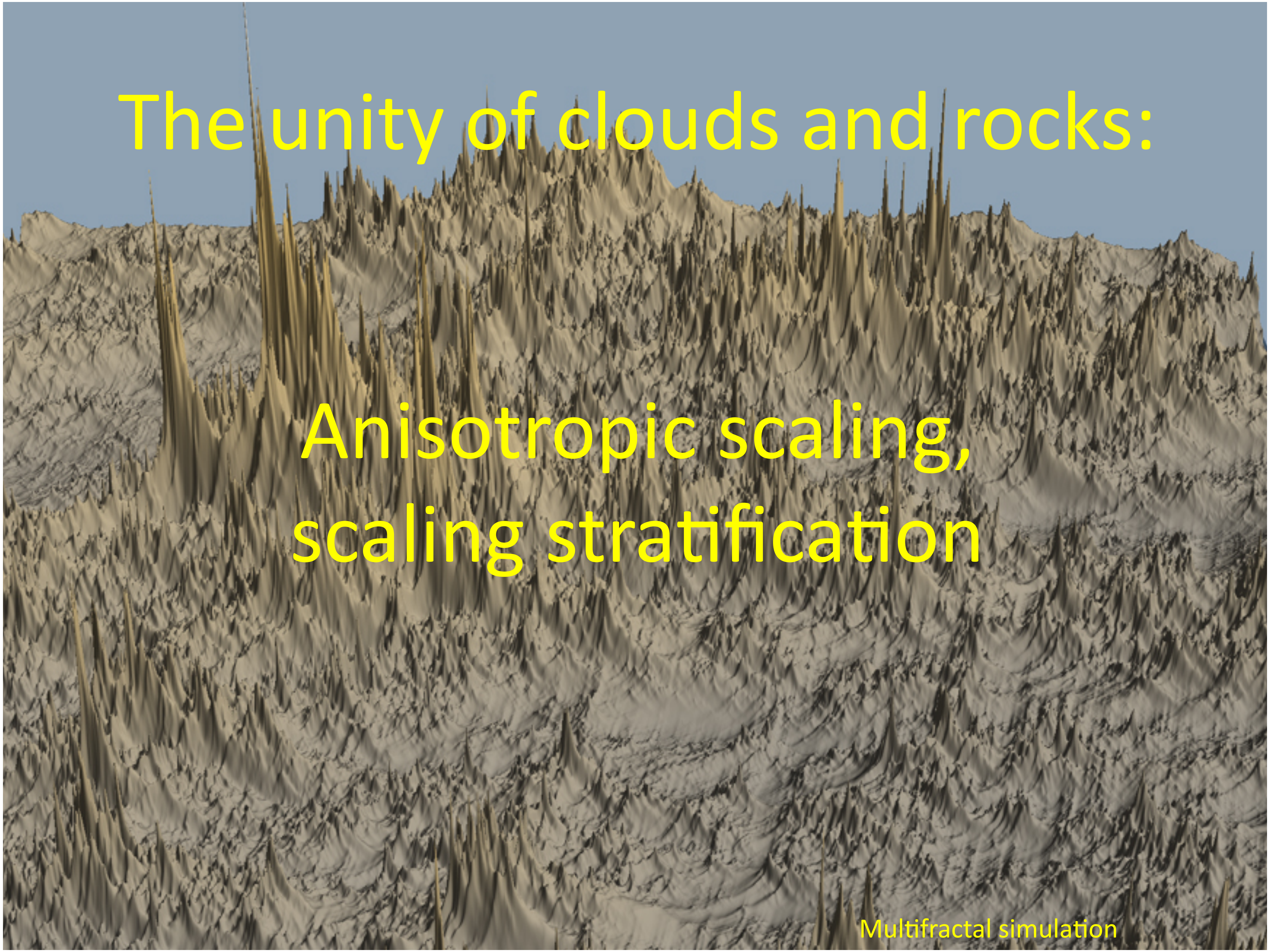
Zoom factor 1000



Vertical cross-section

The unity of clouds and rocks:

Anisotropic scaling, scaling stratification



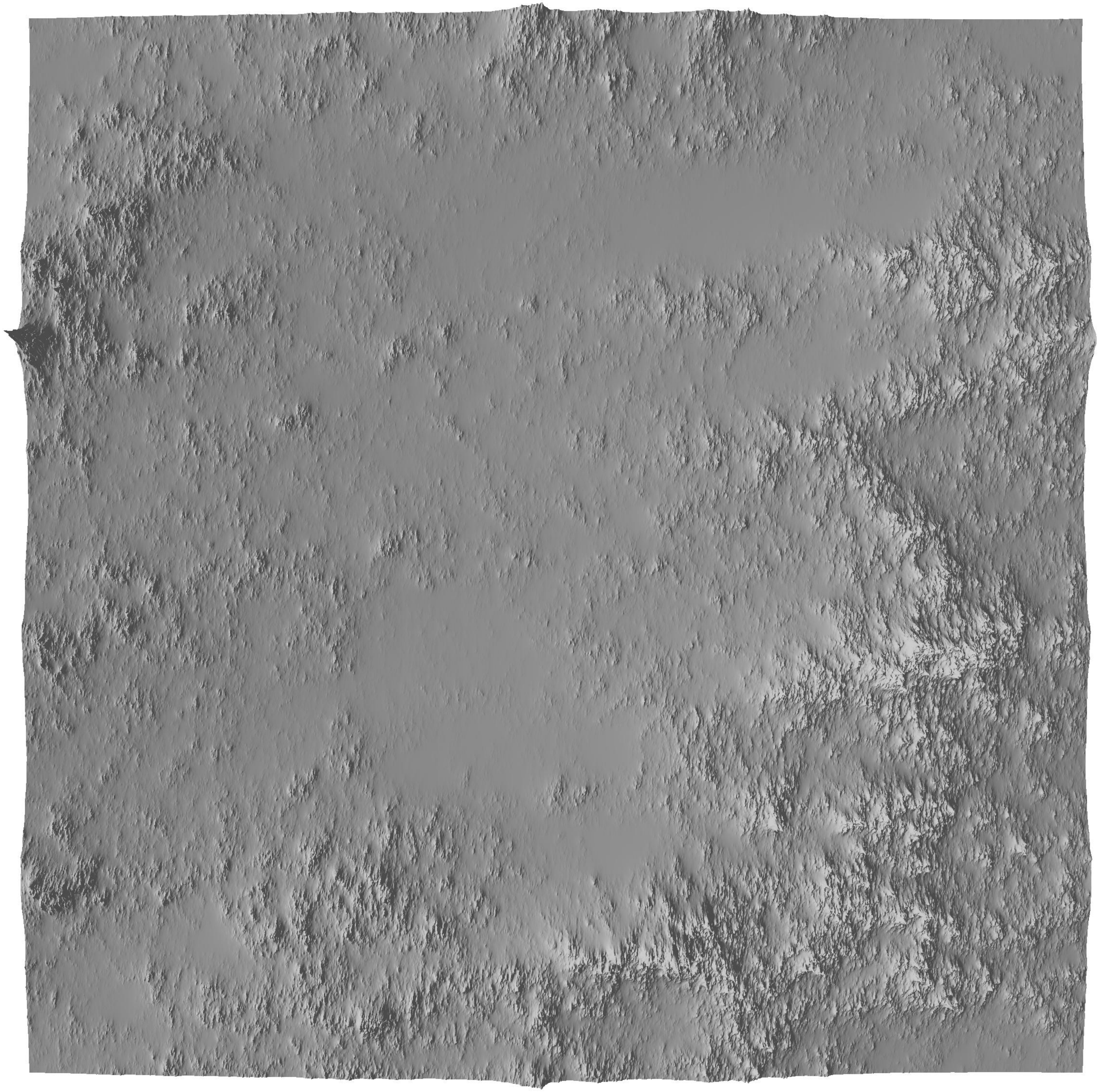
Flyby 1

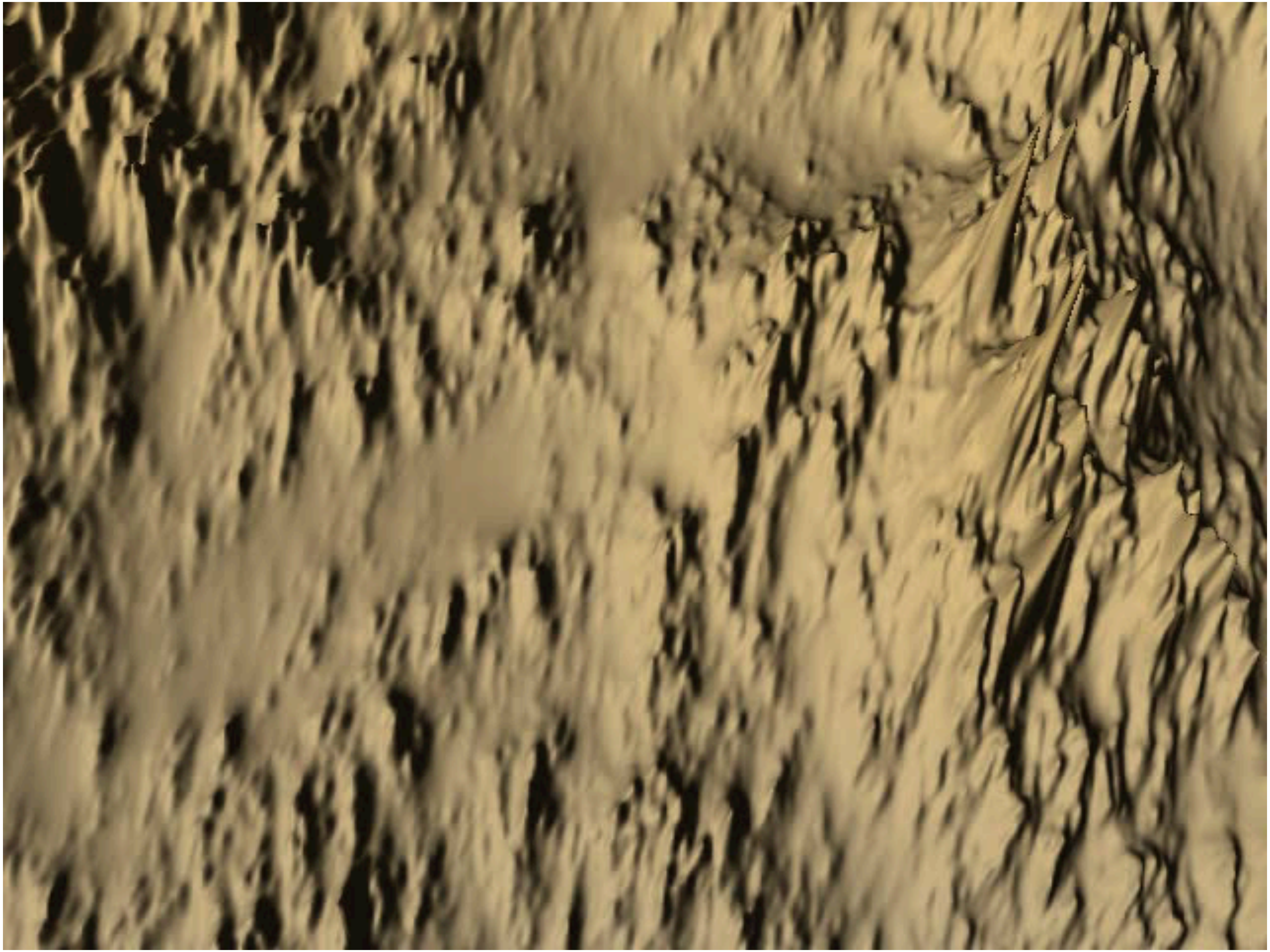
This 4096X4096 simulation is flown over

α=1.8, C_1=0.12, H=0.7

$$G = \begin{pmatrix} 0.65 & -0.1 \\ 0.1 & 1.35 \end{pmatrix}$$

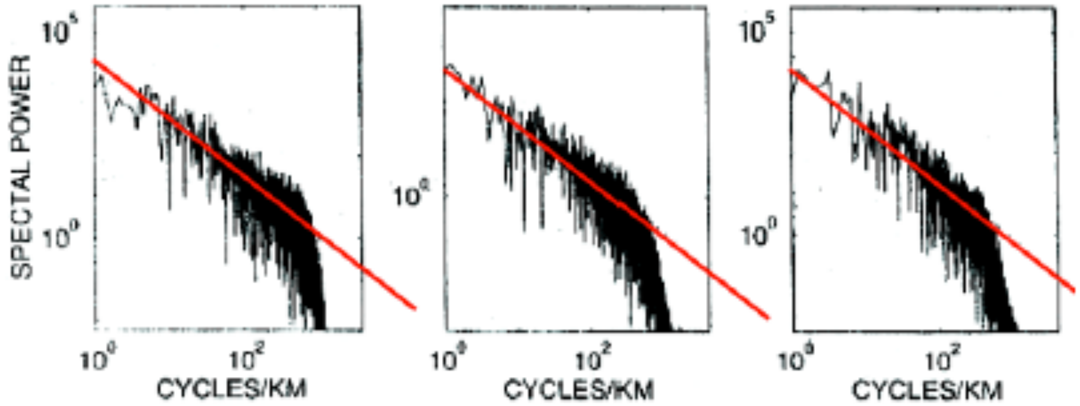
l_s=64 pixels





Horizontal versus vertical borehole rock densities

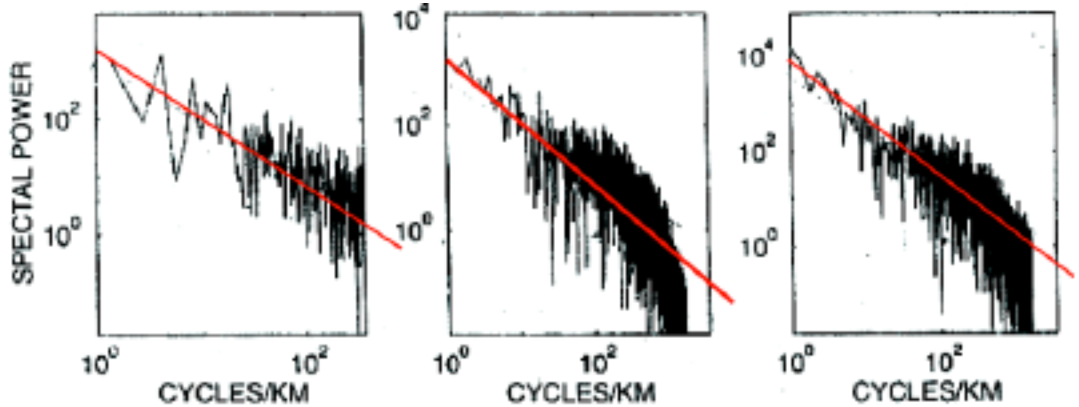
Gamma Density Seismic velocity



Horizontal boreholes (β_h =1.4)

$H_z = (\beta_h -1)/(\beta_v -1) = 2$

Gamma Density Seismic velocity



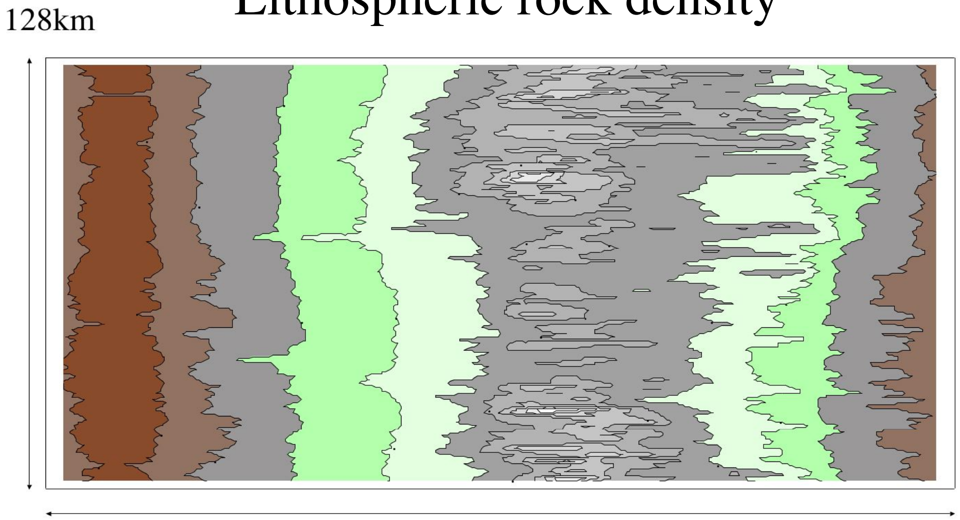
Vertical boreholes (β_v =1.2)

