PHYS 616 Multifractals and Turbulence

Lecture 12:
Generalized Scale Invariance (part 2)

April, 9 2014

Solution to the scale function equation

Isotropic case (G=1): $\|\lambda^{-1}\underline{r}\| = \lambda^{-1}\|\underline{r}\|$ Solution: Vector norm $\|\underline{r}\| = |\underline{r}|\Theta(\hat{r});\quad \Theta(\hat{r}) > 0$ Unit vector

What is the solution of the scale function equation $\|T_\lambda \underline{\Delta r}\| = \lambda^{-1}\|\underline{\Delta r}\|$ With: $T_\lambda = \lambda^{-G} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-H_z} \end{pmatrix}$?

Stratified Case (G diagonal):

Make a nonlinear coordinate transformation:

$$(\Delta x', \Delta z') = \left(\Delta x, sign(\Delta z)|\Delta z|^{1/H_z}\right)$$

Check: $T_\lambda(\Delta x', \Delta z') = \left(\lambda^{-1}\Delta x, sign(\lambda^{-H_z}\Delta z)|\lambda^{-H_z}\Delta z|^{1/H_z}\right) = \lambda^{-1}(\Delta x', \Delta z')$

that transforms the anisotropic scaling equation into the isotropic equation $\|\lambda^{-1}\underline{\Delta r}'\| = \lambda^{-1}\|\underline{\Delta r}'\|$, with $\underline{\Delta r}' = (\Delta x', \Delta z')$. The general solution of this istropic equation has already been given; the general solution of the anisotropic equation is therefore:

$$\|\underline{\Delta r}\| = \Theta(\theta')r';\qquad r' = \left(\Delta x^2 + |\Delta z|^{2/H_z}\right)^{1/2};\quad \tan\theta' = \frac{\Delta z'}{\Delta x'} = \frac{sign(\Delta z)|\Delta z|^{1/H_z}}{\Delta x}$$

where θ’, r’ are the polar angles and radii in the primed system; we have nondimensionalized by the distances by l_s so that the unit scale is also the “sphero-scale”. The canonical solution is therefore obtained by taking $\Theta = 1$. The function Θ defines the unit ball via the polar coordinate equation $\|\underline{\Delta r}\| = 1$:

Unit ball:

$$r' = \Theta(\theta')^{-1}$$

so that Θ>0 is necessary so that the unit ball is closed; structures are spatially localized.

General Scale functions in linear GSI in 2D

(real eigenvalues)

Note, for complex case, a variant of the method works, see the book, p.237.

G is a nondiagonal matrix; it suffices to diagonalize it before applying a (nonlinear) transformation of variables. Consider for simplicity, the two dimensional case, with the two eigenvalues Λ_x, Λ_y. When these are real we have the following coordinate transformations

$$\underline{r}' = \Omega^{-1}\underline{r} \longleftarrow \text{diagonal}$$

$$\underline{r}'' = (x'', y'') = \left(sign(x')|x'|^{1/\Lambda_x}, sign(y')|y'|^{1/\Lambda_y} \right)$$

From the above discussion we see that

$$\left\|\lambda^{-G}\underline{r}\right\| = \lambda^{-1}\left\|\underline{r}\right\| \leftrightarrow \left\|\lambda^{-1}\underline{r}''\right\| = \lambda^{-1}\left\|\underline{r}''\right\|$$

so that the solution of the functional scale equation is the same as for the self-affine (diagonal *G* case) except for the doubly primed variables:

$$\left\|\underline{r}\right\| = \Theta(\theta'')r''; \qquad r'' = \left|\underline{r}''\right| = \left(x'^{2/\Lambda_x} + |y'|^{2/\Lambda_y} \right)^{1/2}; \quad \tan\theta'' = \frac{y''}{x''} = \frac{sign(y')|y'|^{1/\Lambda_y}}{sign(x')|x'|^{1/\Lambda_x}}$$

Once again, the condition that the balls are decreasing with λ (no crossing of balls) is that $\Theta(\theta'')>0$ and the choice of the otherwise arbitrary Θ determines the shape of the unit ball: in polar coordinates its equation is $r'' = 1/\Theta(\theta'')$. $\longleftarrow$ Unit ball

Simulations of anisotropic multifractals

All that needs to be done is to simply replace the vector norms $|\underline{r}|$ everywhere by scale functions $\|\underline{r}\|$ and the spatial dimensions d by elliptical dimensions D_{el}:

$$\varepsilon_\lambda = e^{\Gamma_\lambda}; \quad \Gamma_\lambda(\underline{r}) = C_1^{1/\alpha} N_{D_{el}}^{-1/\alpha} \int_{1 \le \|\underline{r}'\| \le \lambda} \frac{\gamma(\underline{r}')}{\|\underline{r}-\underline{r}'\|^{D_{el}/\alpha}} d^d \underline{r}'$$

Note: in evaluating the integral the following is useful

$$\underline{r}' = \lambda^{-G}\underline{r}$$

$$d\underline{r}' = \det(\lambda^{-G}) d\underline{r} = \lambda^{-Tr(G)} d\underline{r} = \lambda^{-D_{el}} d\underline{r}$$

where the normalization N_{Del} constant is still given by an angle integral:

$$N_{D_{el}} = \Omega_{D_{el}} = \int_{\|\underline{r}\|=1} d^{D_{el}} \underline{r}'$$

the details of the explicit calculation are somewhat technical and are given in appendix 7A.

The statistics of the resulting v field will satisfy the anisotropic extensions of the earlier formulae:

$$\left\langle |\Delta v(\underline{\Delta r})|^q \right\rangle = \|\underline{\Delta r}\|^{\xi(q)}; \quad \xi(q) = qH - K(q)$$

which for any scale function is equivalent to:

$$\left\langle |\Delta v(T_\lambda \underline{\Delta r})|^q \right\rangle = \lambda^{-\xi(q)} \left\langle |\Delta v(\underline{\Delta r})|^q \right\rangle$$

Anisotropic singularities, Generalized Scale Invariance

Schertzer and Lovejoy 1987

Self-similar processses → Anisotropic scaling processes

noise

$$\phi * \left|\underline{x}\right|^{-(D-H)} \longrightarrow \phi * \left\|\underline{x}\right\|^{-(D_{el}-H)}$$

Distance function (norm) → scale function

Scale function equation:

$$\left\|T_{\lambda}\underline{x}\right\| = \lambda^{-1}\left\|\underline{x}\right\|; \quad T_{\lambda} = \lambda^{-G}; \quad D_{el} = TraceG$$