Lovejoy and Schertzer 2007 (adapted from Leary 1997)

Stratified Multifractal Crust, Mantle rock density simulation

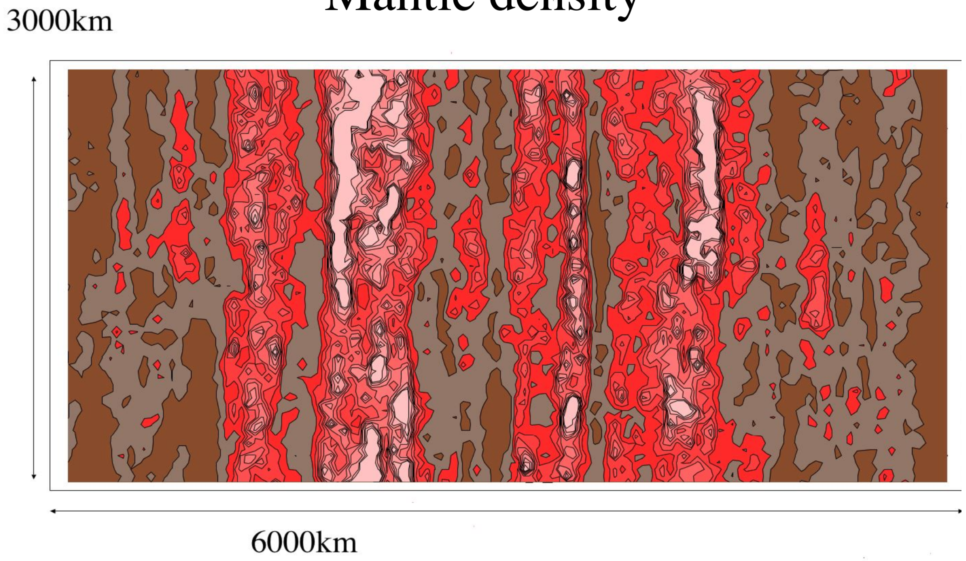
Vertical cross-sections $D_{el}=3$

Lithospheric rock density



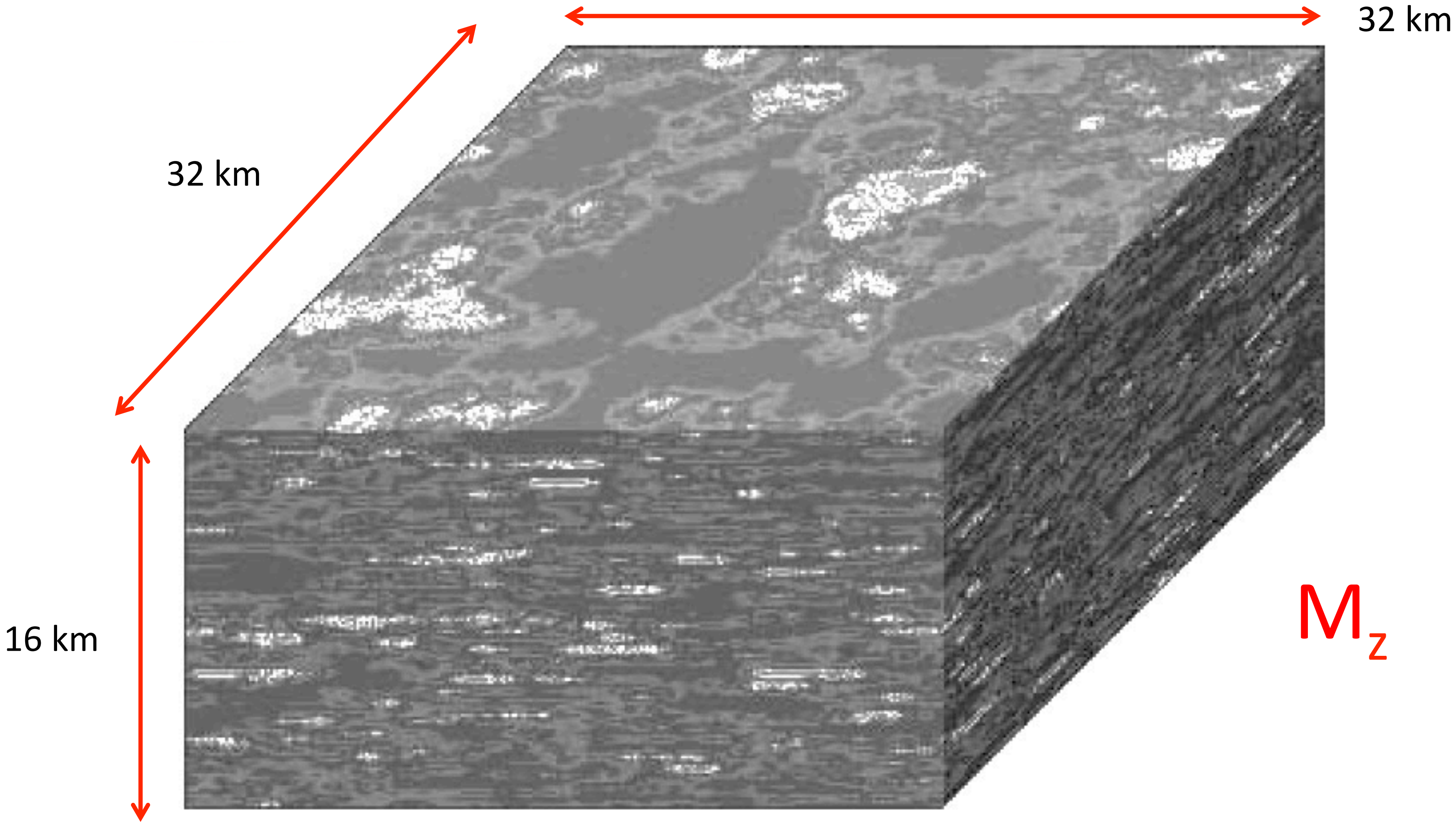
Sphero-scale l_s=256km, with 1 pixel = 1km.

Mantle density



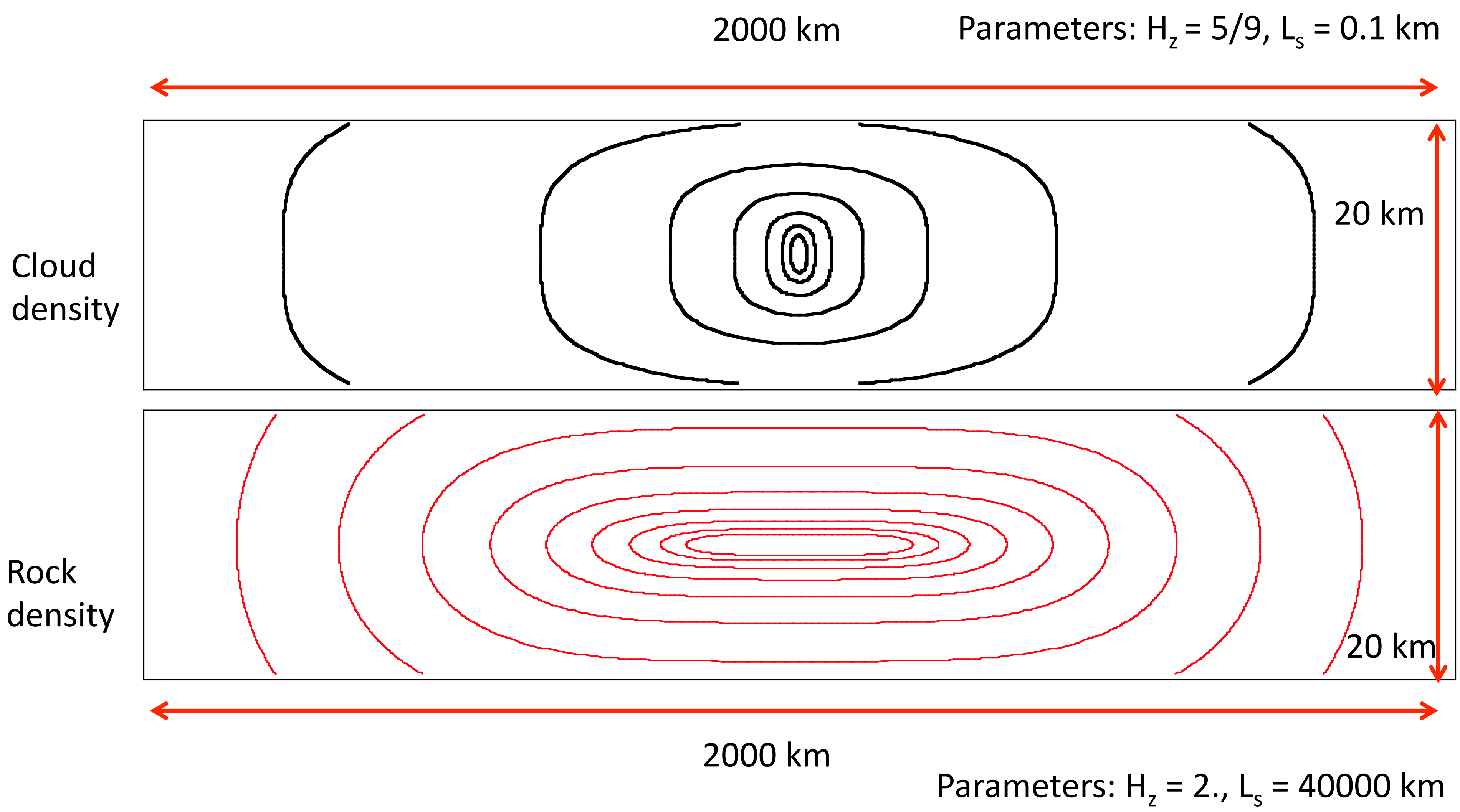
Sphero scale = 1 pixel. Each pixel is 50 km, sphero-scale = 25km. Hot (low density) plumes shown as white/red (this is a model for either density or temperature fluctuations (the two being proportional; we assume constant expansion coefficient). These are for fluctuations with respect to the mean vertical profile

Simulated magnetization field for horizontally isotropic crustal magnetization



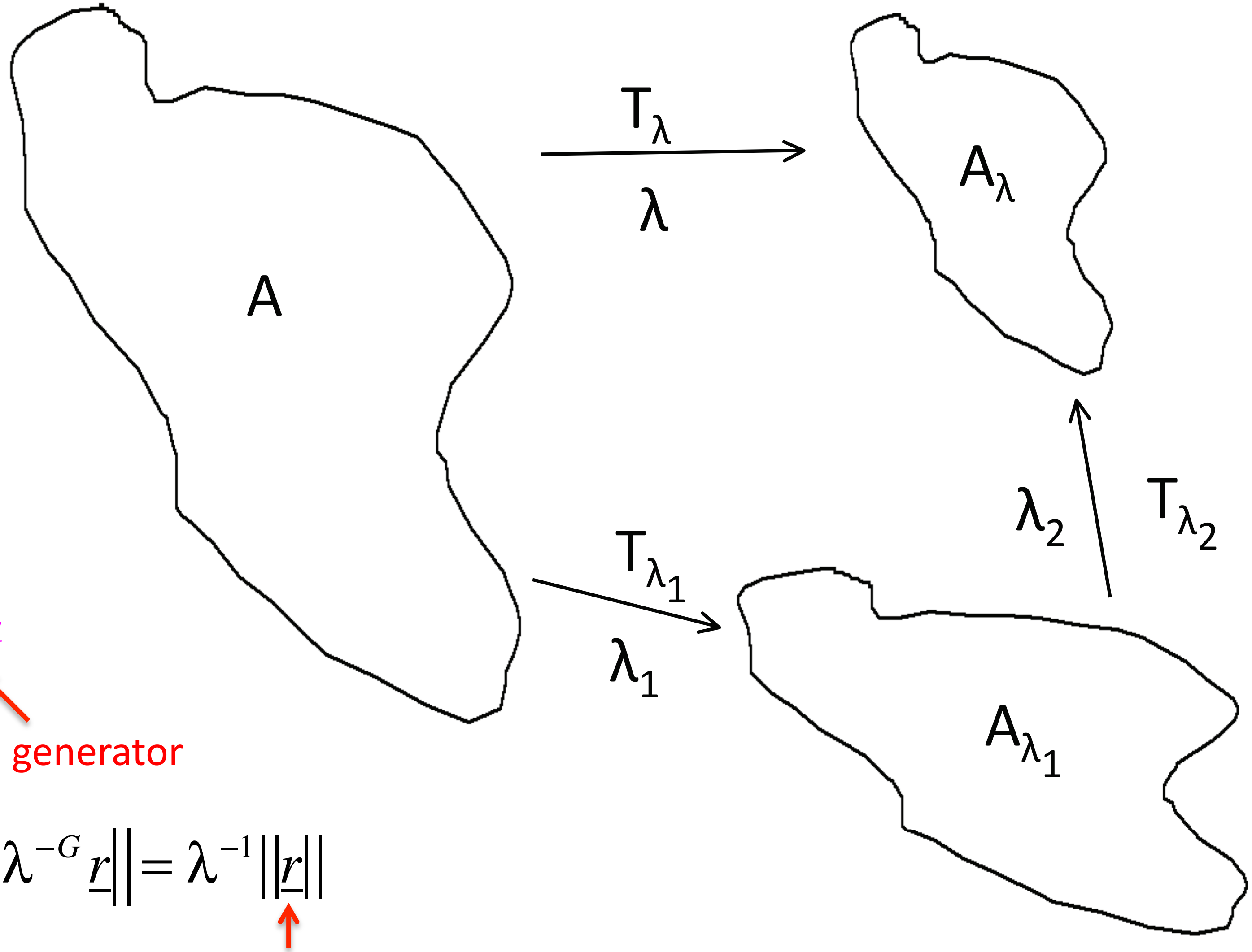
Parameters: are H_z =1.7, s = 4, H = 0.2, α =1.98, C_1 = 0.08, l_s = 2500 km,

The unity of geosciences: clouds and rocks



aspect ratios = 1/5

Generalized (anisotropic) scale invariance



Scale changing operator

$$T_\lambda = \lambda^{-G}$$

generator

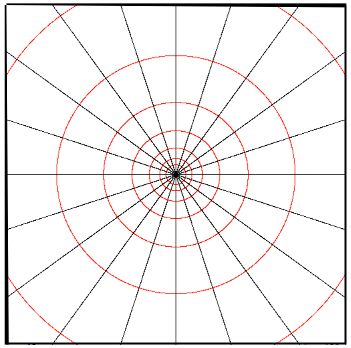
$$\left\|\lambda^{-G}\underline{r}\right\| = \lambda^{-1}\left\|\underline{r}\right\|$$

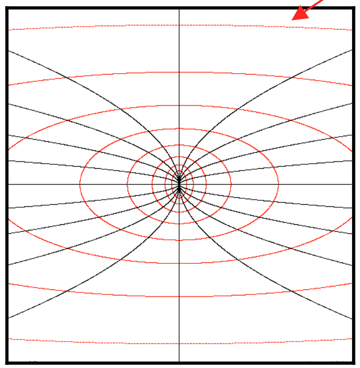
Scale function

Scale functions in linear GSI (position independent)

Scale isolines in red

Isotropic (self similar)





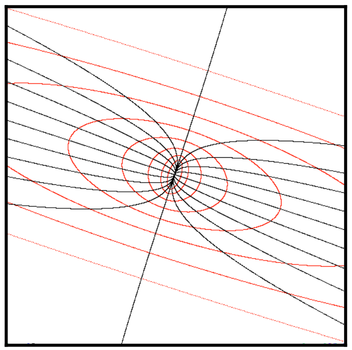
Self-affine

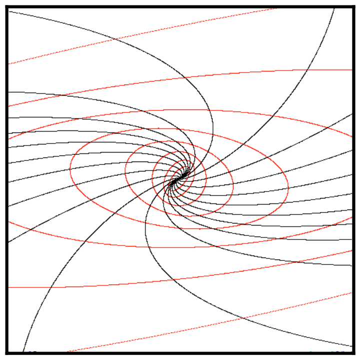
$$T_\lambda = \lambda^{-G}$$

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$G = \begin{pmatrix} 1.35 & 0 \\ 0 & 0.65 \end{pmatrix}$$

Stratification dominant (real eigenvalues)





Rotation dominant (complex eigenvalues)

$$G = \begin{pmatrix} 1.35 & 0.25 \\ 0.25 & 0.65 \end{pmatrix}$$

$$G = \begin{pmatrix} 1.35 & -0.45 \\ 0.85 & 0.65 \end{pmatrix}$$

A generalized blow-down with increasing of the acronym "NVAG". If $G = I$, we would have obtained a standard reduction, with all the copies uniformly reduced converging to the centre of the reduction. Here the parameters determining G are:

$$G = \begin{pmatrix} 1.3 & 1.3 \\ 0.3 & 0.7 \end{pmatrix}$$

and each successive reduction is by 28%.