Reduced scale vector

generator

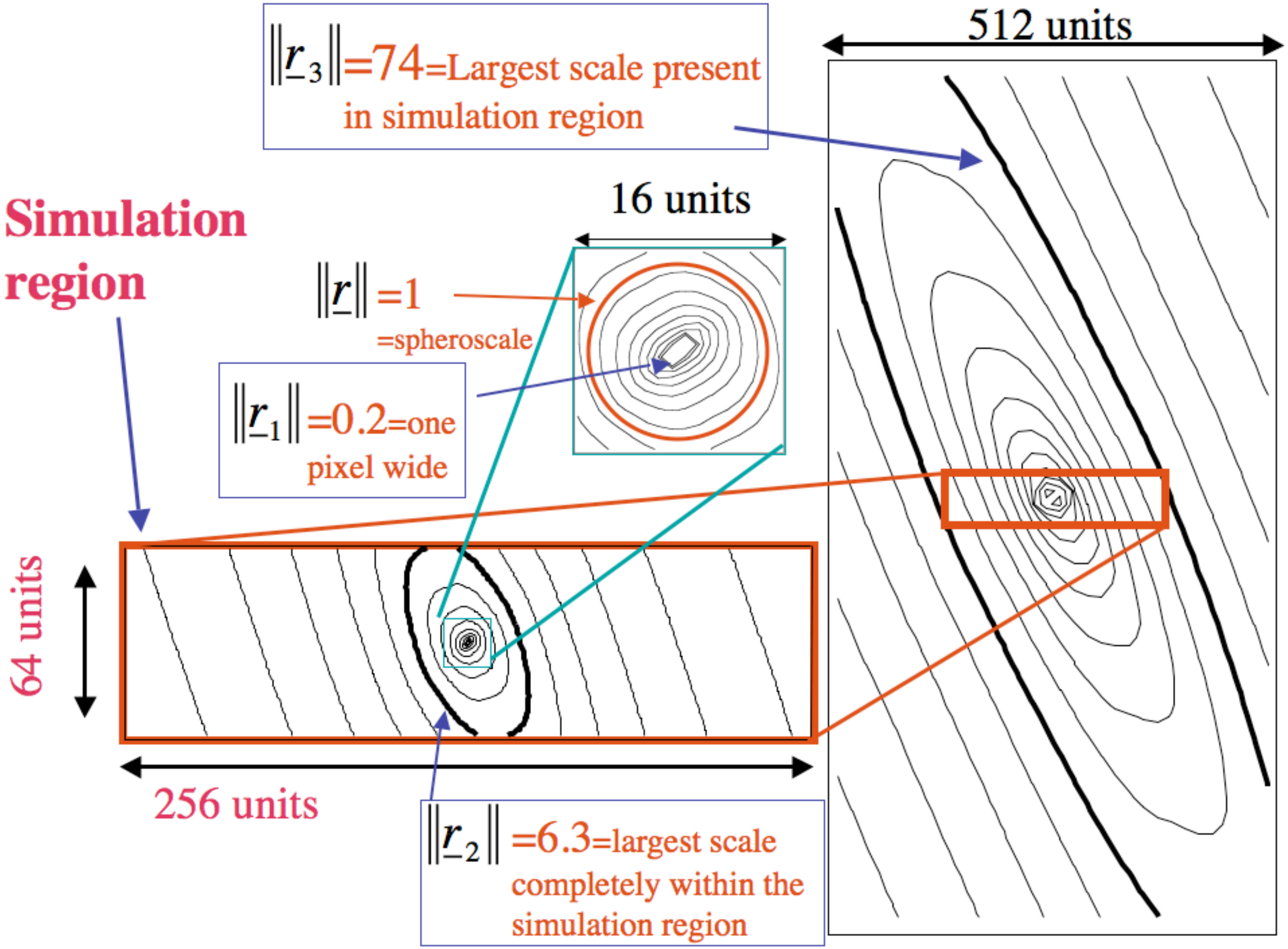
Elliptical dimension

Overall

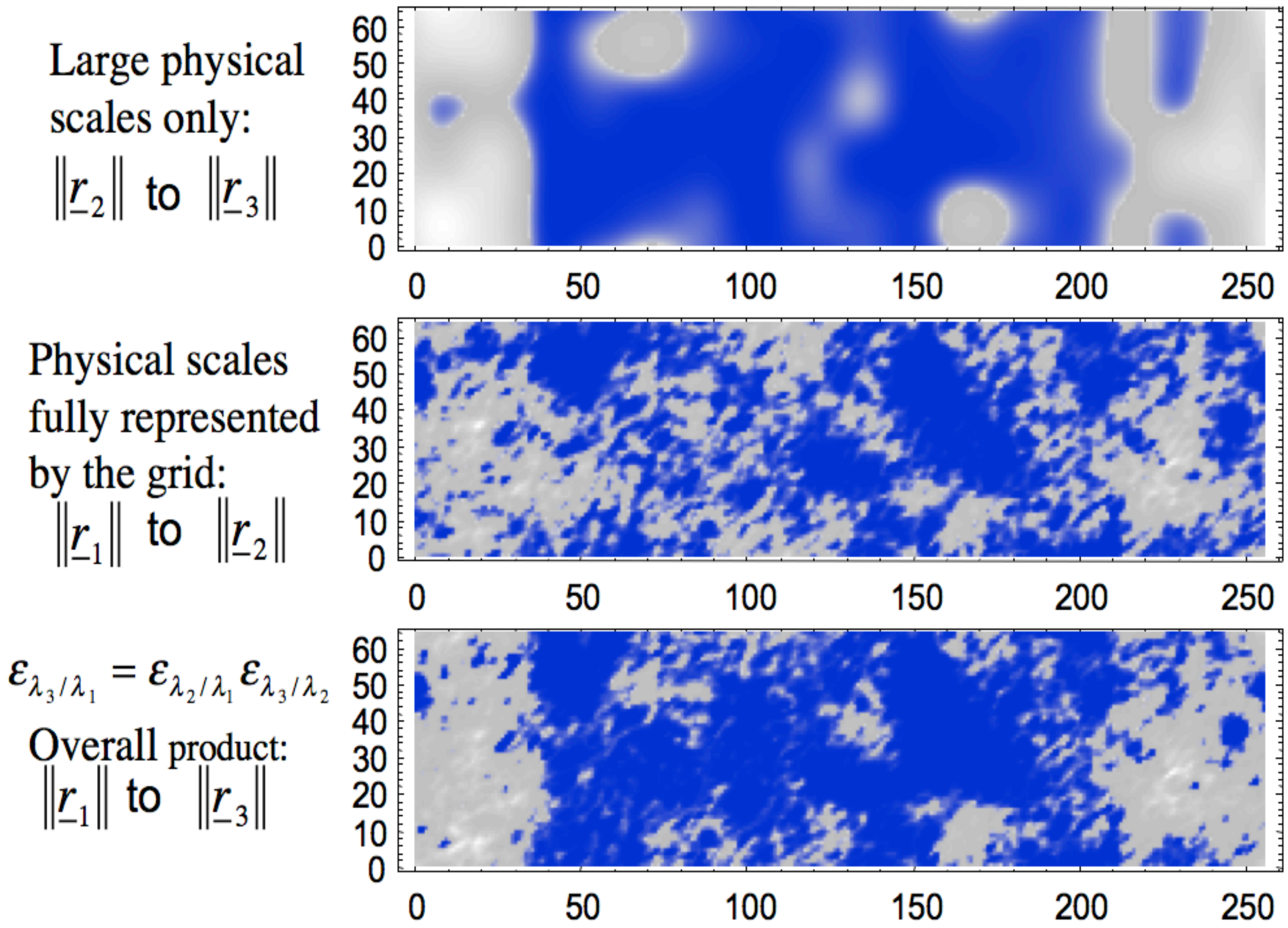
Isotropy $\longrightarrow$ anisotropy

$$\left|\underline{x}\right| \to \left\|\underline{x}\right\|; \quad D \to D_{el}$$

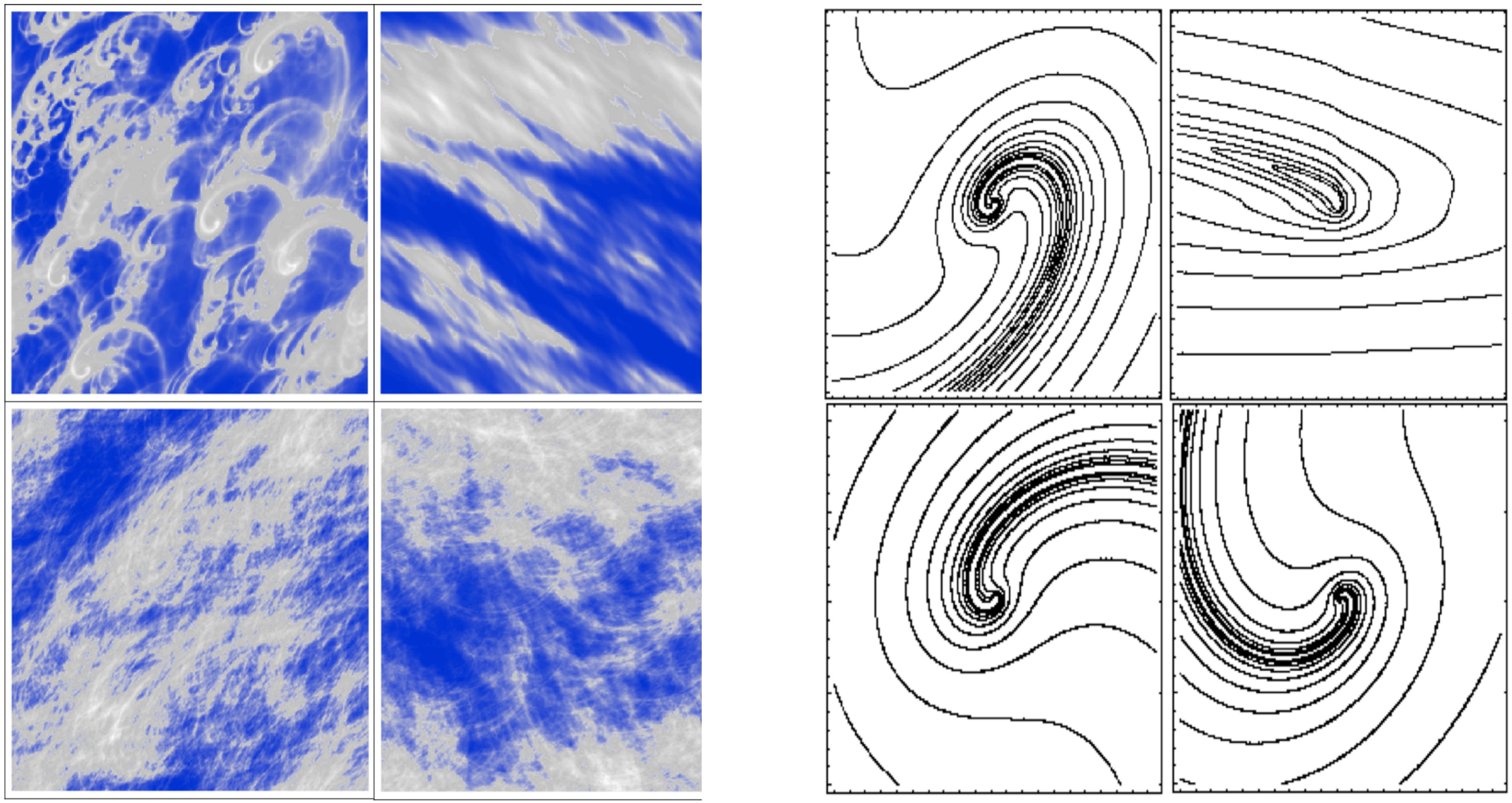
Some practical issues for anisotropic simulations



A schematic diagram showing the balls associated with the canonical system with l_s= 8 and: d=1, c=-0.1, f=-0.2, e =-0.1. =0.2 is the smallest scale which is completely resolved by the 1x1 pixel grid, =6.3 is the largest scale completely resolved by the 64x256 pixel simulation region; =74 is the largest scale that influences the simulation region.