Generalized (anisotropic) scale invariance

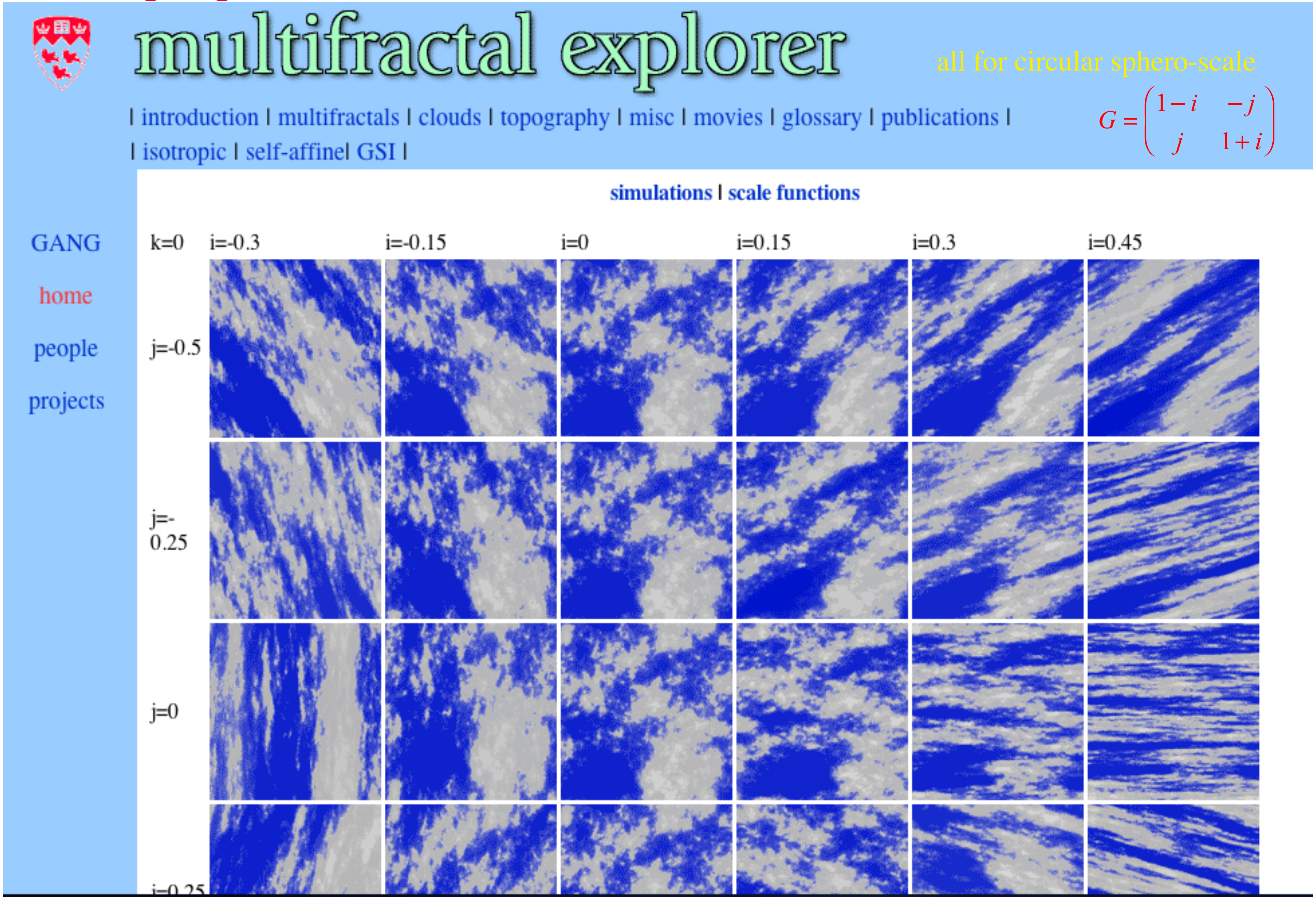
Overall

Isotropy $\longrightarrow$ **anisotropy**

$$\left|\underline{x}\right| \rightarrow \left\|\underline{x}\right\|; \quad D \rightarrow D_{el}$$

Changing G

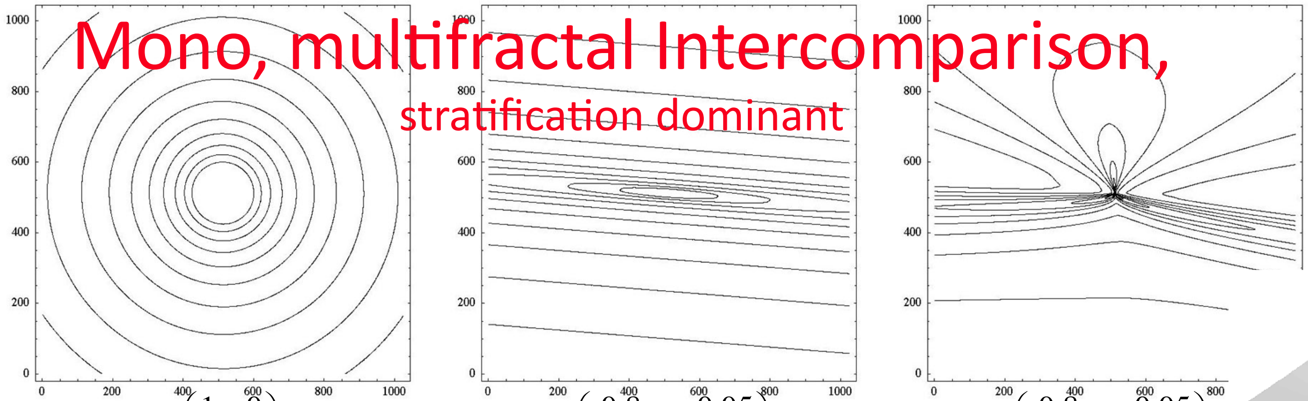
http://www.physics.mcgill.ca/~gang/multifrac/index.htm



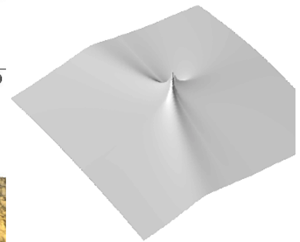
Mono, multifractal Intercomparison,

stratification dominant

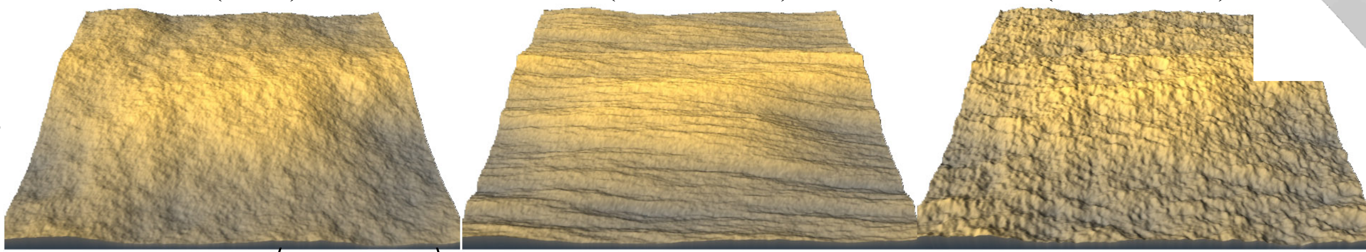
Contours of the s
functions



$$G=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad G=\begin{pmatrix}0.8 & -0.05\\ 0.05 & 1.2\end{pmatrix} \qquad G=\begin{pmatrix}0.8 & -0.05\\ 0.05 & 1.2\end{pmatrix}$$

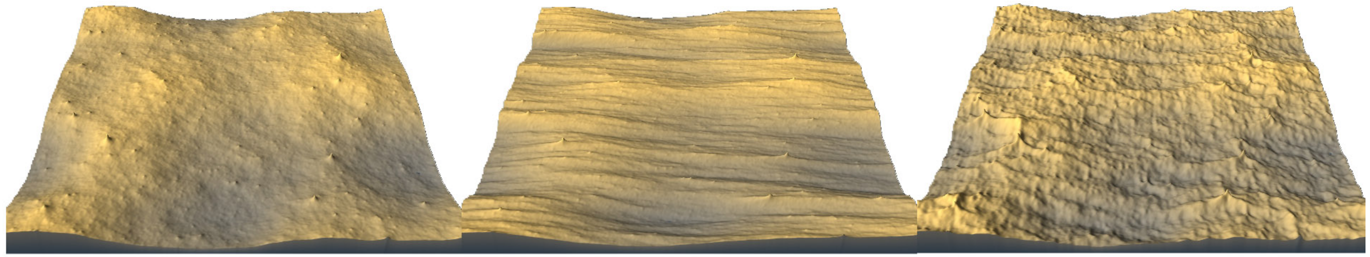


Fractional
Brownian motion,
H=0.7



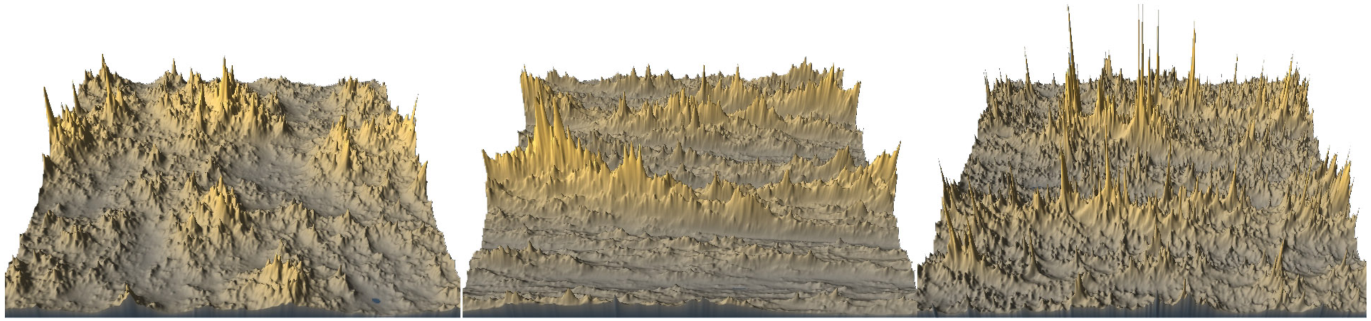
$$\langle \Delta h(\Delta r)^q \rangle \approx \Delta r^{qH-K(q)} \qquad K(q)=0$$

Fractional Levy
motion,
H=0.7, α =1.8



Multifractal FIF
H=0.7, α =1.8,
C_1=0.12

$$K(q)=\frac{C_1}{\alpha-1}\left(q^\alpha - q\right)$$

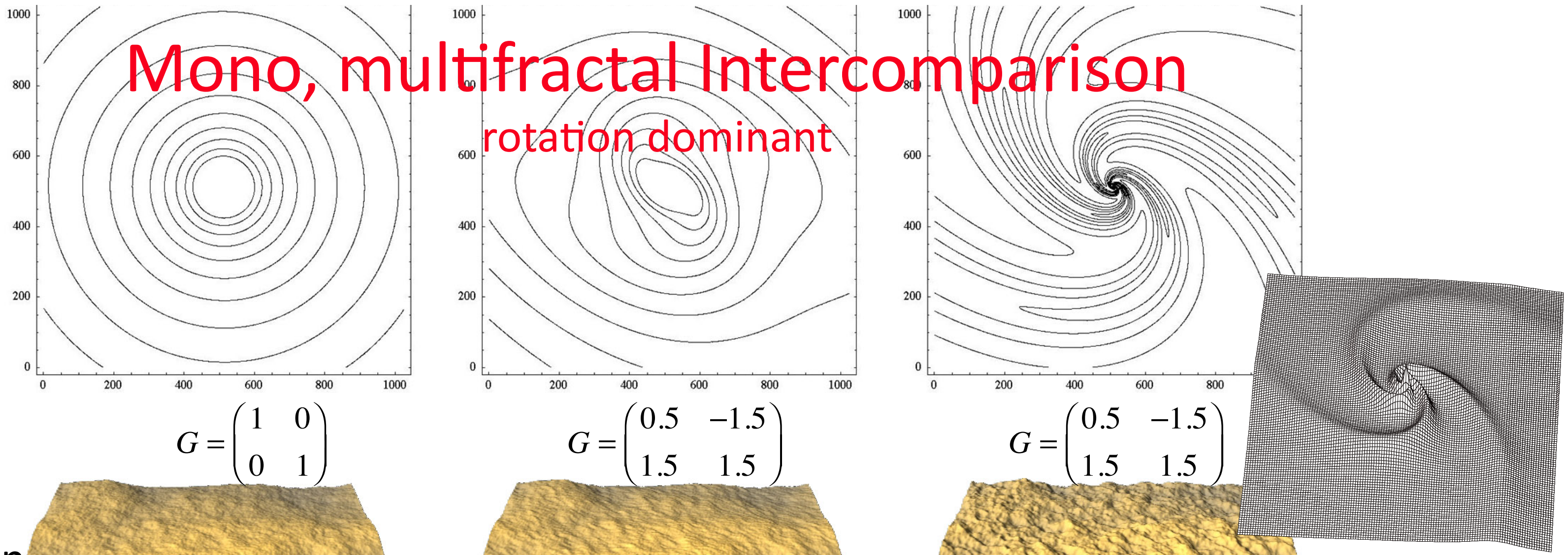


isotropic — Anisotropic no trivial anisotropy — Anisotropic with trivial anisotropy