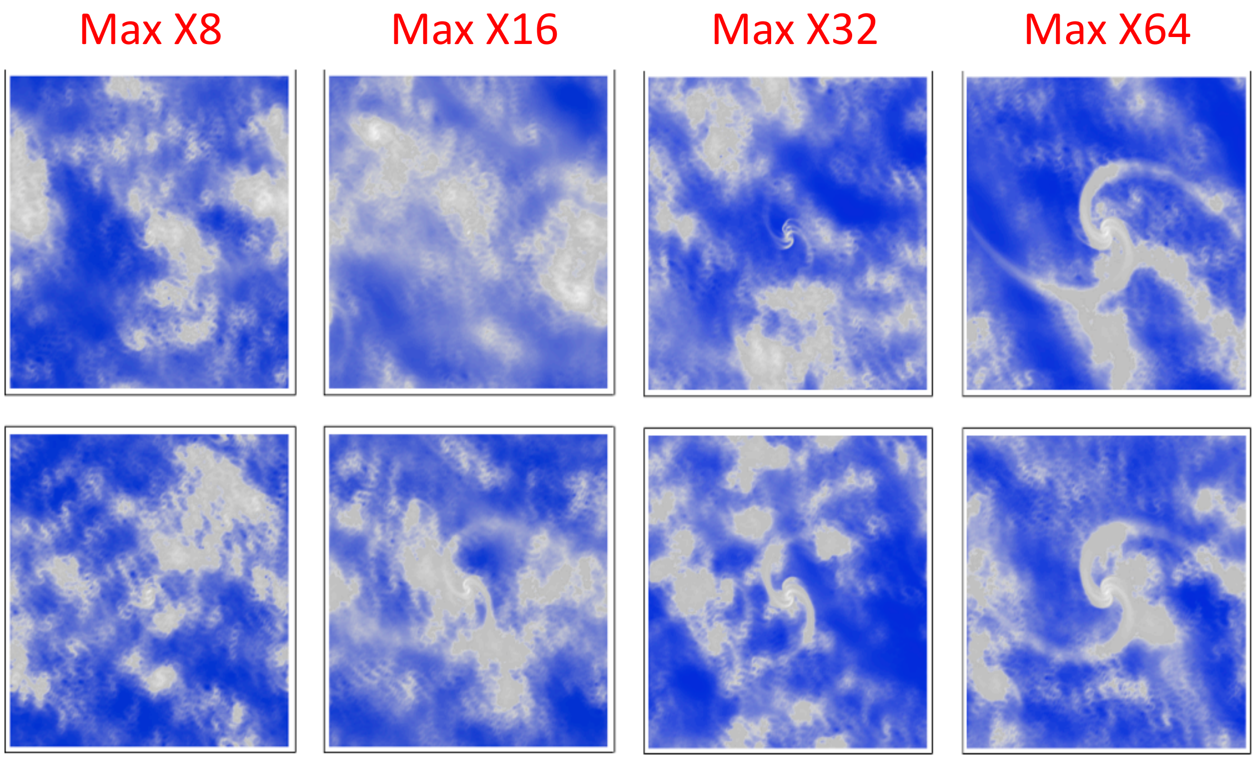


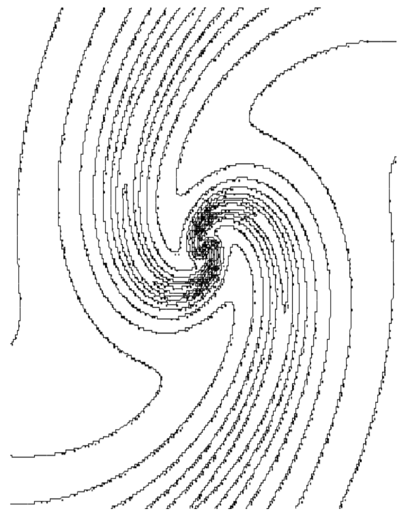
This shows the contributions from the fully resolved band (scales $\|\underline{r}_1\|$ to $\|\underline{r}_2\|$) and the partially resolved band $\|\underline{r}_2\|$ to $\|\underline{r}_3\|$ to the total simulation; a = 1.6, C_1 = 0.1 (same G as in previous).



Examples of 2D simulations on 512x512 pixel grids with $\alpha = 1.8$, $C_1 = 0.1$, $H = 0.333$, $d = 1$, $f = 0$. Upper left: $c = 0.8$, $e = 2$, $l_s = 512$, x = 1.3 ($2^k = r_{max}/r_{min} \approx 54$), upper right: $c = -2/7$, $e = 0.1$, $l_s = 32$, $2^k \approx 5$, lower left: $c = 0.3$, $e = 1.2$, $l_s = 32$, $2^k \approx 800$, lower right: $c = 0.3$, $e = 1.2$, $l_s = 1$, $2^k \approx 800$.

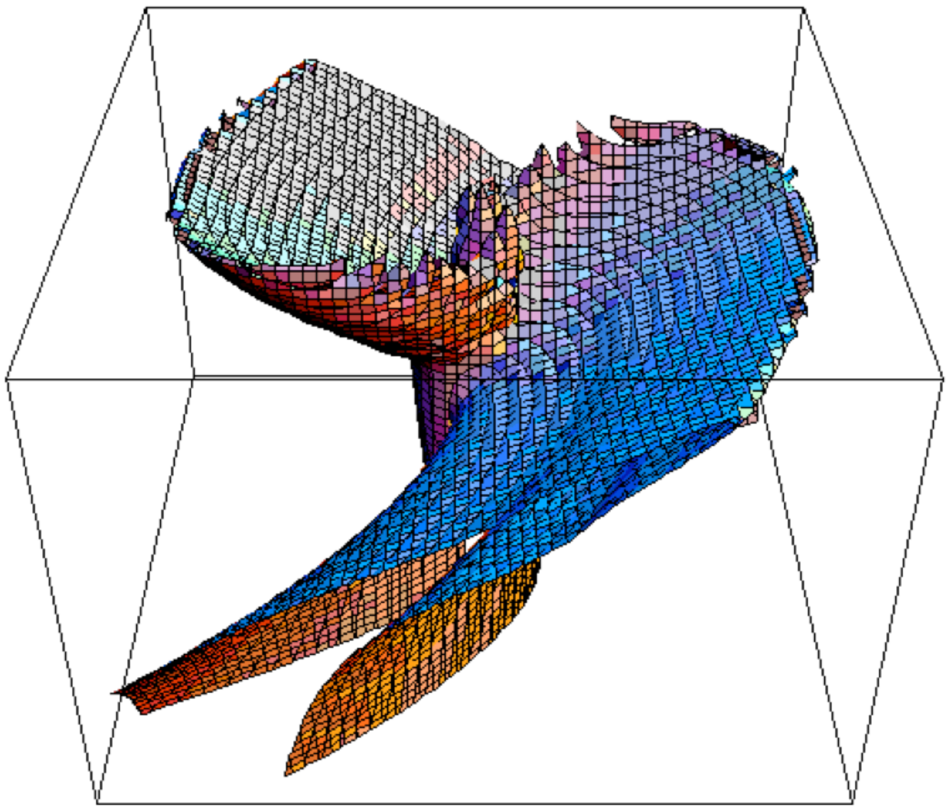
Order emerging from chaos





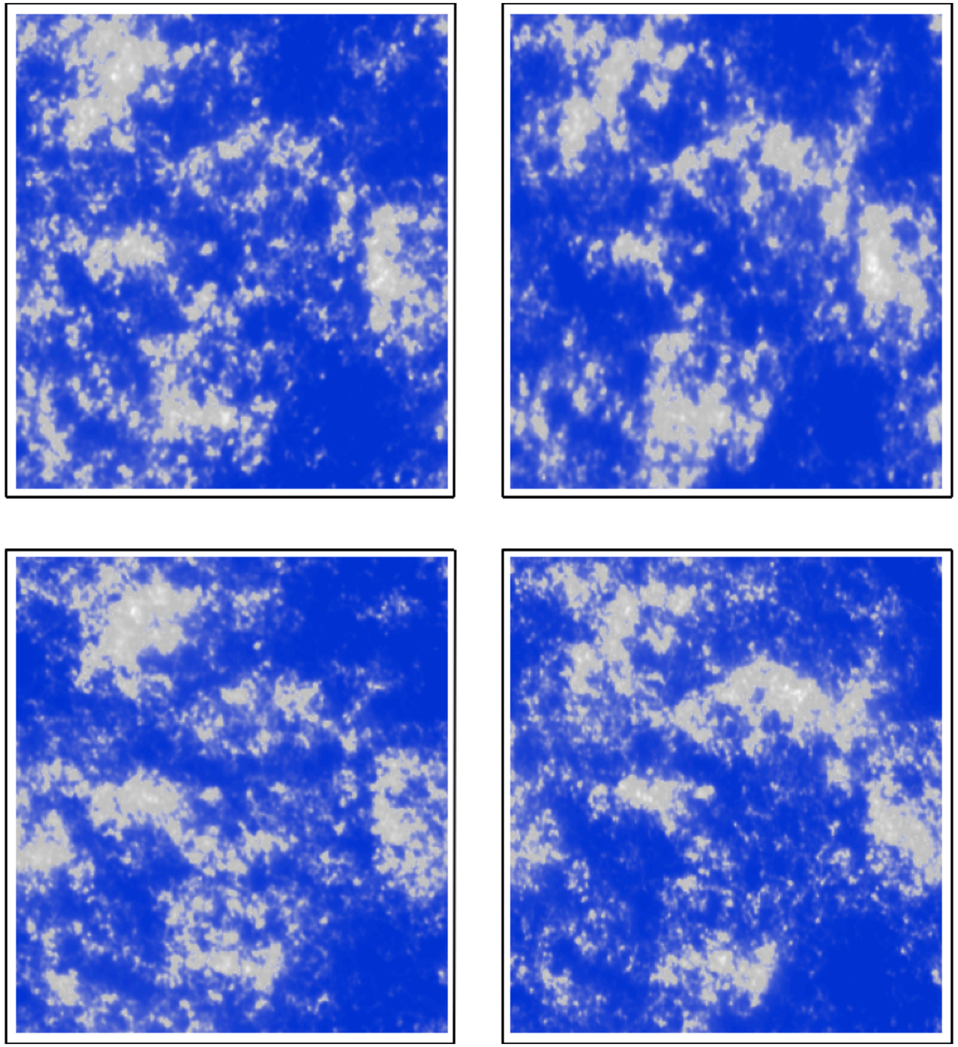
Each row shows a realization of a random multifractal process with a single value of of the subgenerator $\gamma(\underline{r})$ at the centre of a 512X512 grid replaced by the maximum of $\gamma(\underline{r})$ over the field boosted by factors of N increasing by 2 from left to right (from 8 to 64) in order to simulate very rare events (α = 1.8, C_1 = 0.1, H = 0.333). The scaling is anisotropic with complex eigenvalues of G, the scale function is shown at right.

Simulations in three dimensions, rendering with simulated radiative transfer



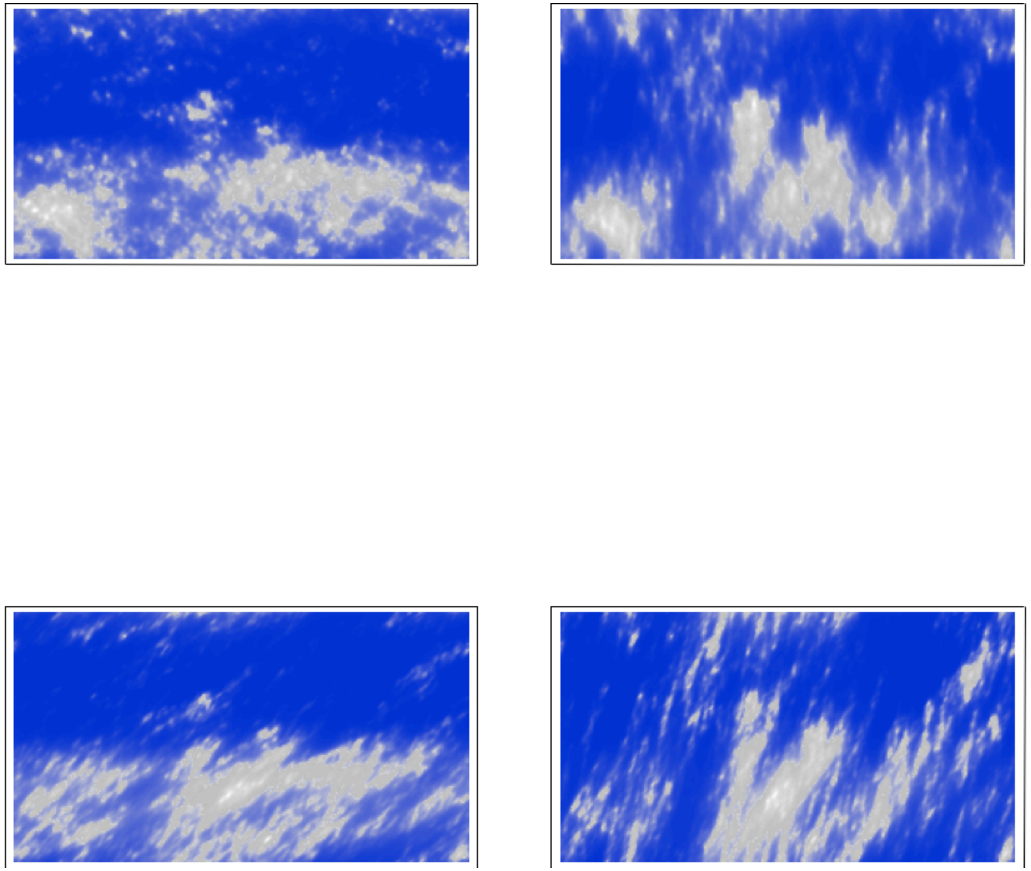
This is a contour of the scale function corresponding to a single scale; this is a strongly rotationally dominant case with n = 2, $x_q = x_f$ = 1.4, d =1, c = 0.5, e = 1, f = 0, H_z = 0.8, l_s =64,

Cloud tops (densities)

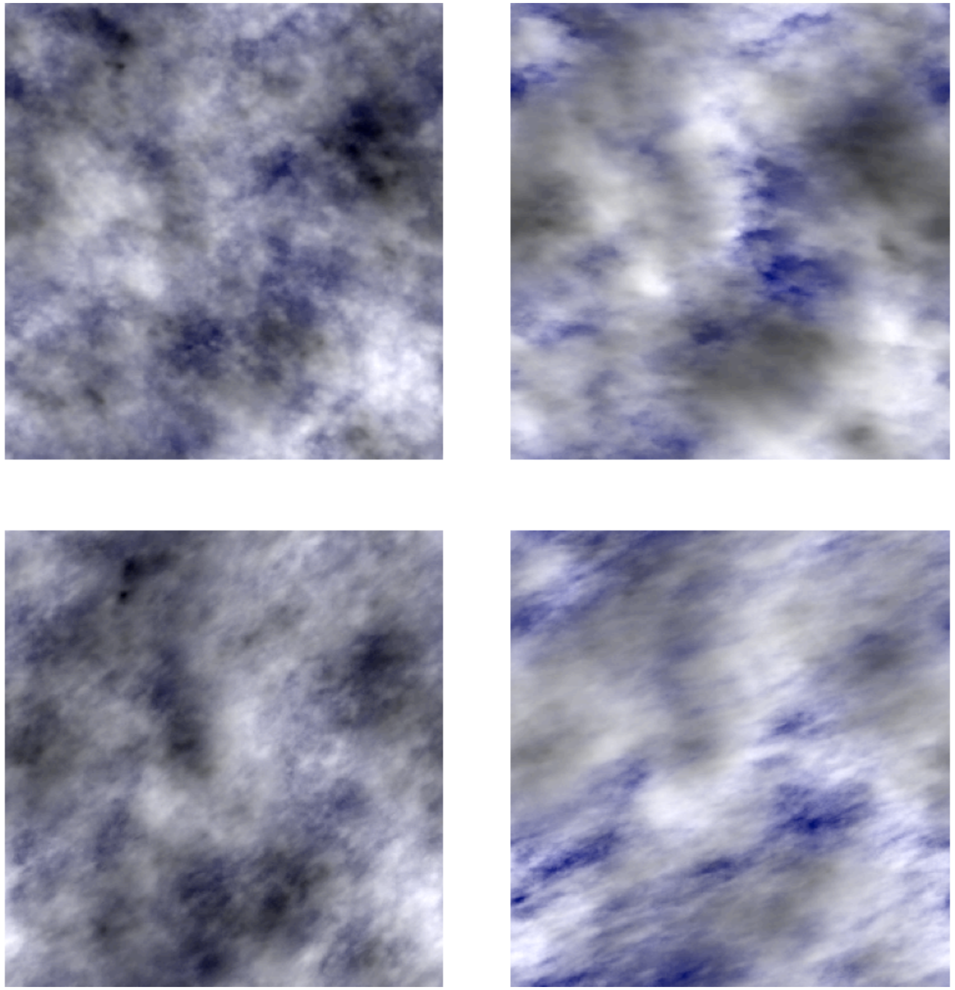


This shows the top layers of three dimensional cloud liquid water density simulations (false colours) all have d = 1, c = 0.05, e = 0.02, f = 0, H_z = 0.555, α = 1.8, C_1 = 0.1, H = 0.333 and are simulated on a 256x256x128 point grid (a^2>0; stratification dominant in the horizontal). The simulations in the top row have l_s = 8 pixels, (left column), 64 pixels (right column), k=0, k=32 (bottom row). Note that in these simulations, the l_s =8, 64 applies to both vertical and horizontal cross-sections (i.e. $l_s = l_{sz}$). Show an example with IR scattering?

Sides, same clouds (densities)

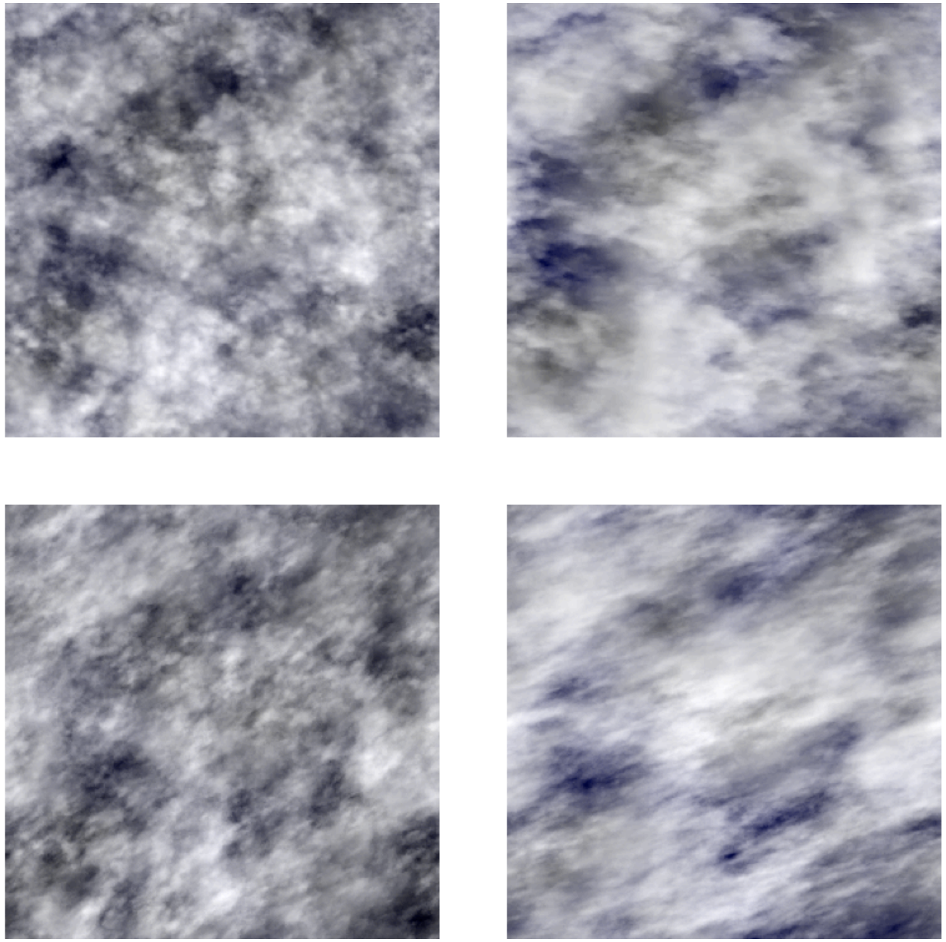


Same clouds radiative transfer, top view



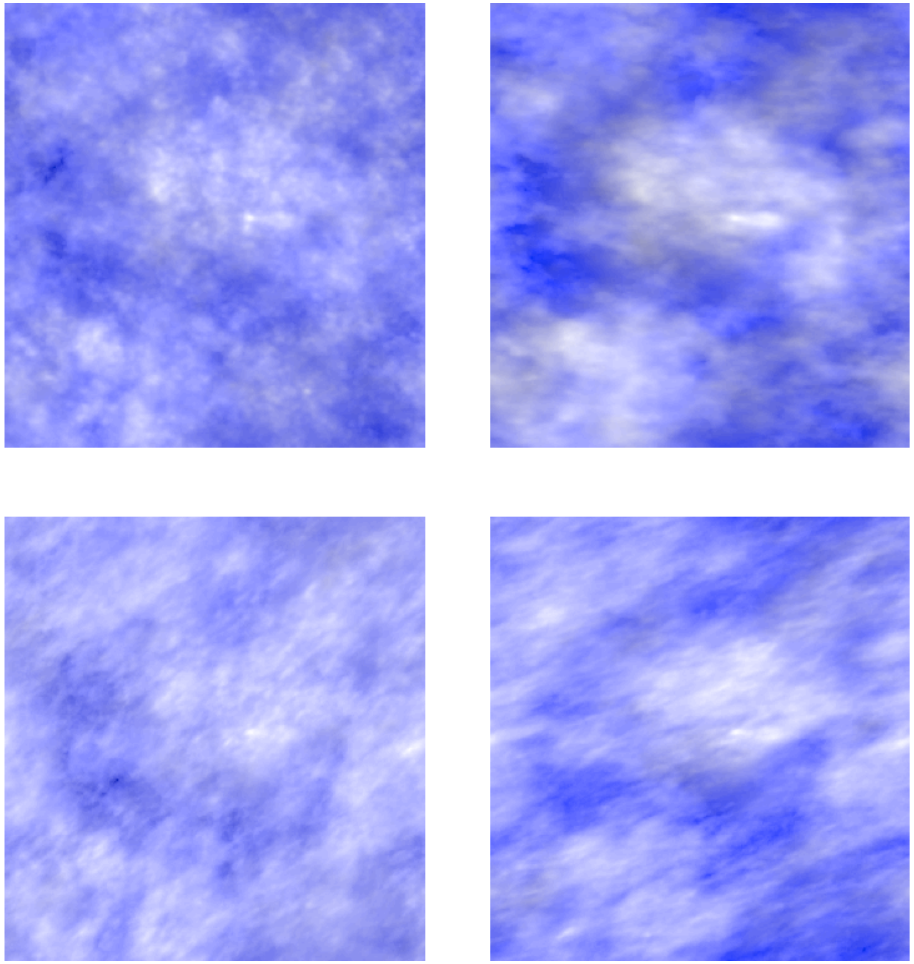
The top view with single scattering radiative transfer; incident solar radiation at 45° from the right, mean vertical optical thickness = 50