Mono, multifractal Intercomparison

rotation dominant

Contours of the scale functions

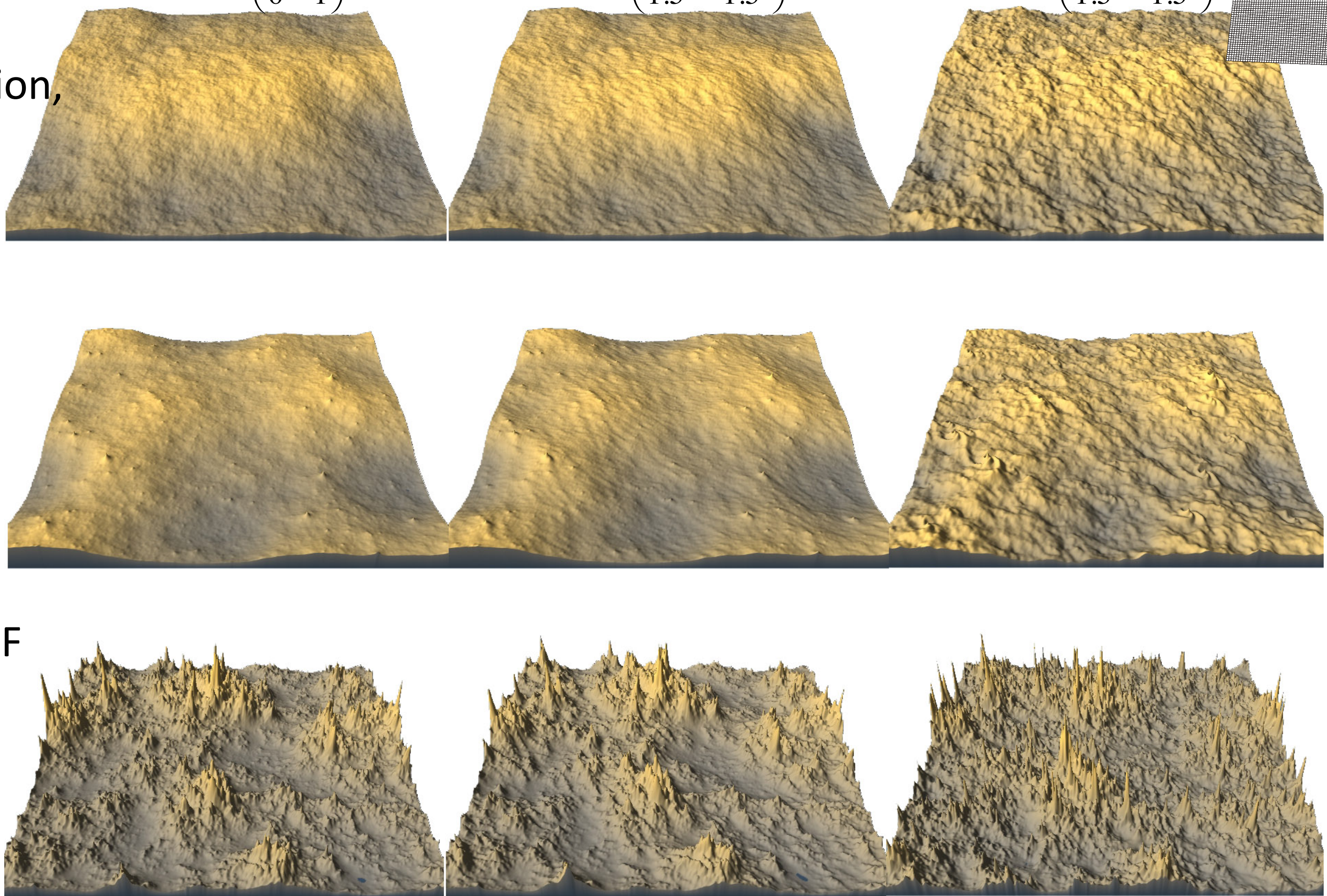


$$G=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad G=\begin{pmatrix}0.5 & -1.5\\ 1.5 & 1.5\end{pmatrix} \qquad G=\begin{pmatrix}0.5 & -1.5\\ 1.5 & 1.5\end{pmatrix}$$

Fractional Brownian motion, H=0.7

Fractional Levy motion, H=0.7, α=1.8

Multifractal, FIF H=0.7, α =1.8, C_1=0.12



isotropic

Anisotropic no trivial anisotropy

Anisotropic with trivial anisotropy

Effect of varying C_1, H (self-affine, l_s=1)

C_1=0.05

All:

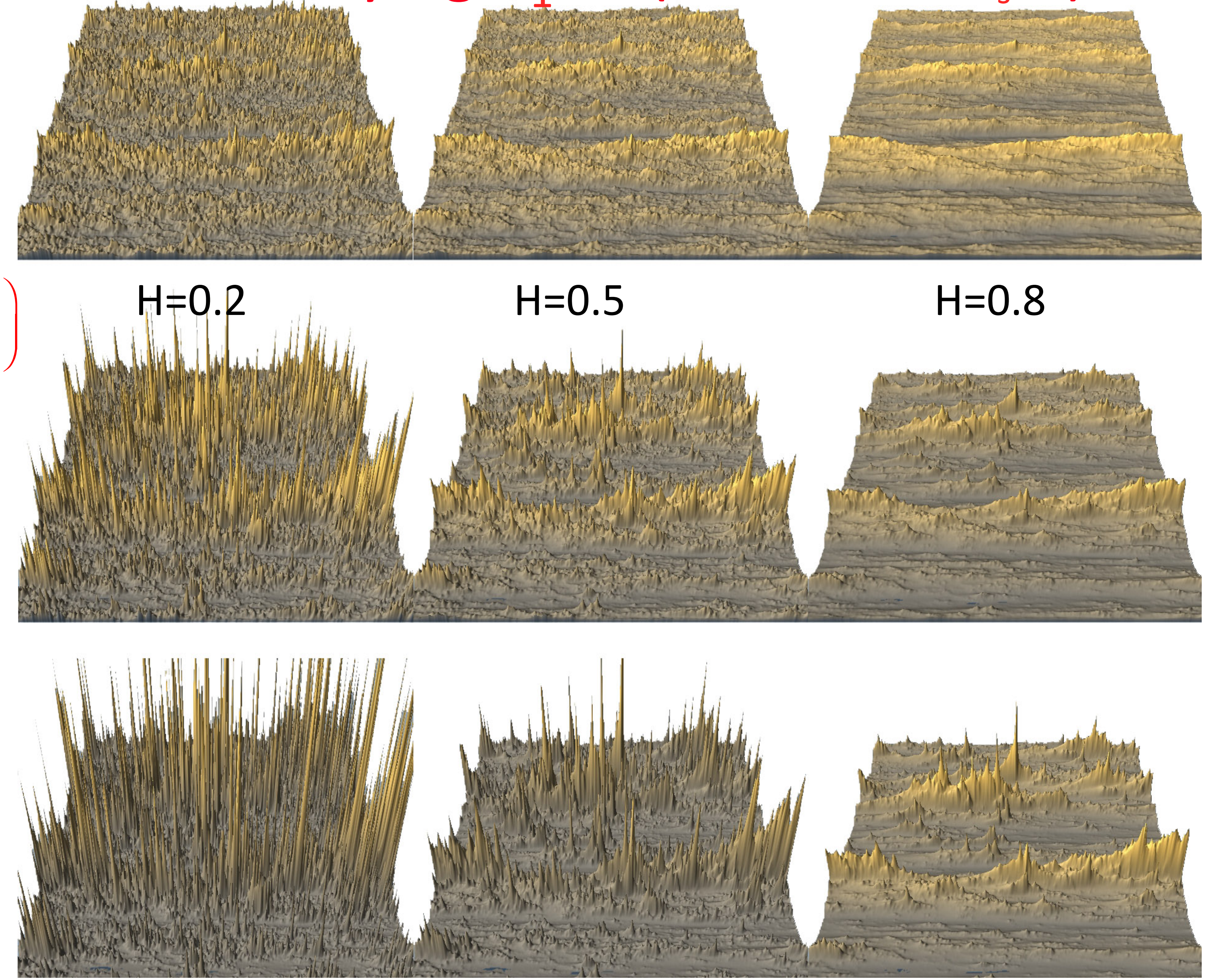
α=1.8

$$G = \begin{pmatrix} 0.8 & 0 \\ 0 & 1.2 \end{pmatrix}$$

H=0.2 H=0.5 H=0.8

C_1=0.15

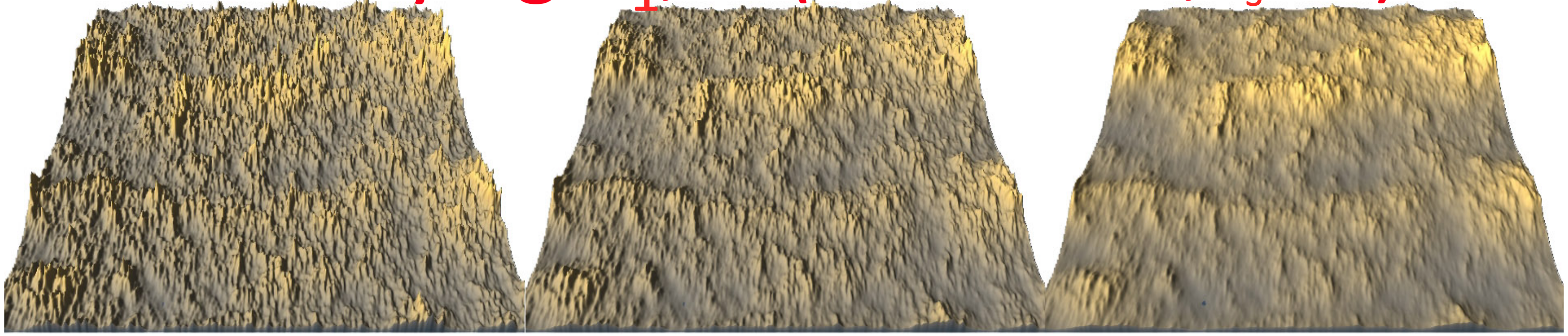
C_1=0.25



Effect of varying C_1, H (self-affine, l_s=64)

C_1=0.05

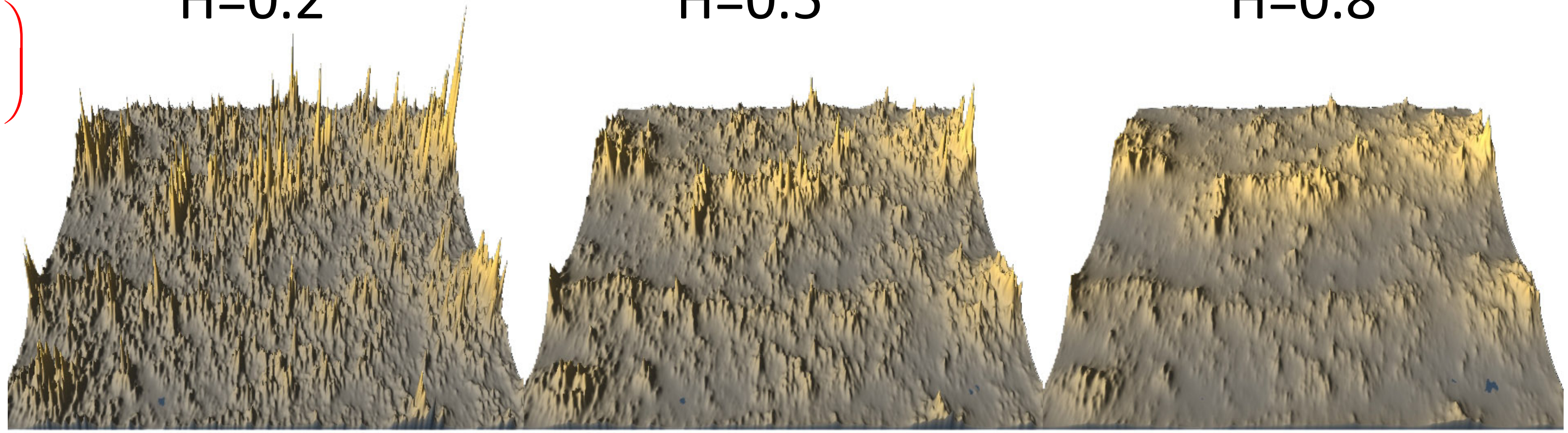
All:
α=1.8



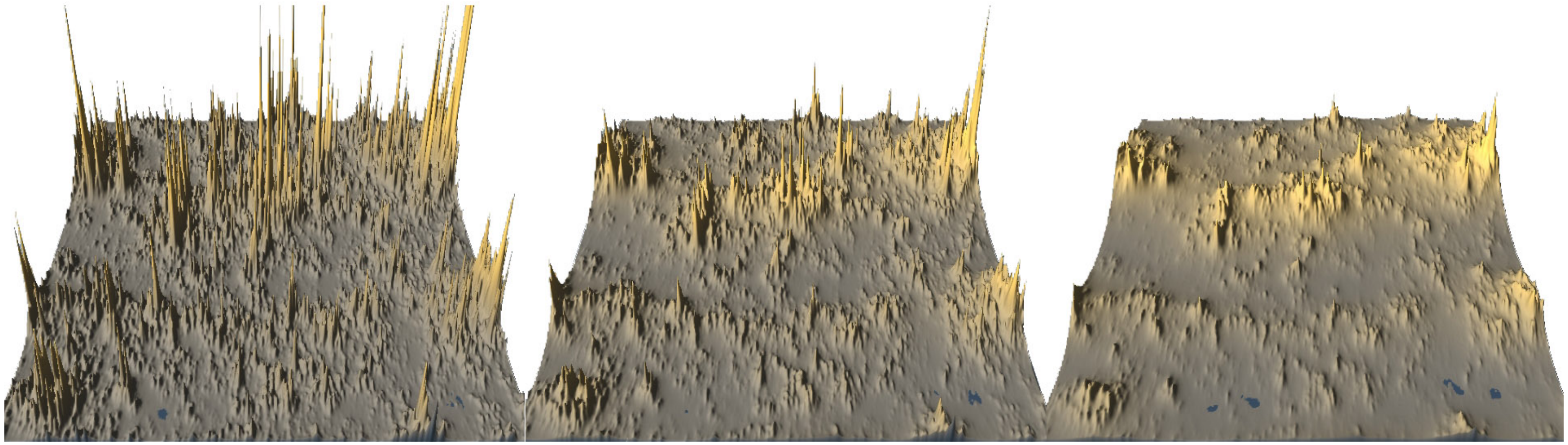
H=0.2 H=0.5 H=0.8

$$G = \begin{pmatrix} 0.8 & 0 \\ 0 & 1.2 \end{pmatrix}$$

C_1=0.15



C_1=0.25



Monofractal sets

↓

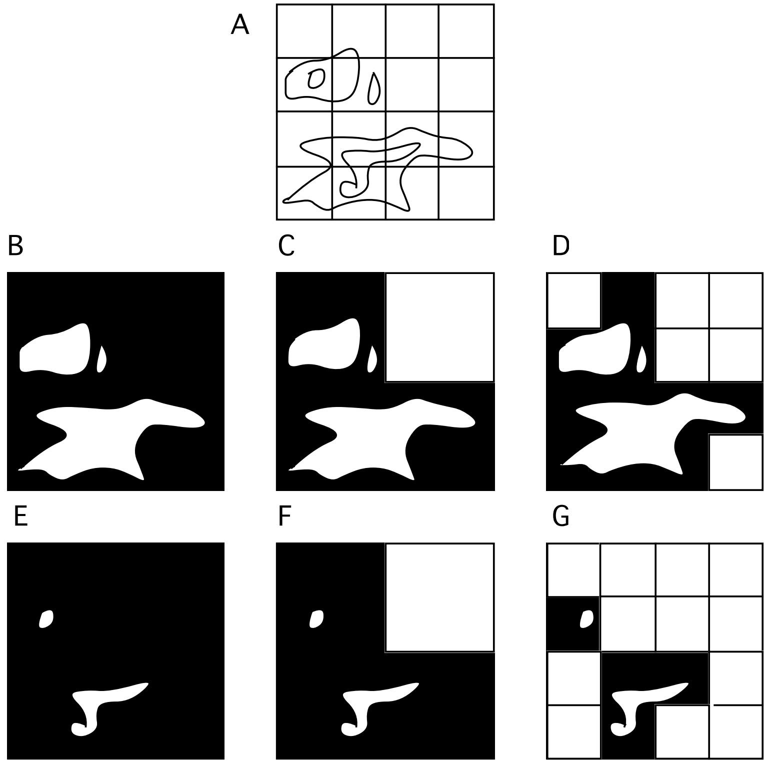
(singular) multifractal fields....