Same clouds radiative transfer, bottom view

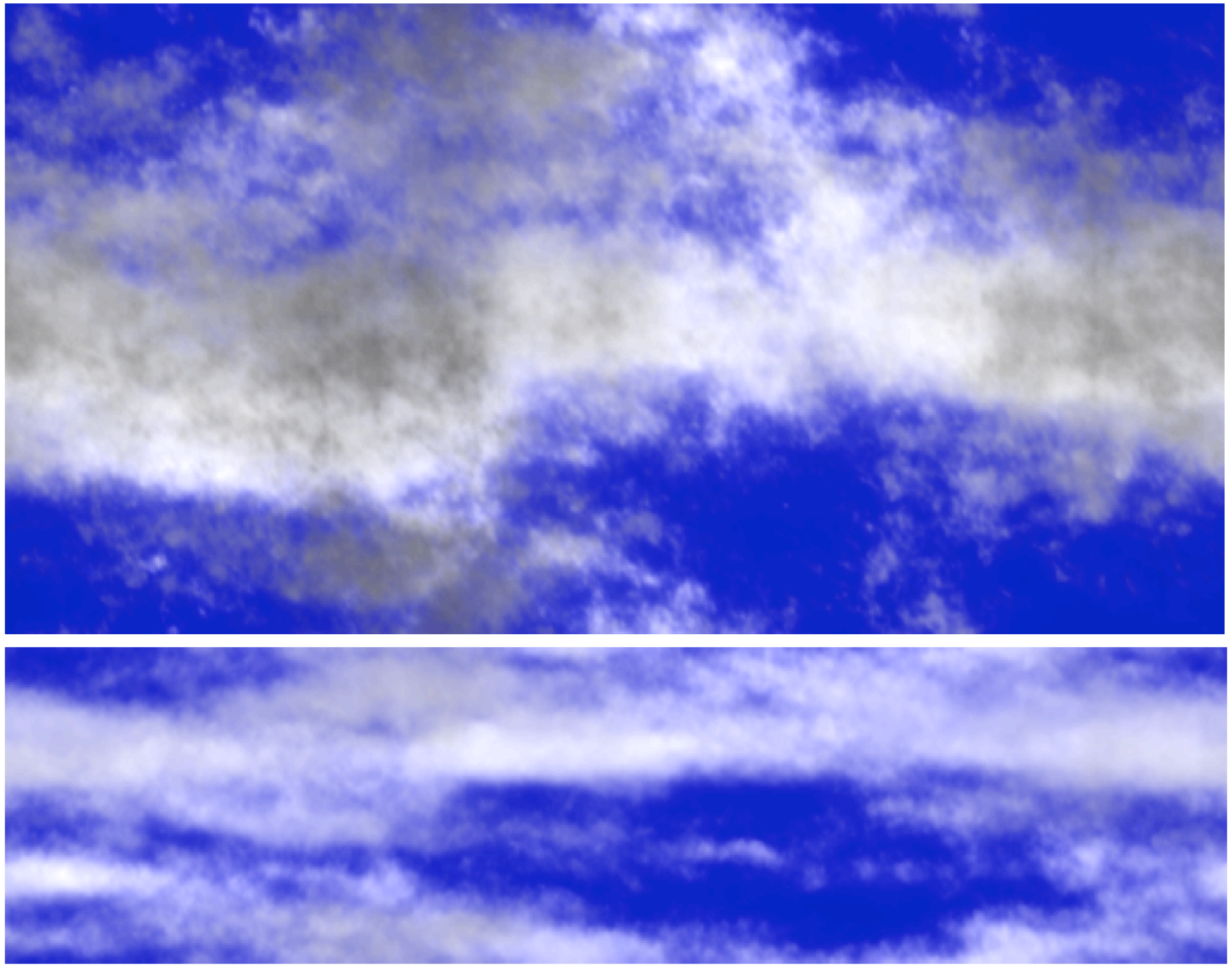


The same except viewed from the bottom.

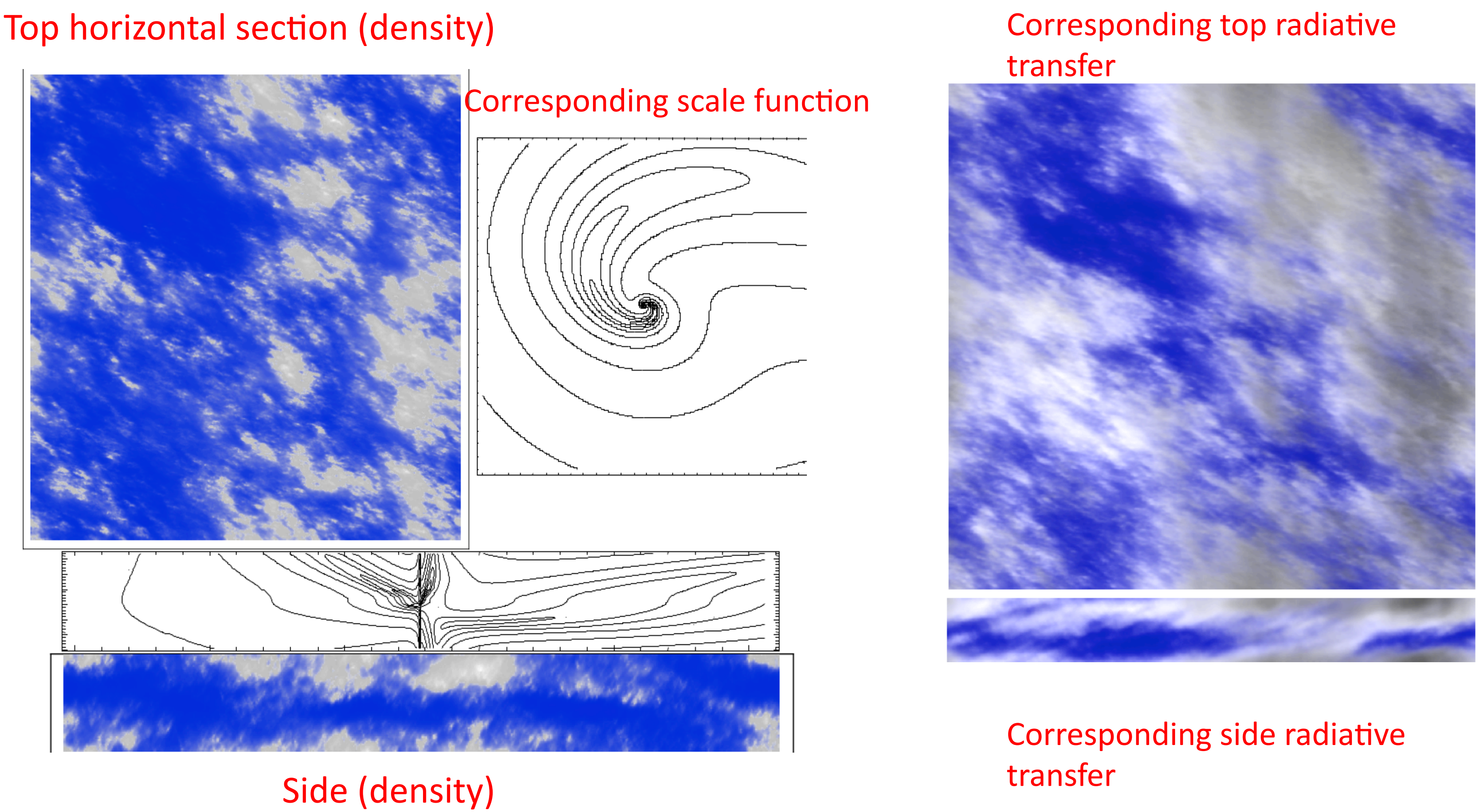
Same clouds Infra red emission, top view



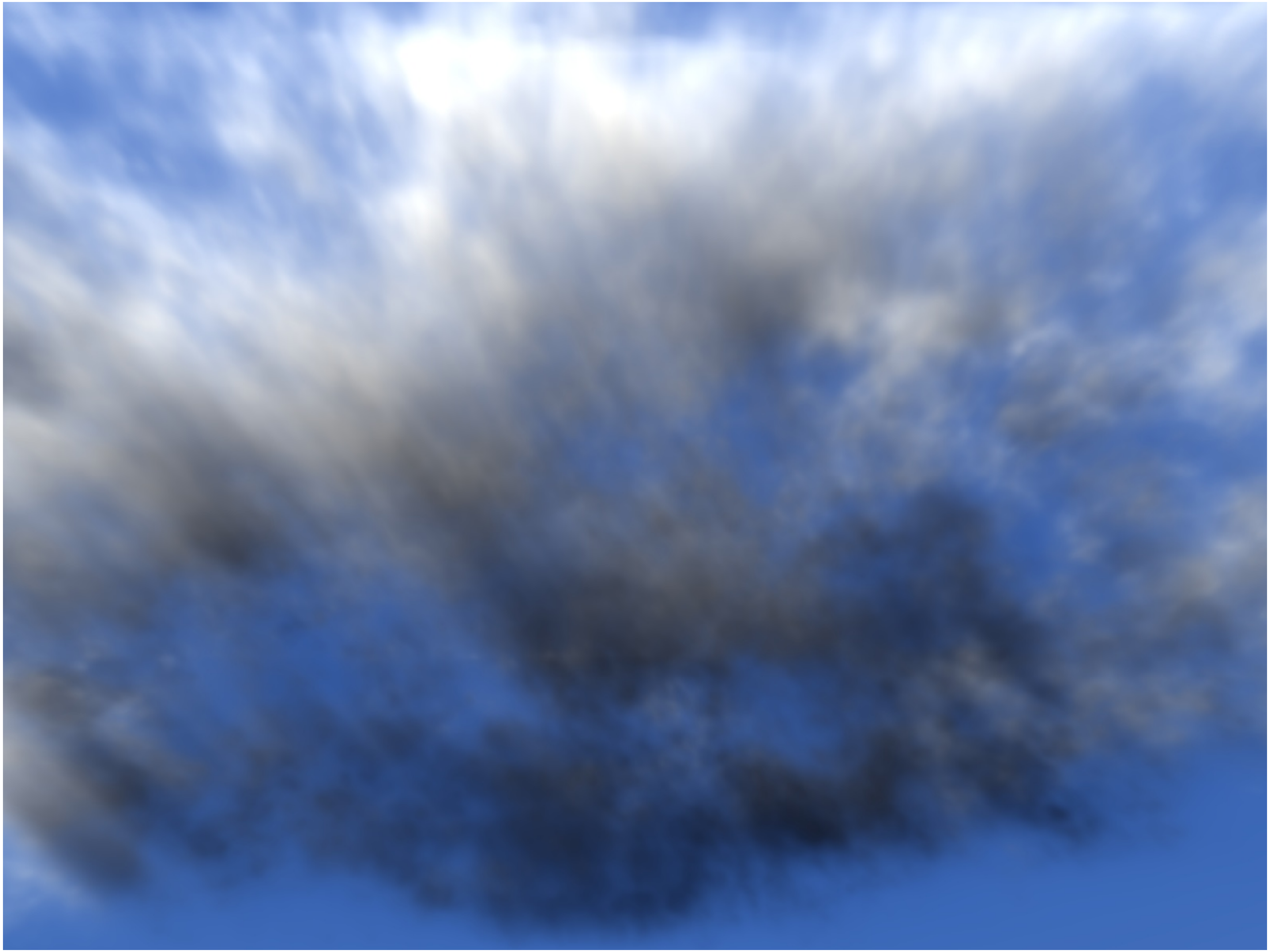
The same as the previous except for a false colour rendition of a thermal infra red field (assuming a constant extinction coefficient and a linear vertical temperature profile).

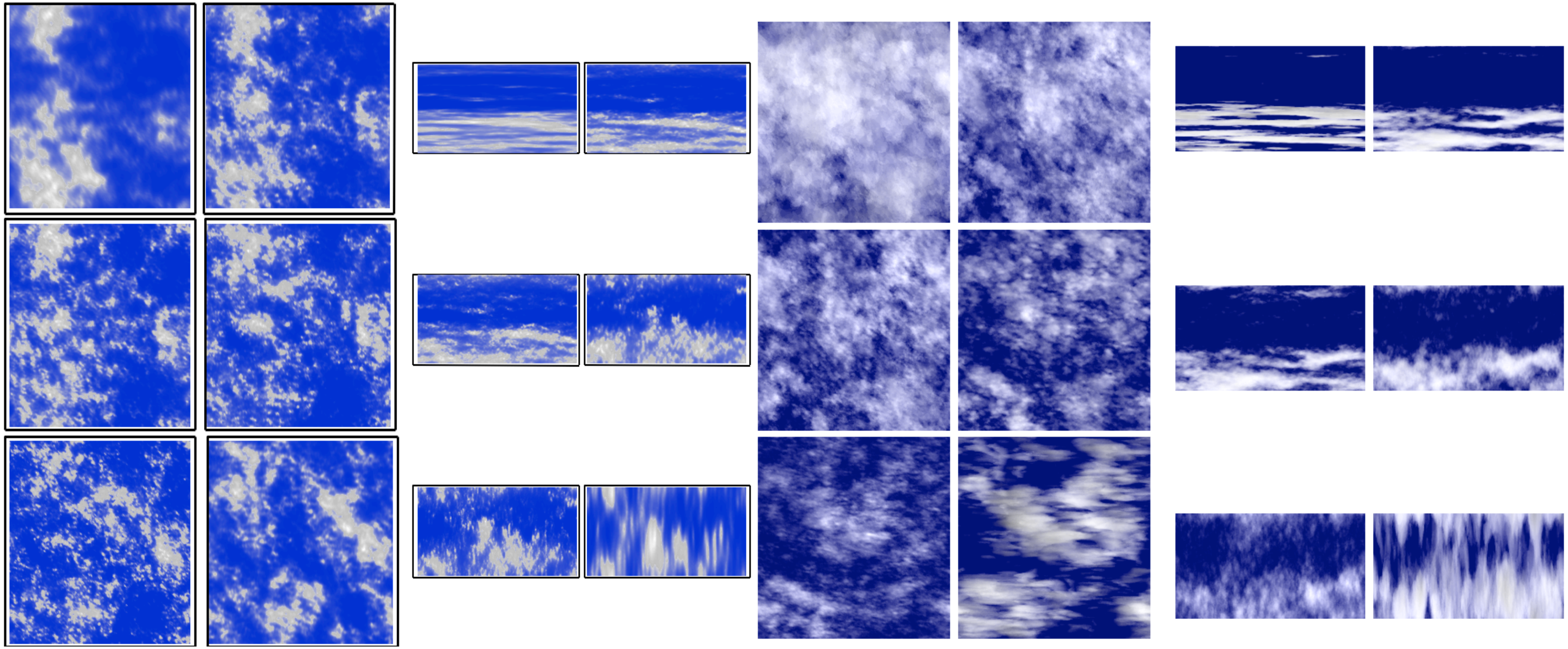


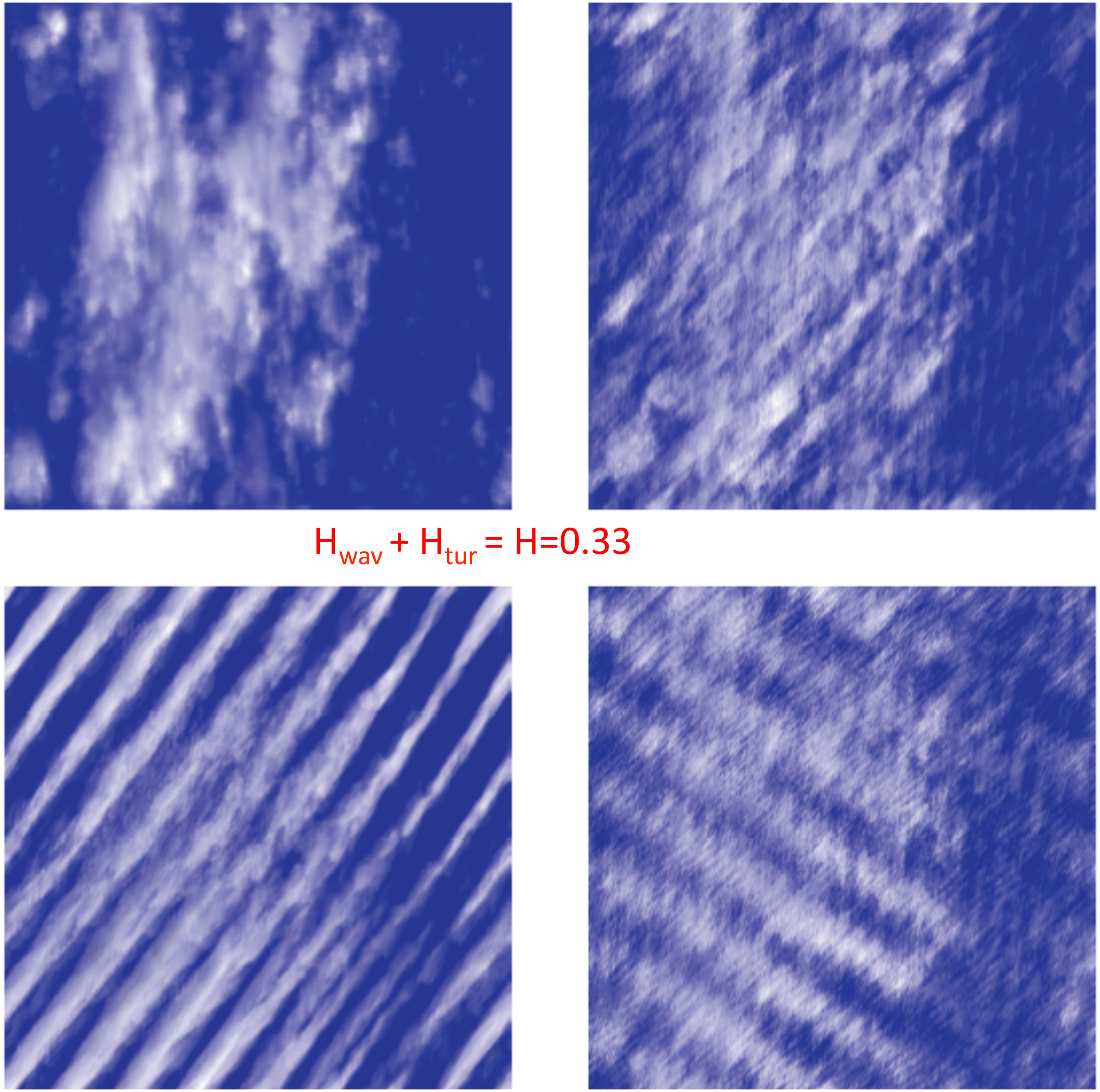
The top is the visible radiation field (corresponding to previous) looking up (sun at 45^{o} from the right); the bottom is a side radiation fields (one of the 512x128 pixel sides), average optical thickness =10, single scattering only.



An example with a = 1.8, C_1 = 0.1, H = 0.333, on a 512x512x64 grid (the latter is the thickness). The parameters are n_q =1, n_f = 2, x_q = 0.3, x_f = 0.8, c = 0.2, e = 0.5, f = 0.2 (rotation dominant), H_z = 0.555 with l_s = 128, l_{sz} = 32. The upper left is the liquid water density field, top horizontal section, to the right is the corresponding central hrizontal cross section of the scale function. The bottom row shows one of the sides (512x64 pixels) with corresponding central part of the vertical cross section.