Multifractality and Functional Box Counting

$$N_T(L) \approx L^{-D(T)}$$

Box counting low threshold (large D)

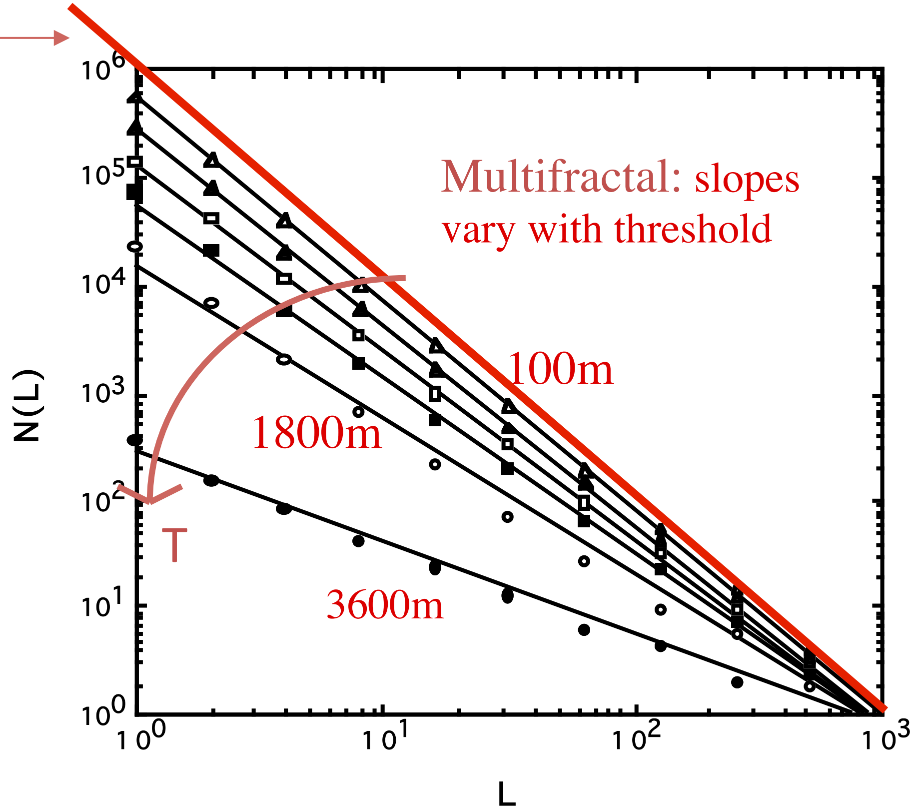
Box counting high threshold (low D)



-Monofractal: D(T) <2 , constant
-Multifractal: D(T)<2, decreasing

Functional box counting on French topography: 1 -1000km

Slope =2
(required for classical geostatistics - regularity of Lebesgue measures)



$N_T(L) \approx L^{-D(T)}$

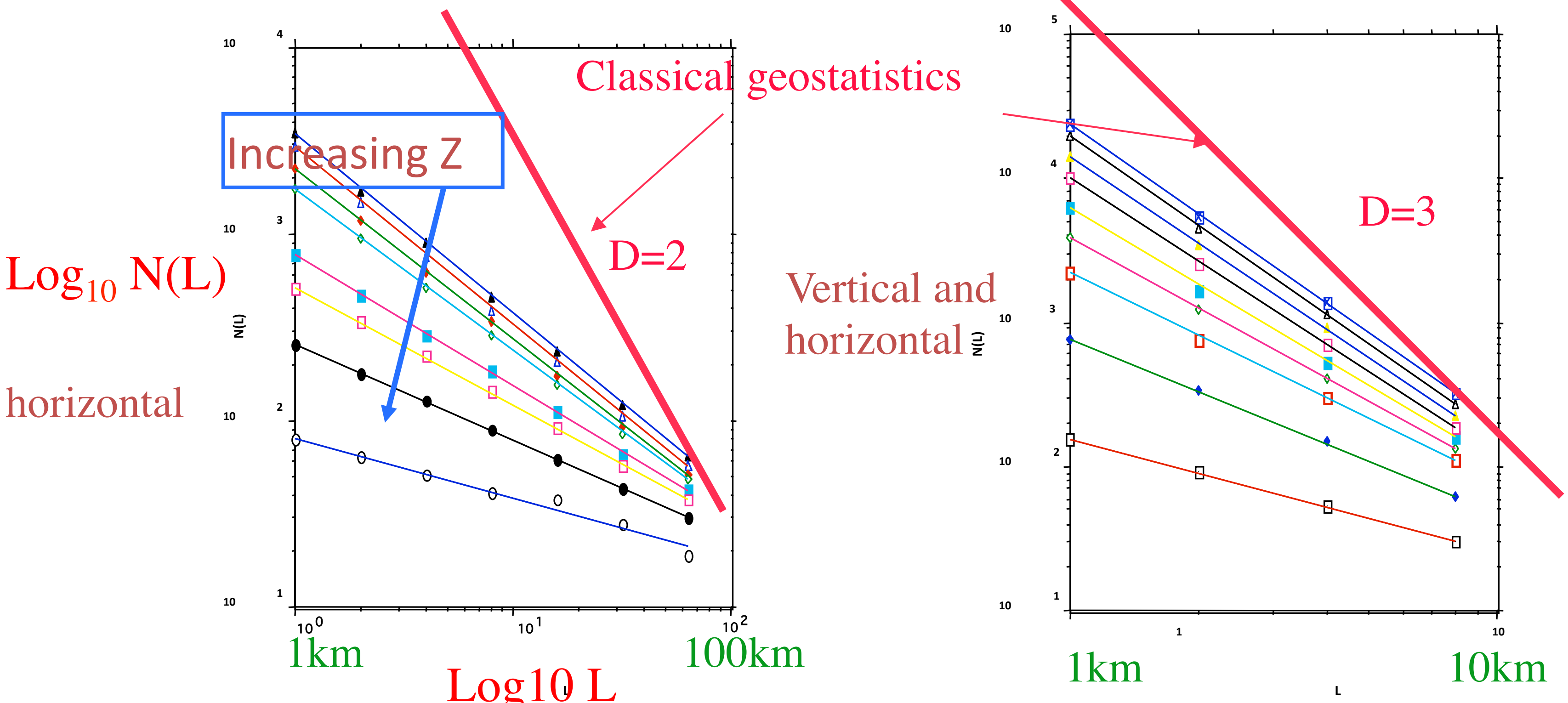
Systematic resolution dependence

km

N(L) = number of covering boxes for exceedance sets at various altitudes.
The dimensions d increase from 0.84 (3600m) to 1.92 (at 100m).

Lovejoy and Schertzer 1990

Functional Box counting on 3D radar rain scans



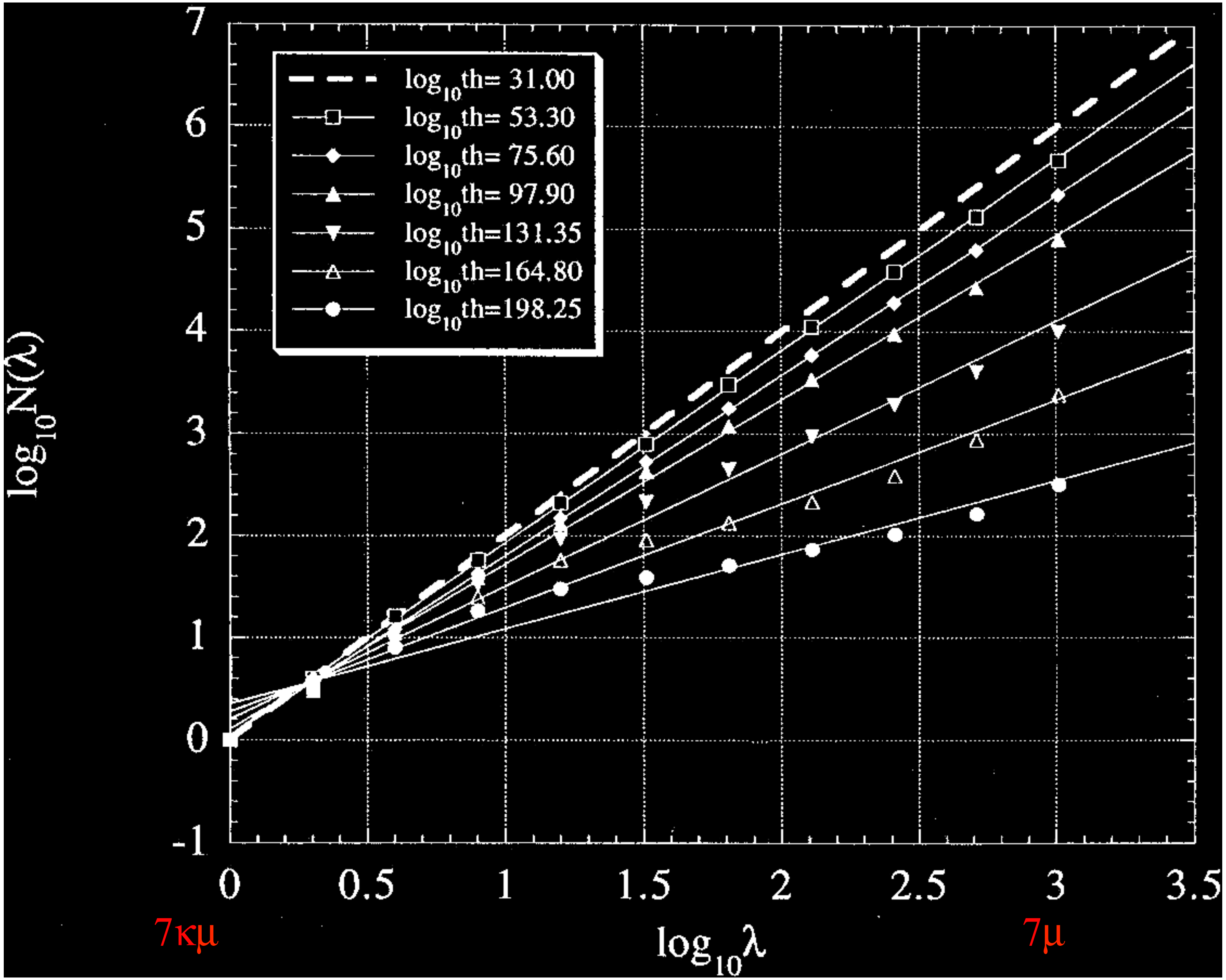
$$N_T(L) \approx L^{-D(T)}$$

reflectivity thresholds increasing (top to bottom) by factors of 2.5 (dat from Montreal).

Science: Lovejoy, Schertzer and Tsonis 1987

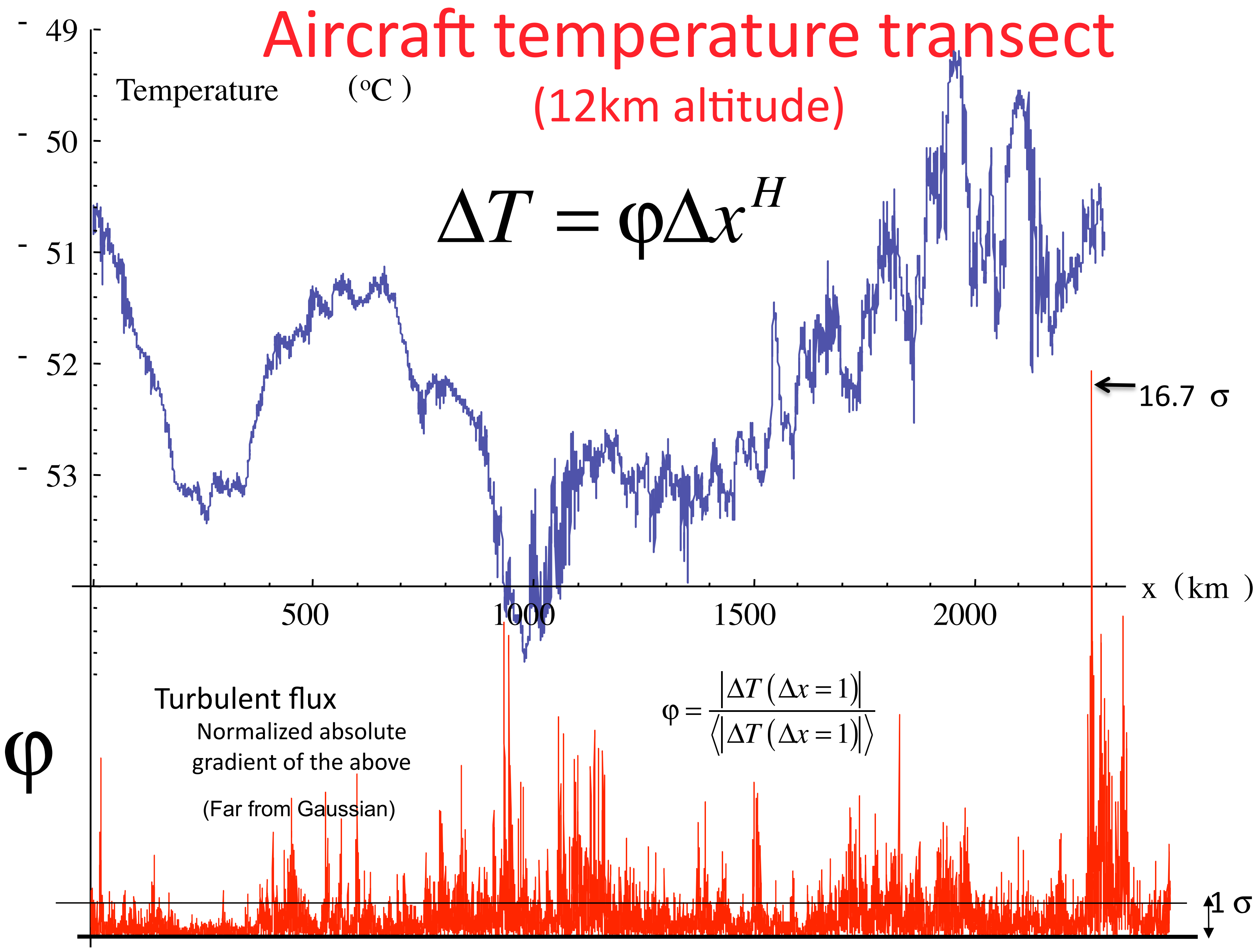
Functional box counting of ocean colour data

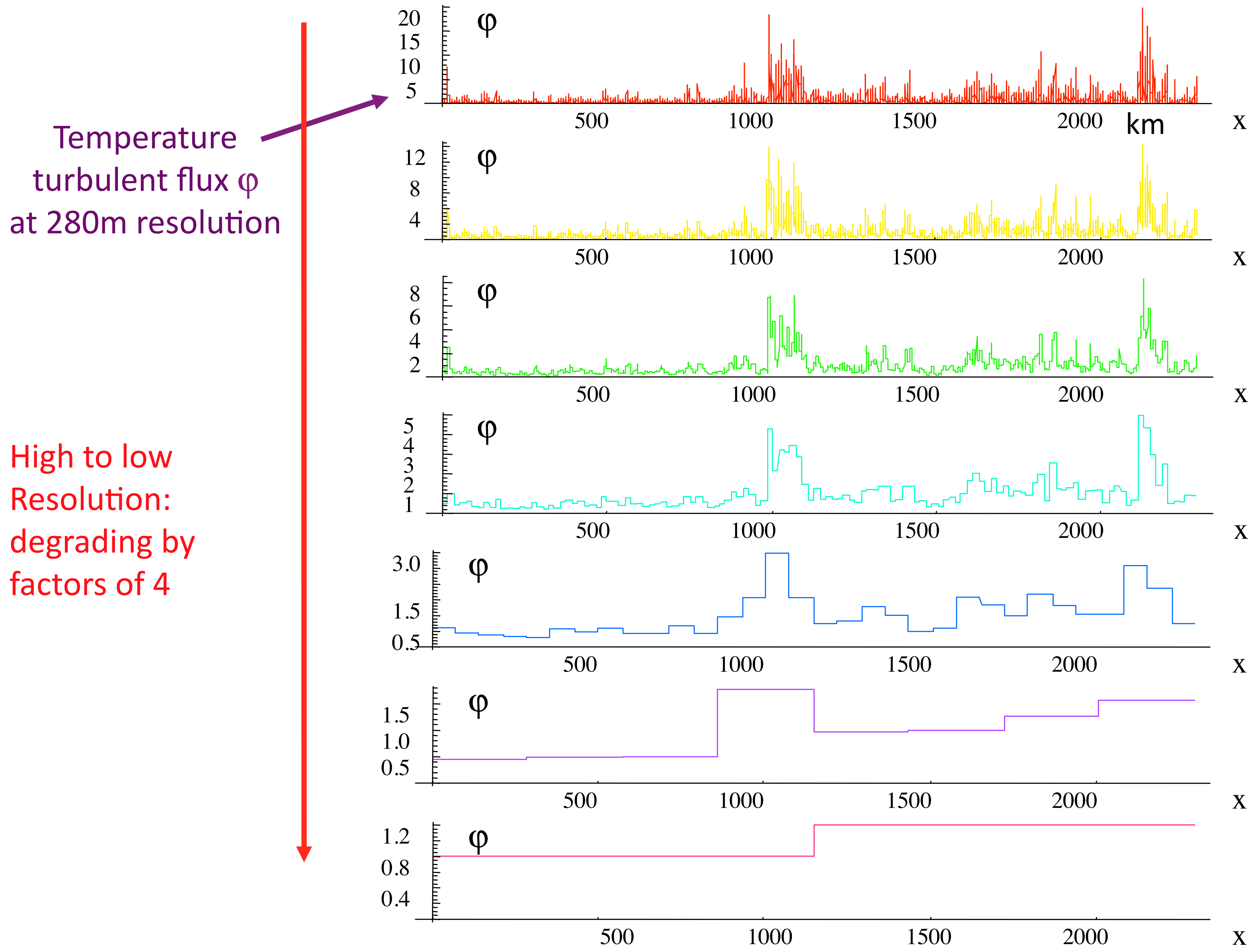
$$N(\lambda) \approx \lambda^{D_T}; \quad \lambda = L_0 / L$$