$H_{wav} = 0$

$H_{wav} = 0.33$

$H_{wav} + H_{tur} = H = 0.33$

$H_{wav} = 0.52$

$H_{wav} = 0.38$

Fly by of anisotropic (multifractal, cascade) cloud

Rocks

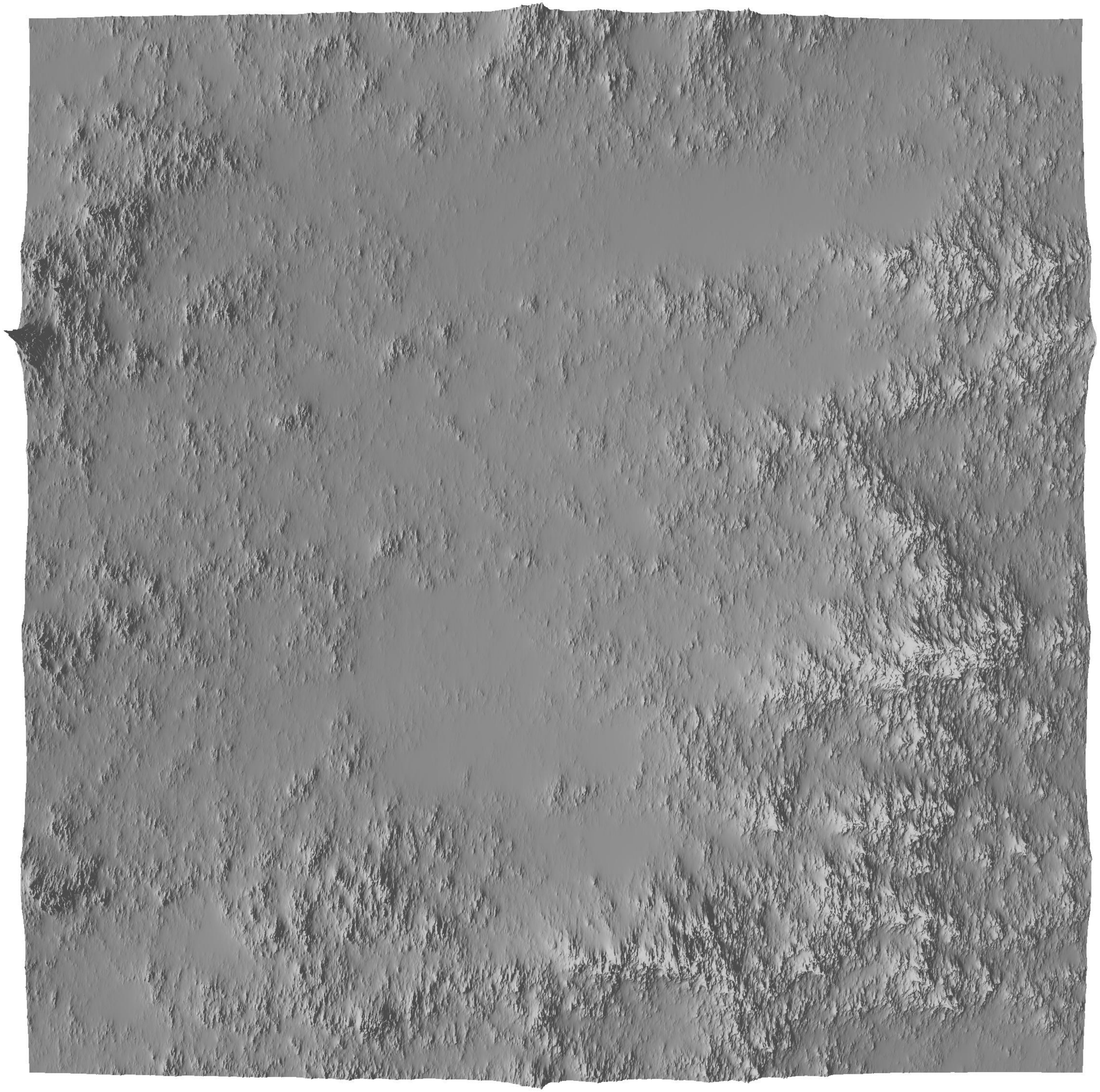
Flyby 1

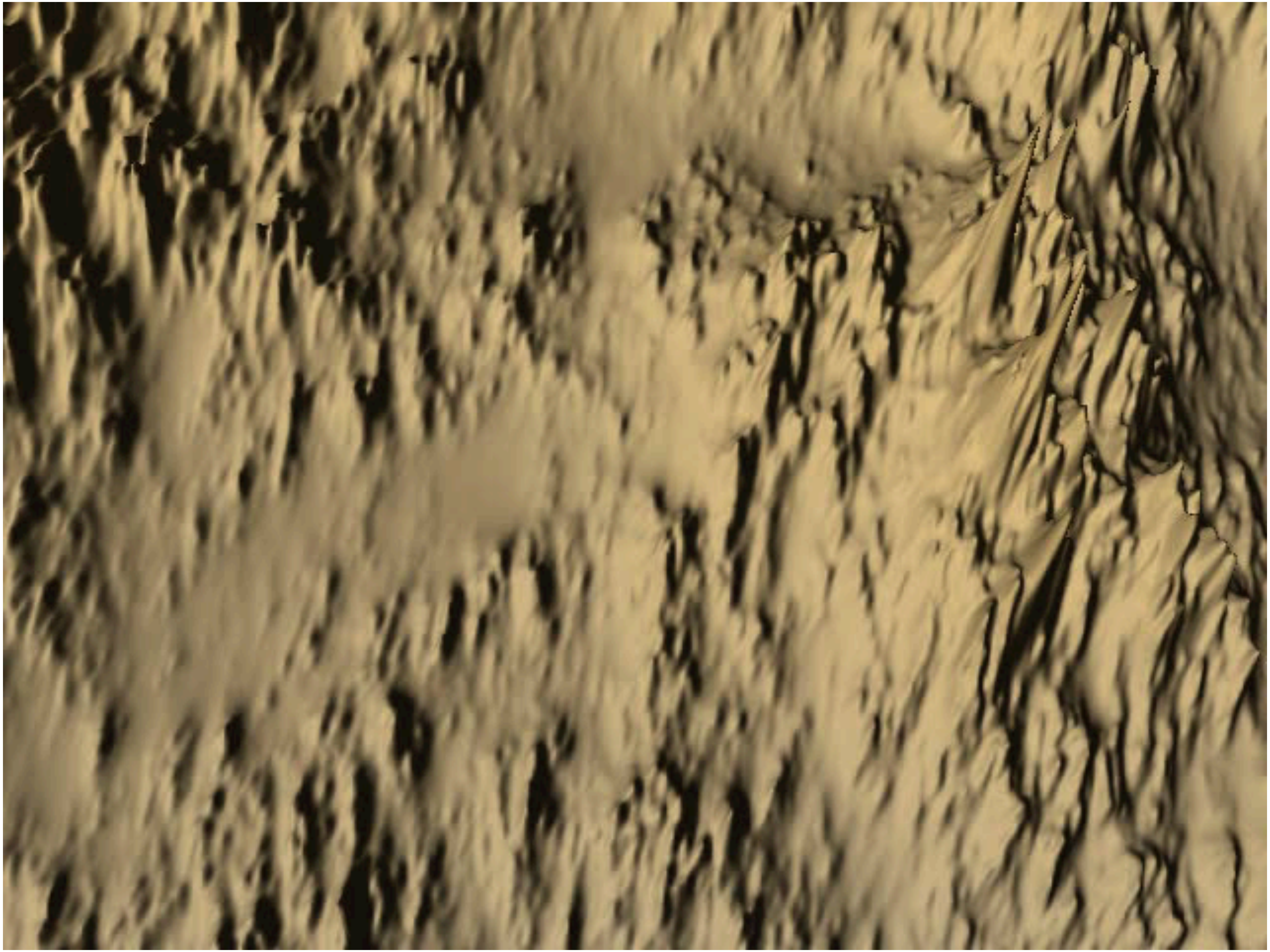
This 4096X4096 simulation is flown over

α=1.8, C_1=0.12, H=0.7

$$G = \begin{pmatrix} 0.65 & -0.1 \\ 0.1 & 1.35 \end{pmatrix}$$

l_s=64 pixels



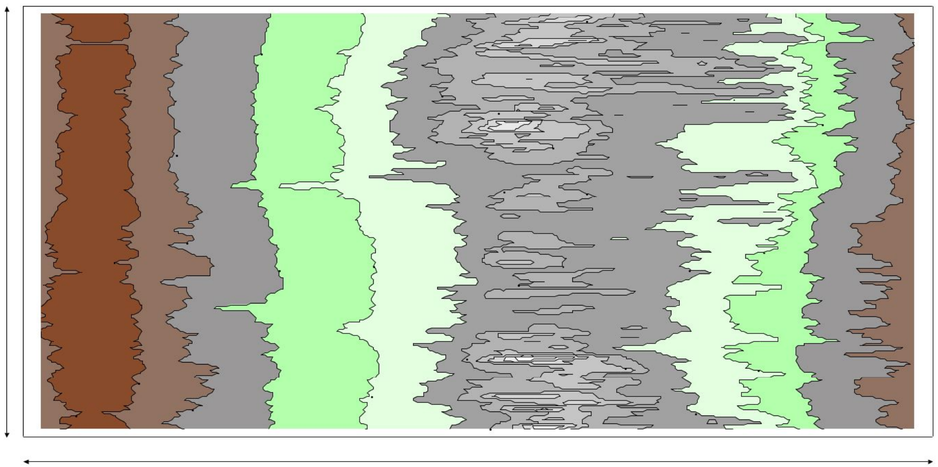


Stratified Multifractal Crust, Mantle rock density simulation

Vertical cross-sections $D_{el}=3$

Lithospheric rock density

128km

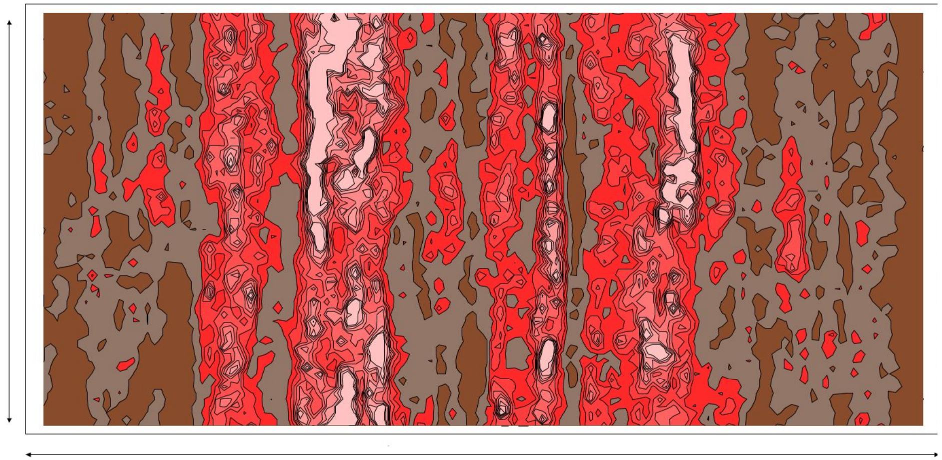


512km

Sphero-scale l_s=256km, with 1 pixel = 1km.