Lovejoy et al 2001

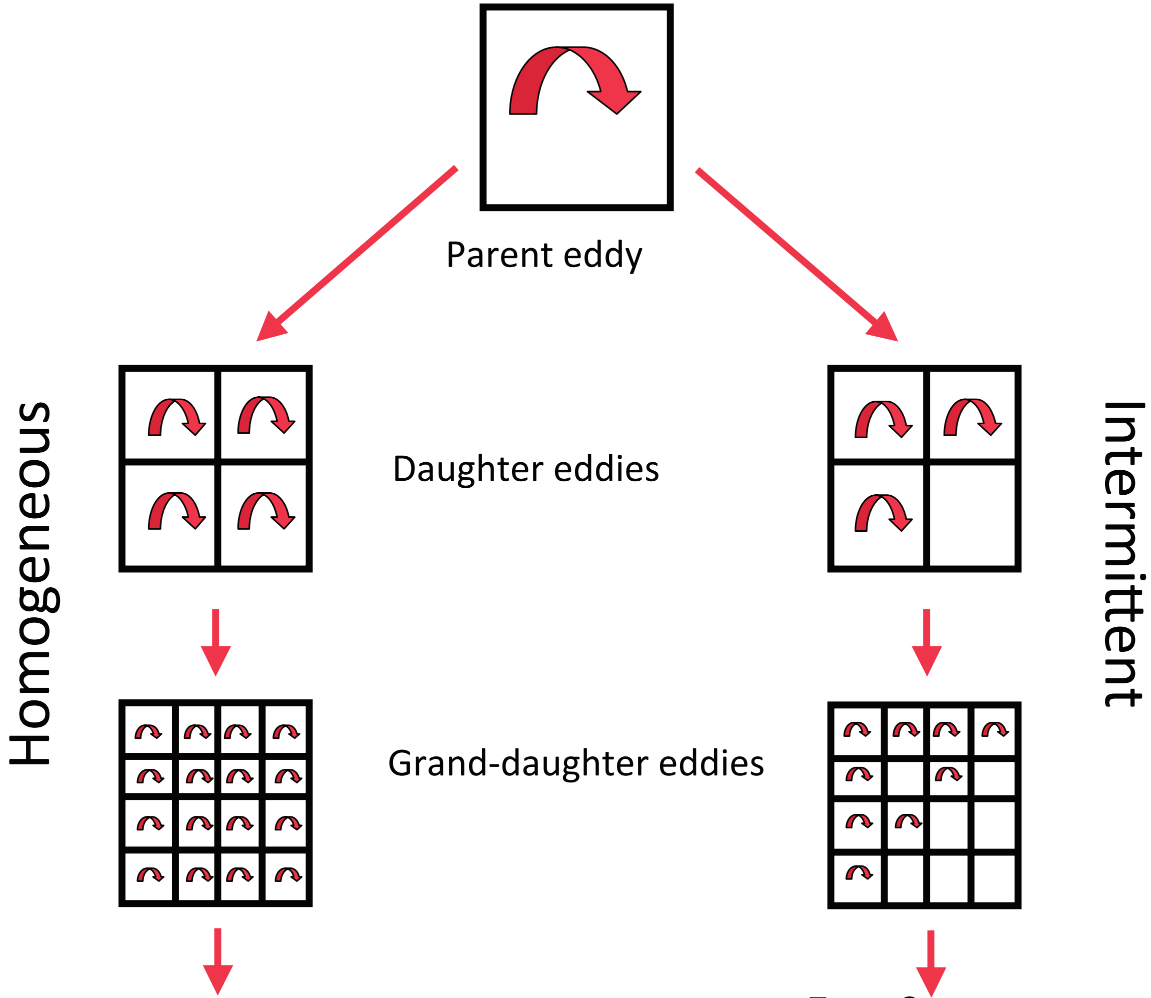
Cascades and Multifractals


Aircraft temperature transect
(12km altitude)
Temperature (°C)
49
50
51
52
53
$\Delta T = \varphi \Delta x^H$
16.7 σ
x (km)
500
1000
1500
2000
Turbulent flux
Normalized absolute
gradient of the above
(Far from Gaussian)
$\varphi = \frac{|\Delta T(\Delta x = 1)|}{\langle |\Delta T(\Delta x = 1)| \rangle}$
φ
1 σ



Temperature
turbulent flux φ
at 280m resolution
High to low
Resolution:
degrading by
factors of 4
φ
km
x
500
1000
1500
2000

Cascades



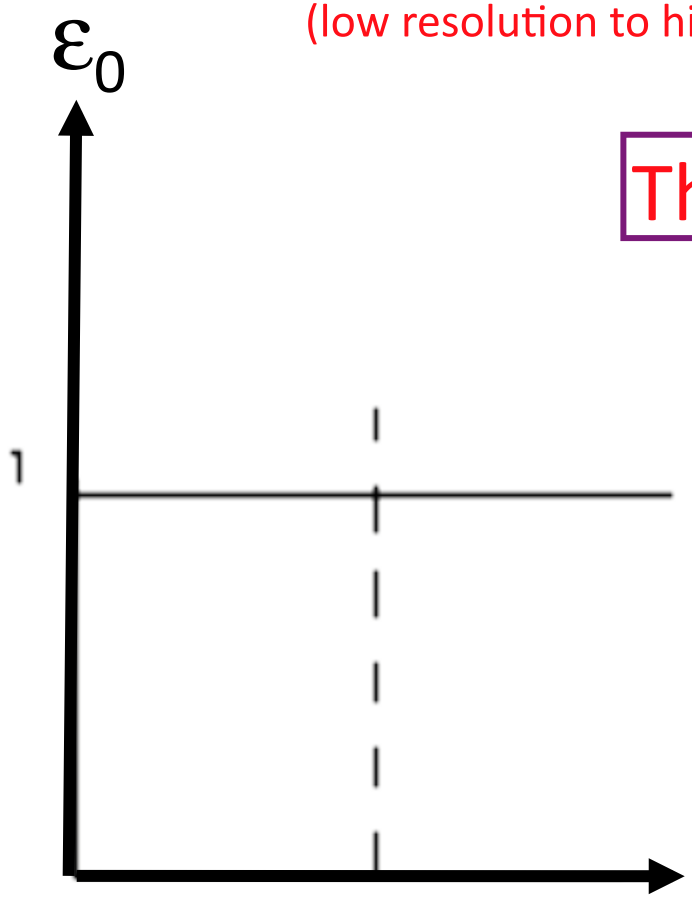
$$D = \frac{Log 4}{Log 2} = 2$$

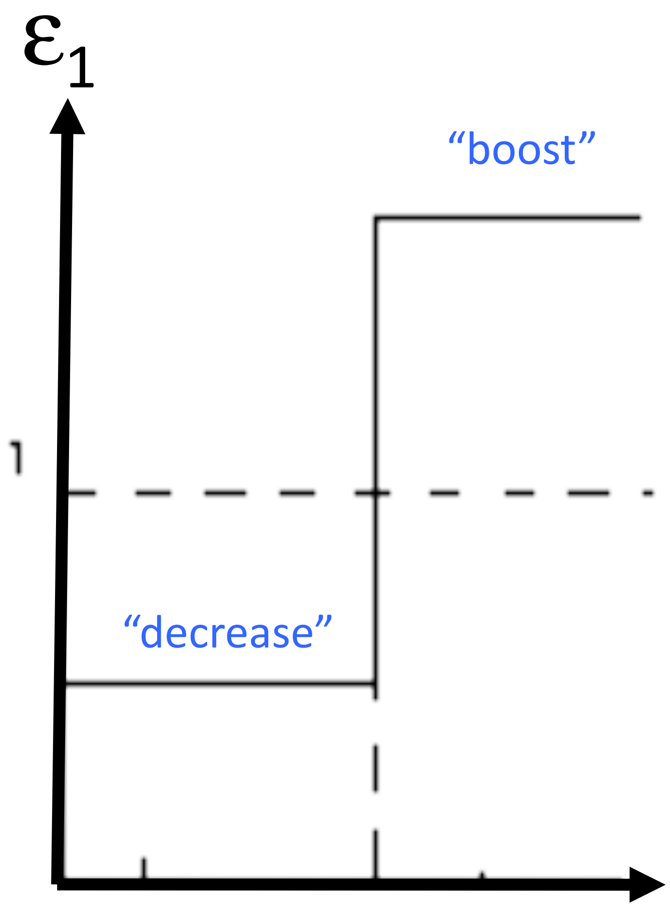
$$D = \frac{Log 3}{Log 2} = 1.58...$$

Cascades and Multifractals

Simulations: **multiplicative** introduction of small scale details (low resolution to high)

The α model





S+L 1983

Multiplicative Cascades

Generic statistical behaviour:

scaling

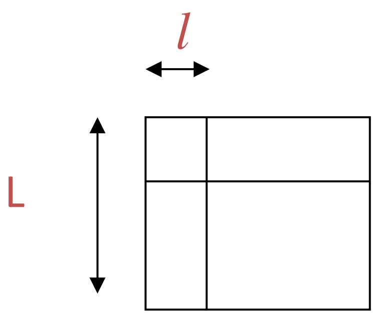
Scale invariant

Turbulent flux

$$\left\langle \varepsilon_\lambda^{q} \right\rangle \approx \lambda^{K(q)}$$

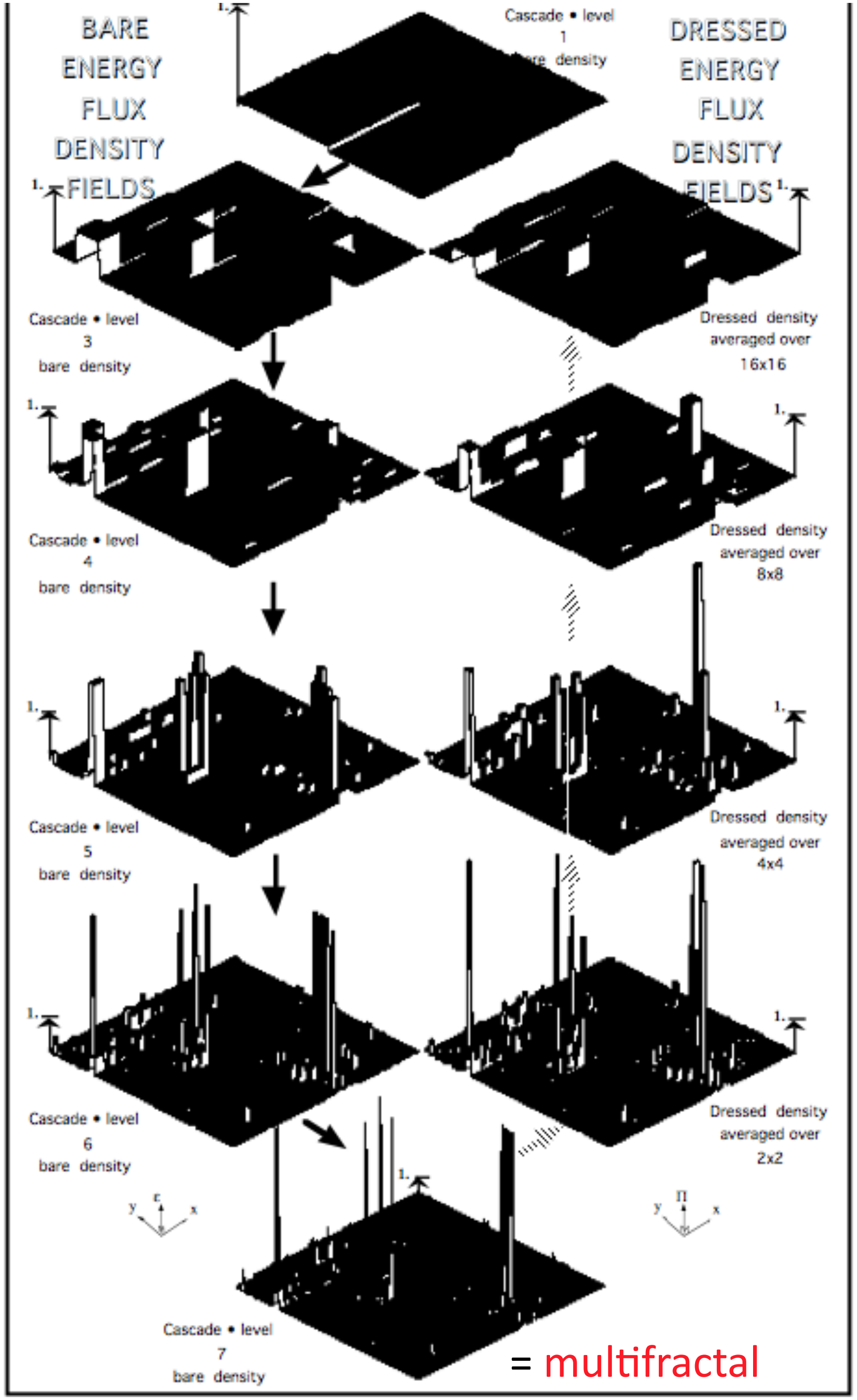
Statistical averaging

Resolution: ratio $\lambda = L/l$



Probabilities:

$$\Pr\left(\varepsilon_\lambda > \lambda^{\gamma}\right) \approx \lambda^{-c(\gamma)}$$



Early evidence of cascades: **Precipitation 1987**

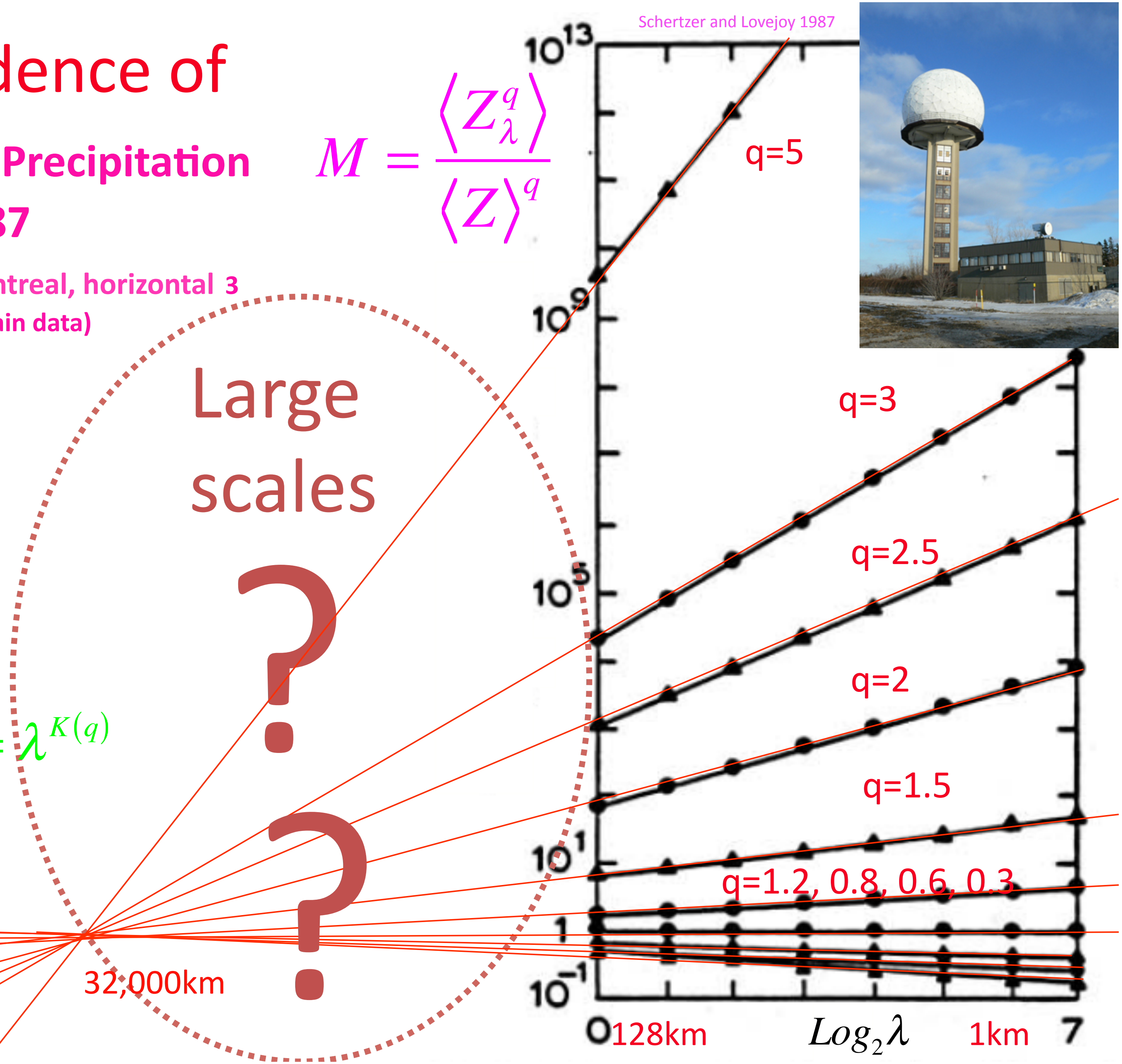
(70 Radar Scans, Montreal, horizontal 3 weeks of rain data)

$$M = \frac{\langle Z_\lambda^q \rangle}{\langle Z \rangle^q}$$

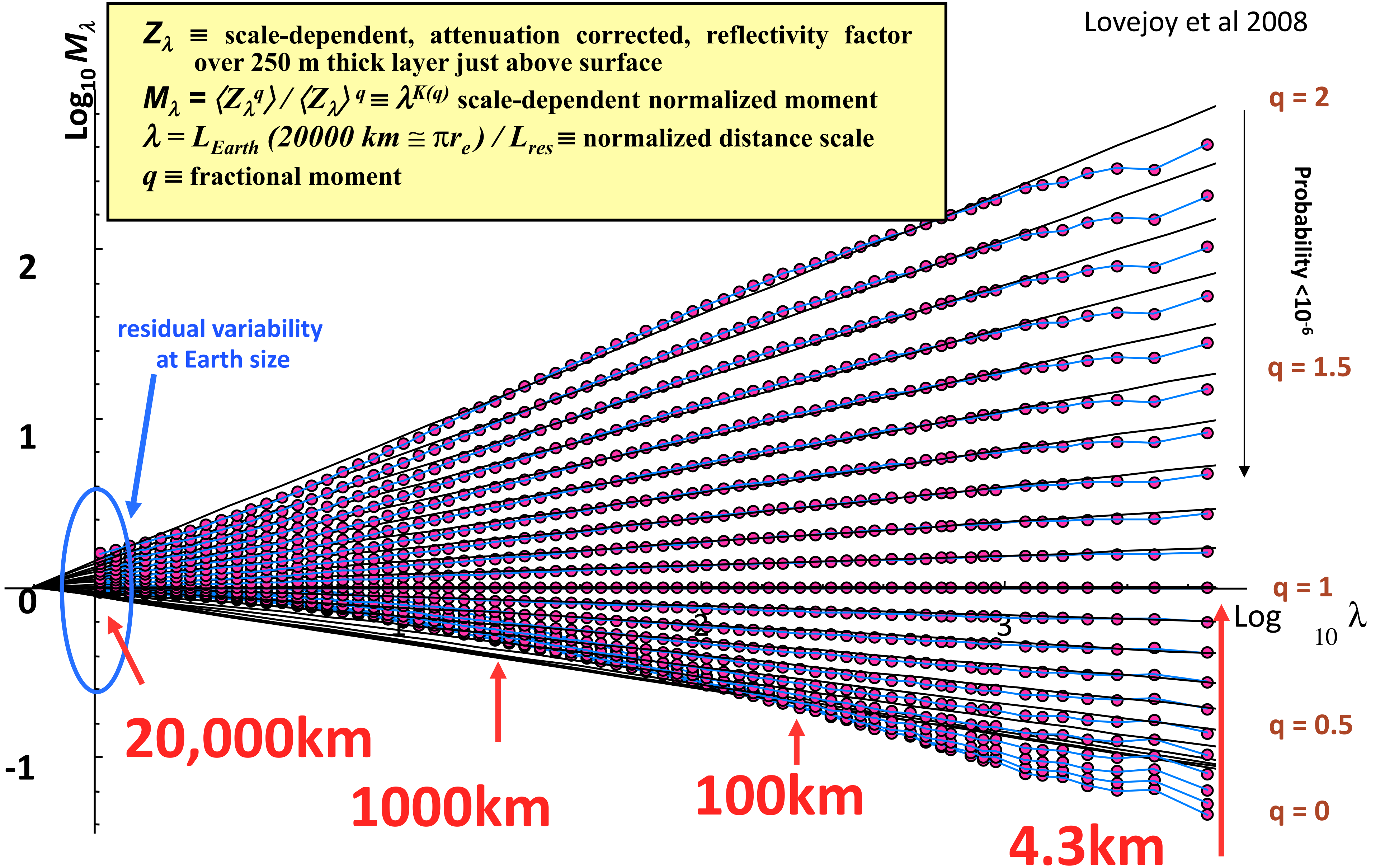
Cascade prediction:

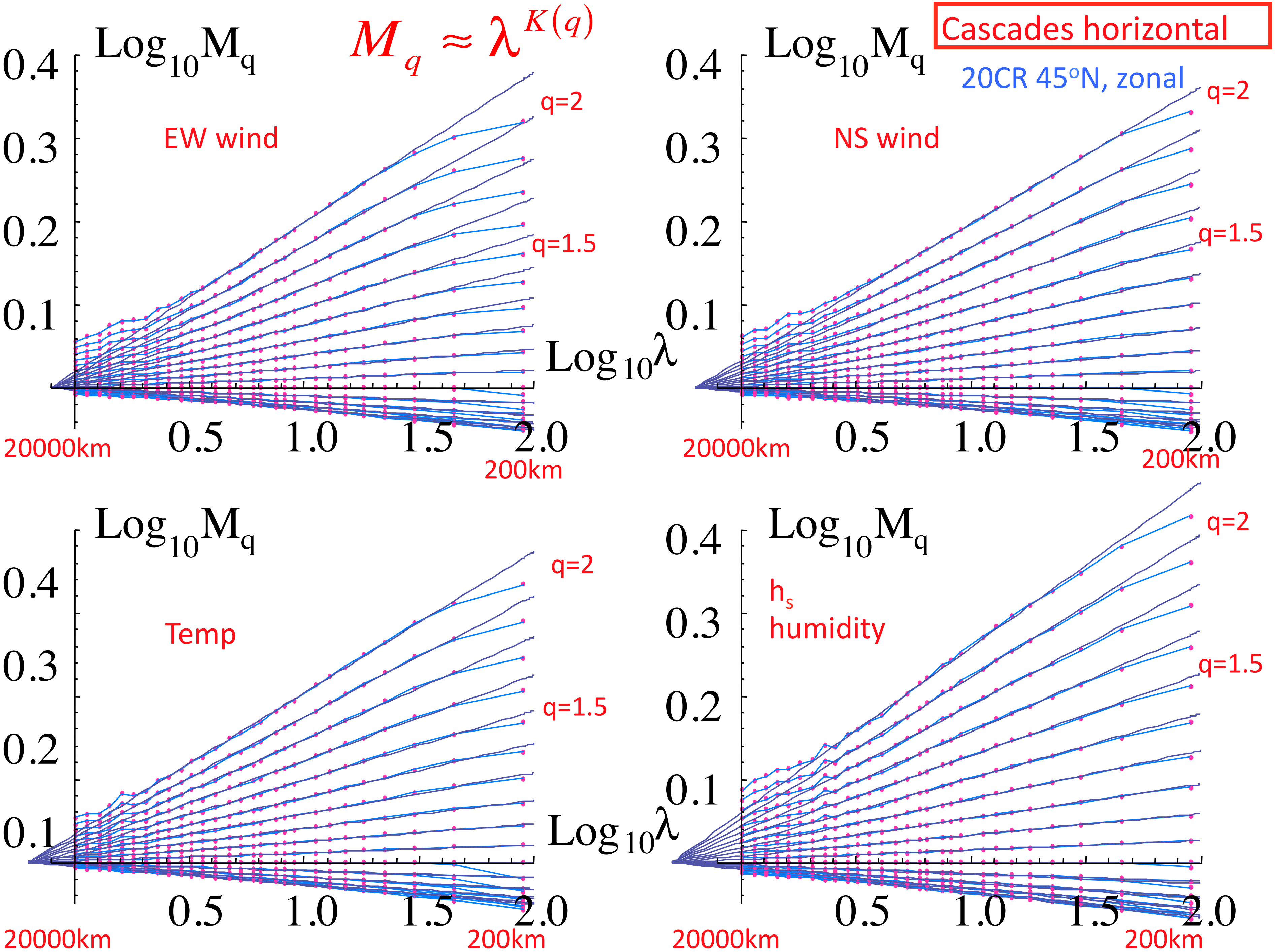
$$\langle Z_\lambda^q \rangle / \langle Z_1 \rangle^q = \lambda^{K(q)}$$

$$\lambda = L_{eff} / L_{res}$$



Scale-dependent TRMM PR Attenuation Corrected Reflectivity Factor [Z_λ] (1176 consecutive orbits -- ~70 days)





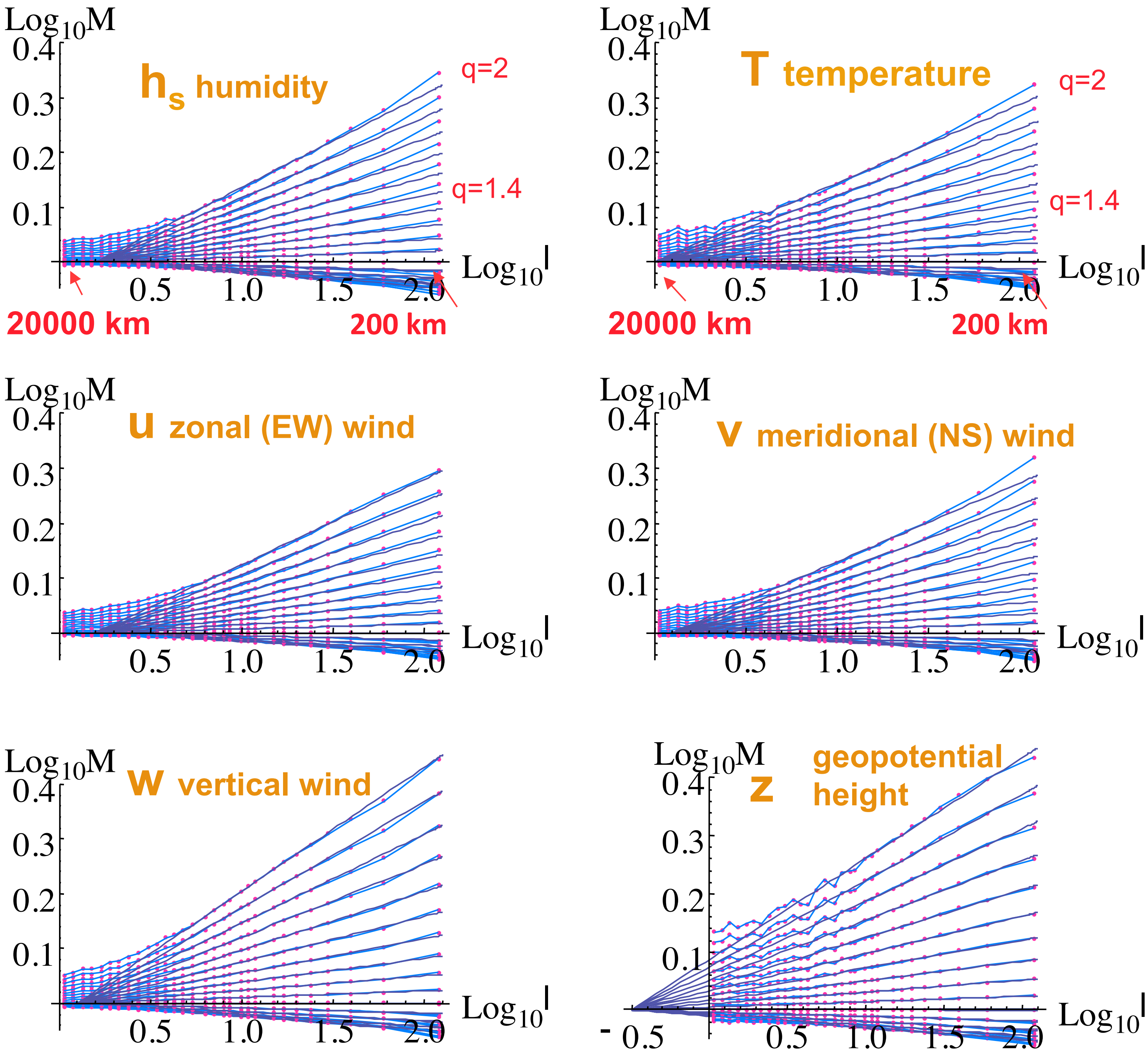
$M_q \approx \lambda^{K(q)}$
Cascades horizontal
20CR 45°N, zonal
Log10Mq
EW wind
NS wind
Temp
hs humidity
q=2
q=1.5
0.4
0.3
0.2
0.1
Log10λ
0.5
1.0
1.5
2.0
20000km
200km

$M = \langle \phi_\lambda^q \rangle / \langle \phi \rangle^q$

ECMWF reanalysis

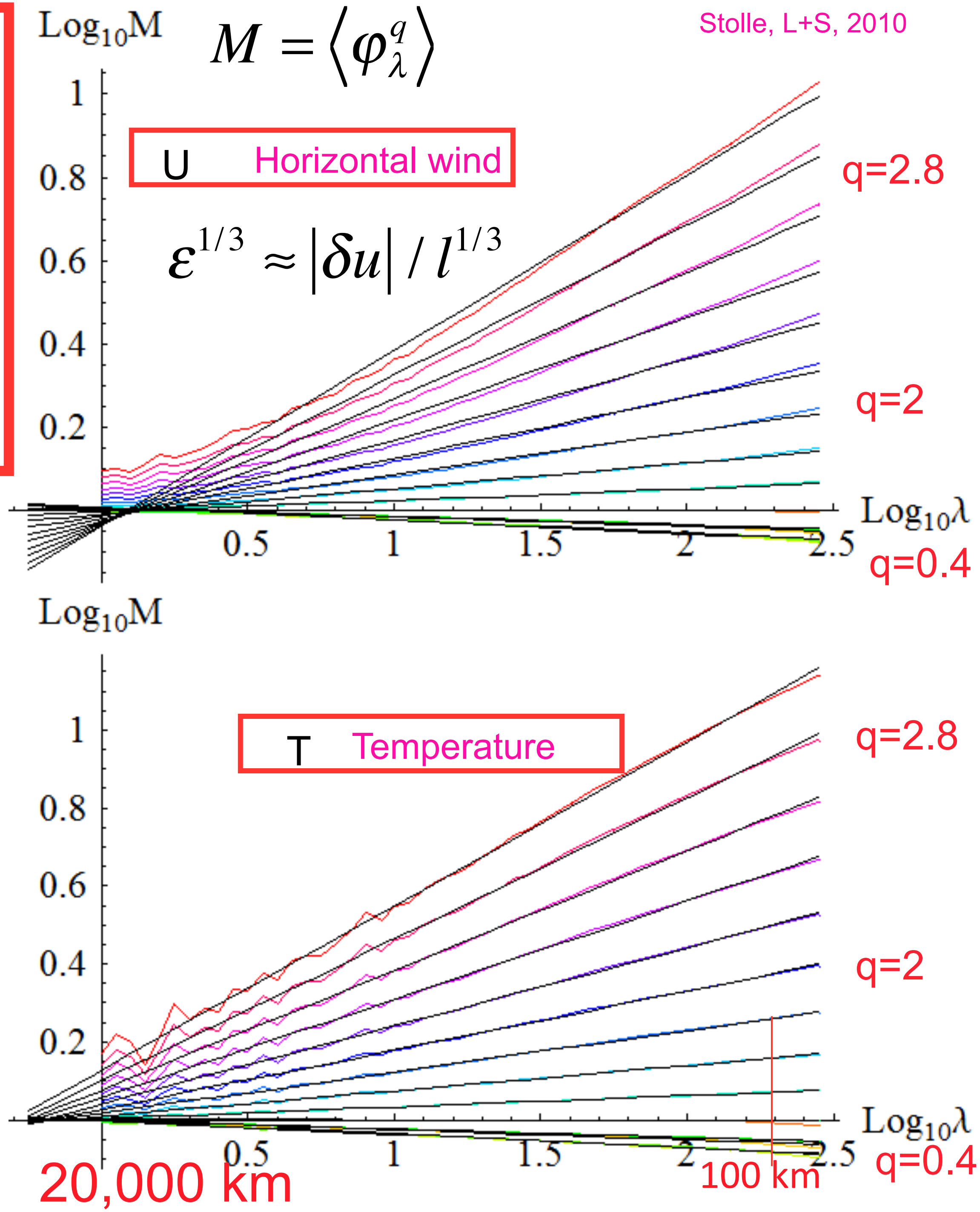
East-West

(2006, 0Z, 700 mb)



Global GEMS Model 00h

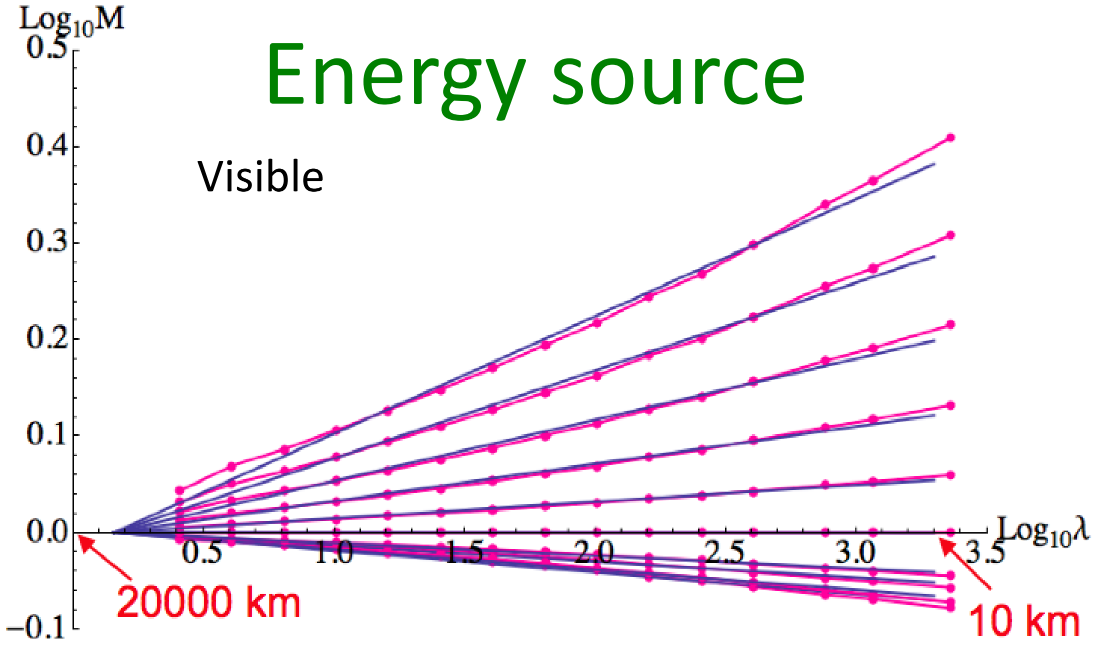
Analysis of four months
U,T at 1000 mb

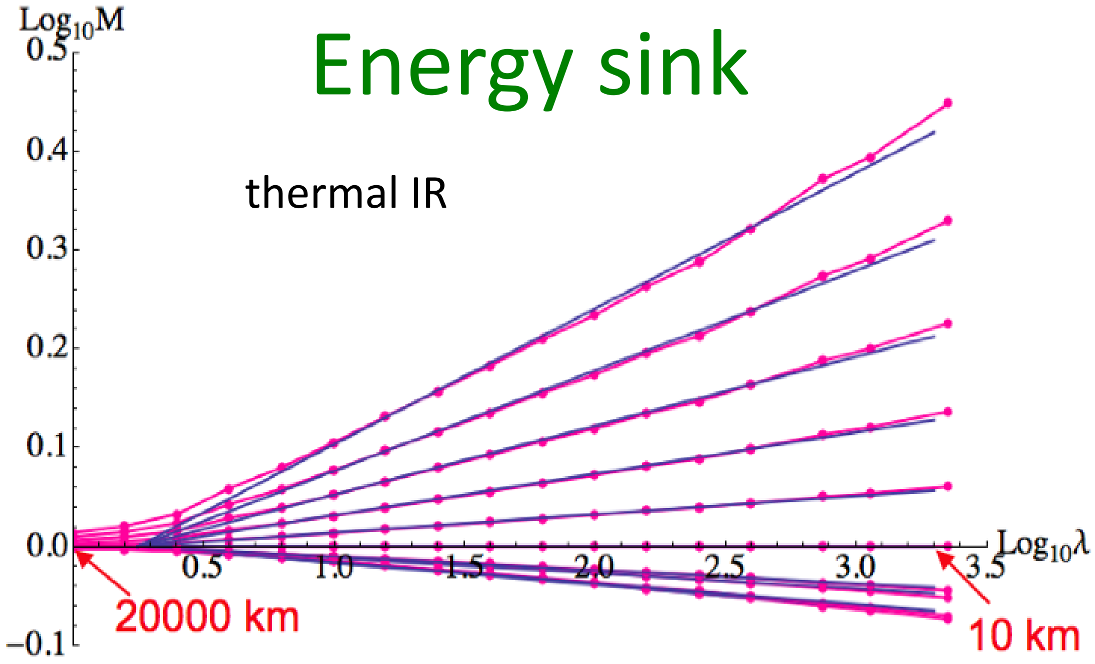


(48 h forecasts are almost the same)

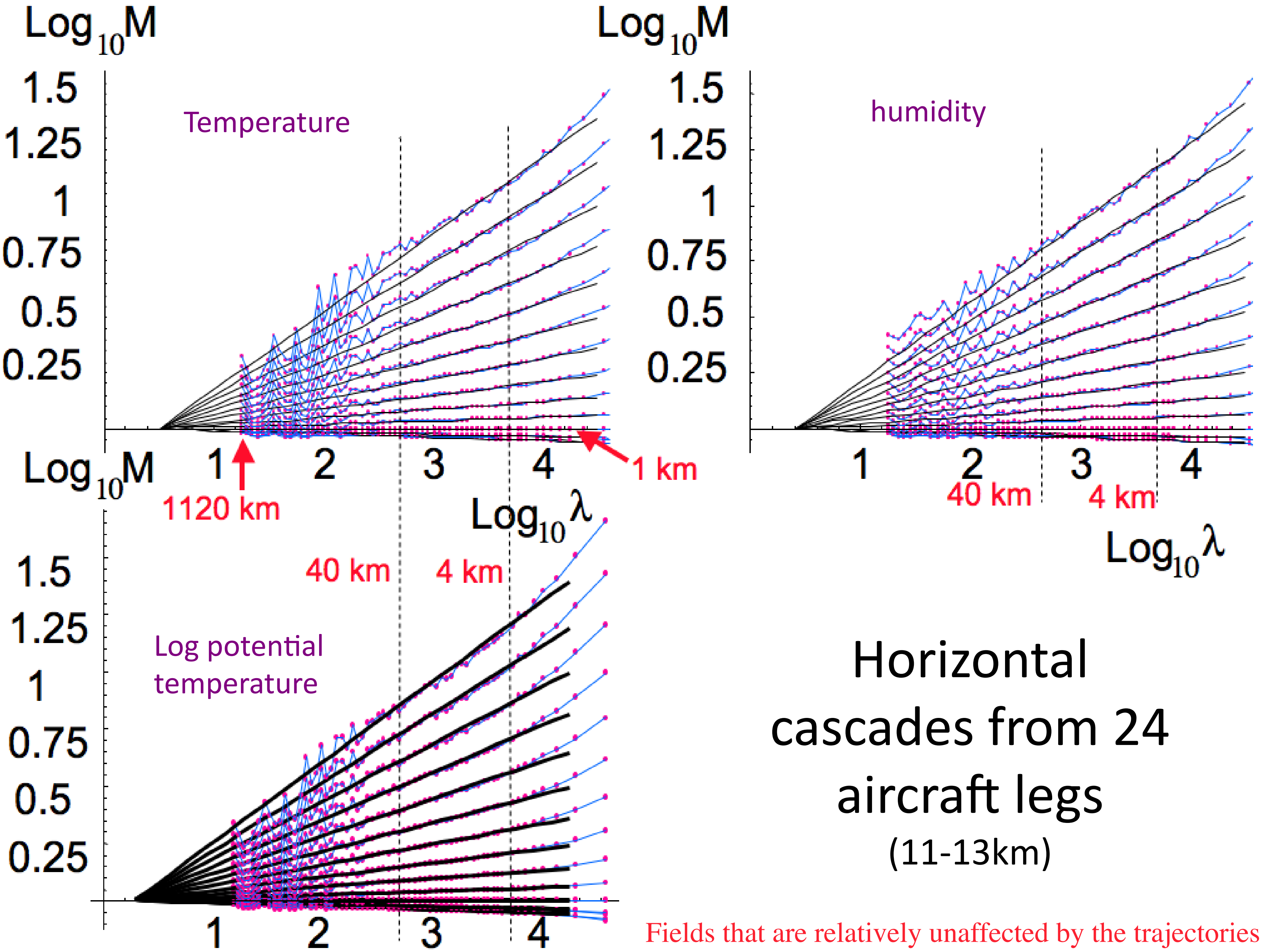
Energy budget

TRMM satellite data, ≈1000 orbits





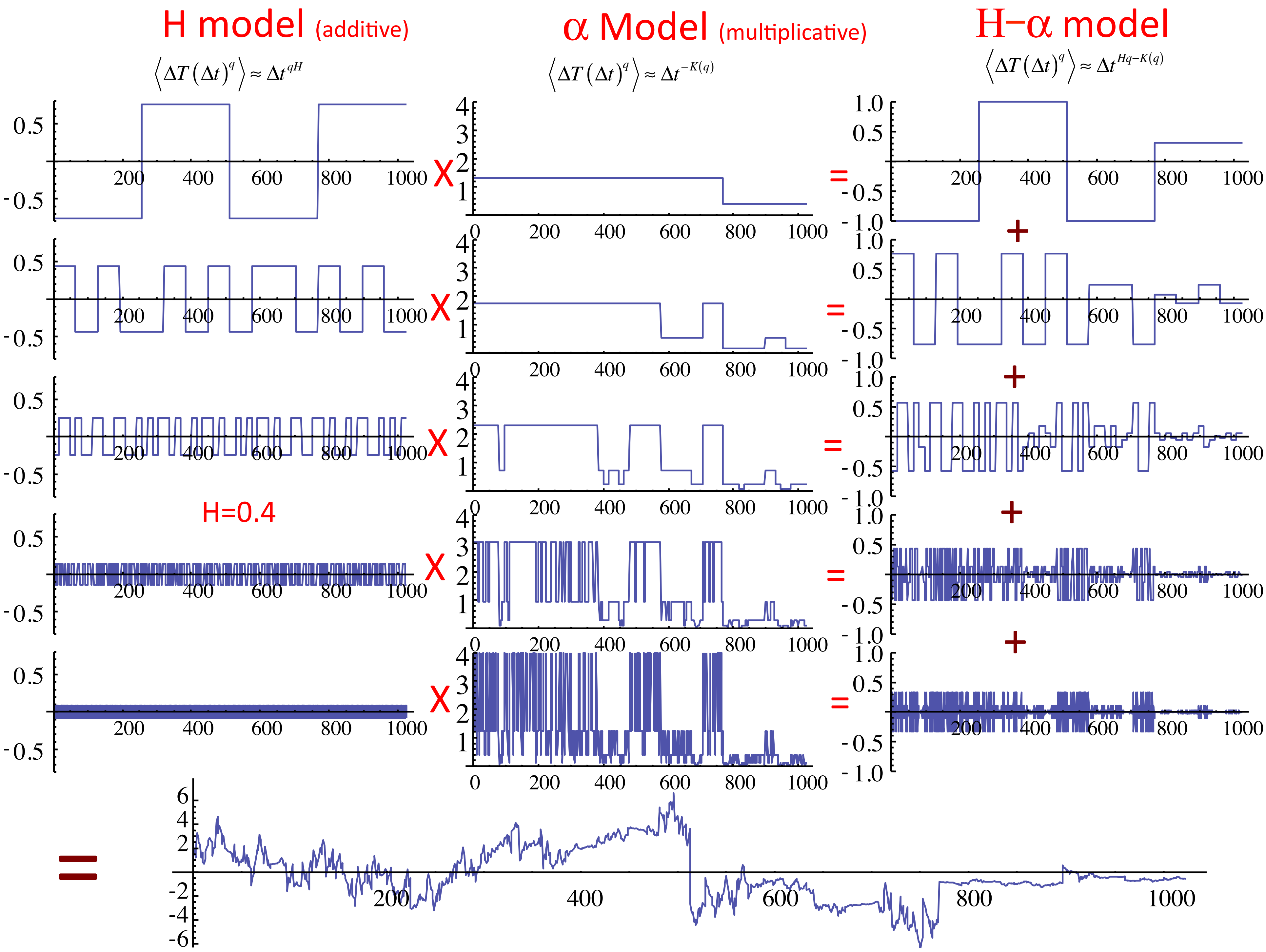
Lovejoy et al 2009



Horizontal cascades from 24 aircraft legs
(11-13km)

Fields that are relatively unaffected by the trajectories

Observables:
additive and multiplicative processes


H model (additive)
$\langle \Delta T(\Delta t)^q \rangle \approx \Delta t^{qH}$
α Model (multiplicative)
$\langle \Delta T(\Delta t)^q \rangle \approx \Delta t^{-K(q)}$
H−α model
$\langle \Delta T(\Delta t)^q \rangle \approx \Delta t^{Hq-K(q)}$
H=0.4
X
=
+