Mantle density

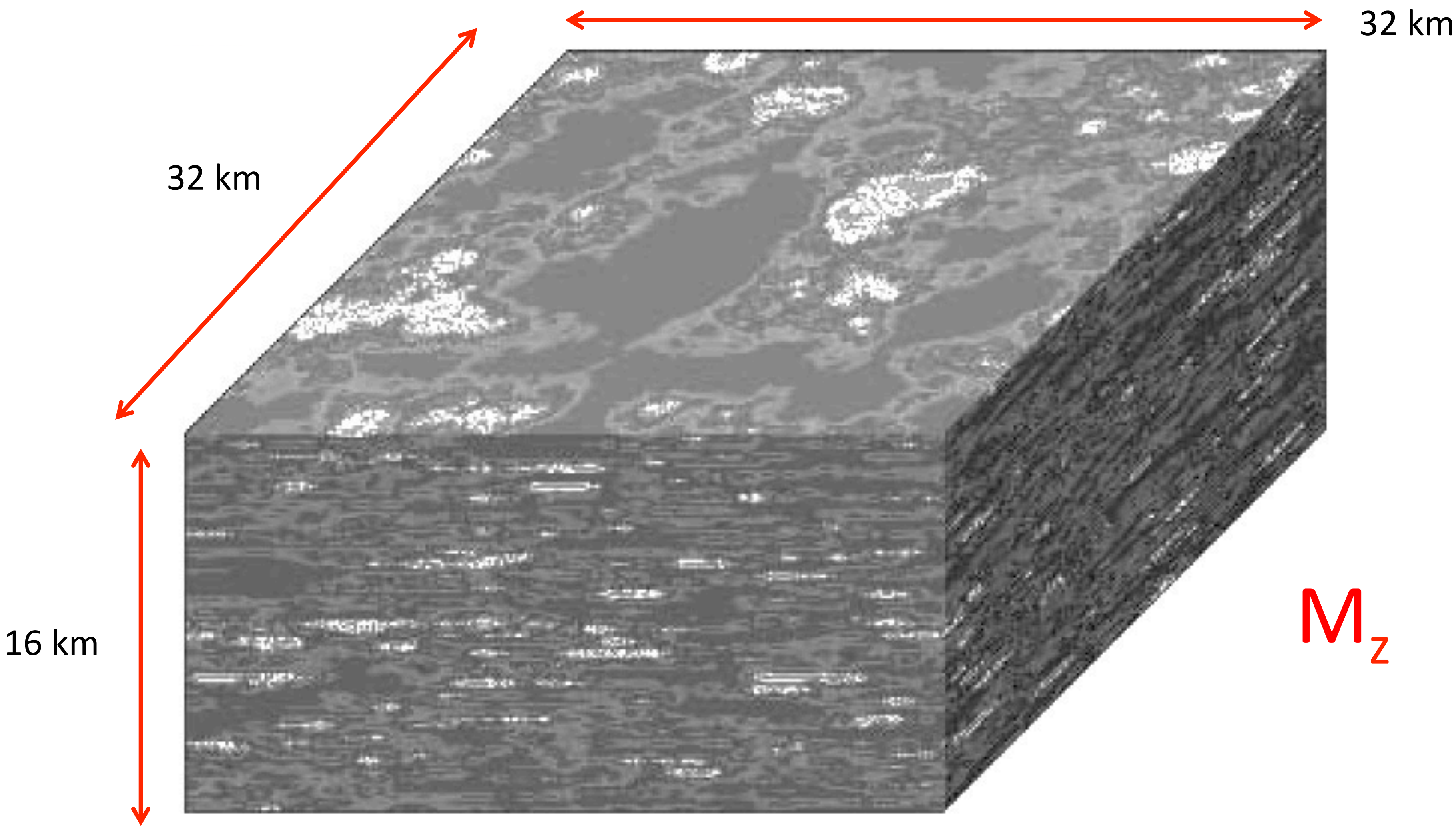
3000km



6000km

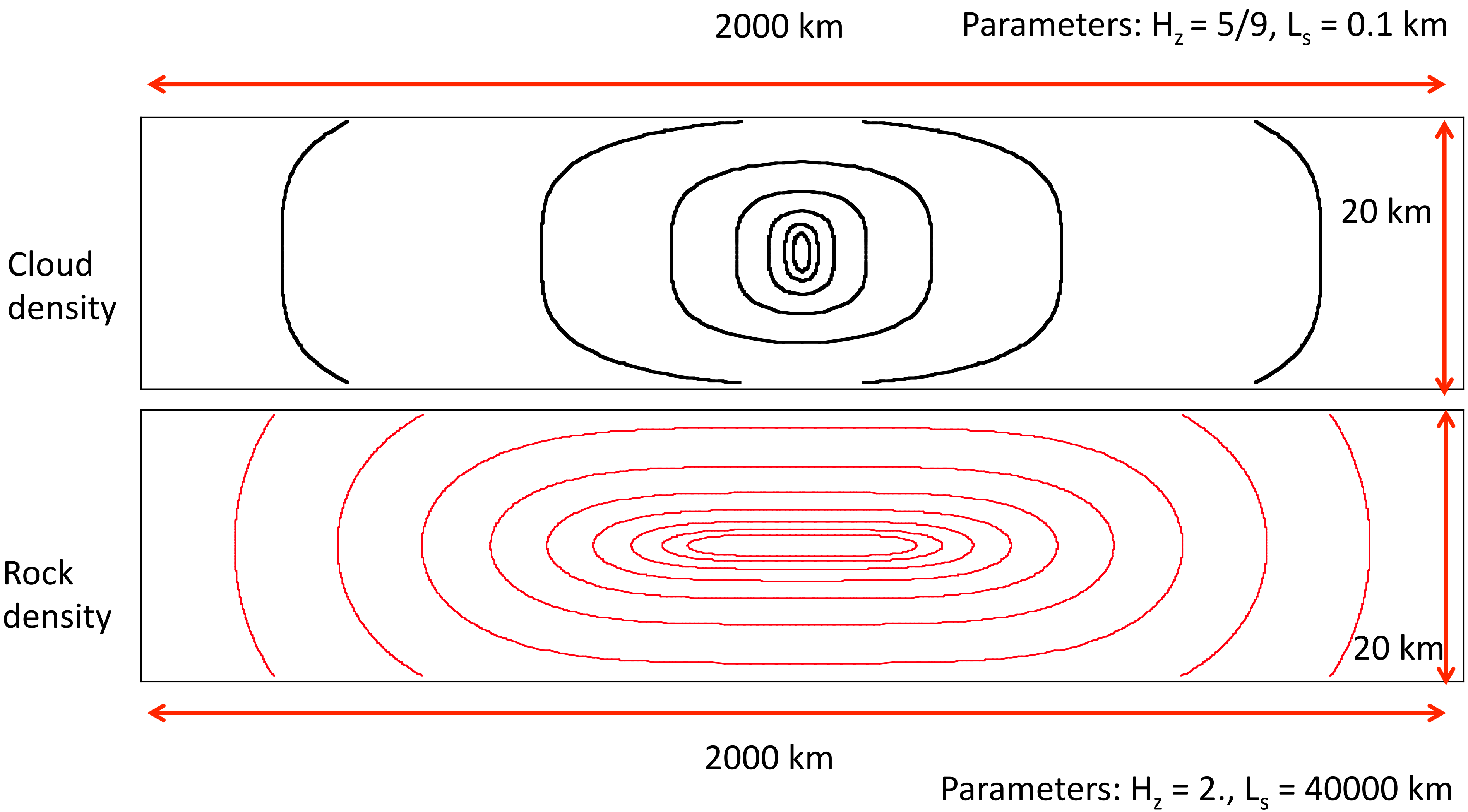
Sphero scale = 1 pixel. Each pixel is 50 km, sphero-scale = 25km. Hot (low density) plumes shown as white/red (this is a model for either density or temperature fluctuations (the two being proportional; we assume constant expansion coefficient). These are for fluctuations with respect to the mean vertical profile

Simulated magnetization field for horizontally isotropic crustal magnetization



Parameters: are H_z =1.7, s = 4, H = 0.2, α =1.98, C_1 = 0.08, l_s = 2500 km,

The unity of geosciences: clouds and rocks

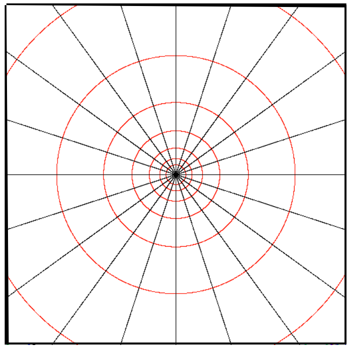


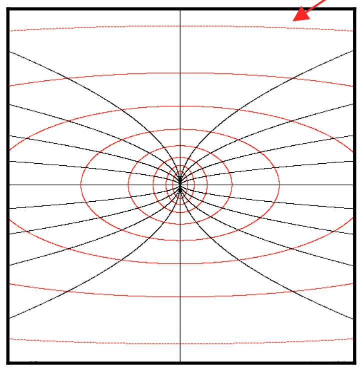
aspect ratios = 1/5

Scale functions in linear GSI (position independent)

Scale isolines in red

Isotropic (self similar)





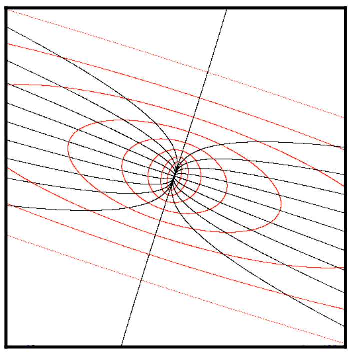
Self-affine

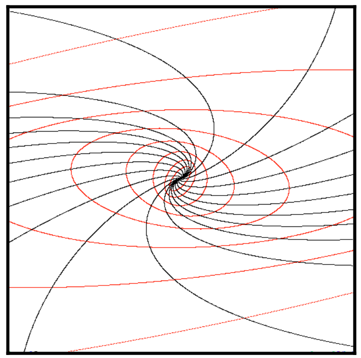
$$T_\lambda = \lambda^{-G}$$

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$G = \begin{pmatrix} 1.35 & 0 \\ 0 & 0.65 \end{pmatrix}$$

Stratification dominant (real eigenvalues)





Rotation dominant (complex eigenvalues)

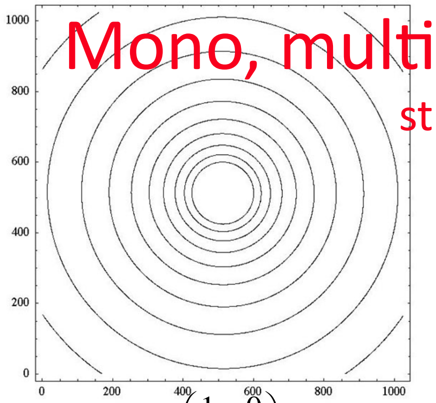
$$G = \begin{pmatrix} 1.35 & 0.25 \\ 0.25 & 0.65 \end{pmatrix}$$

$$G = \begin{pmatrix} 1.35 & -0.45 \\ 0.85 & 0.65 \end{pmatrix}$$

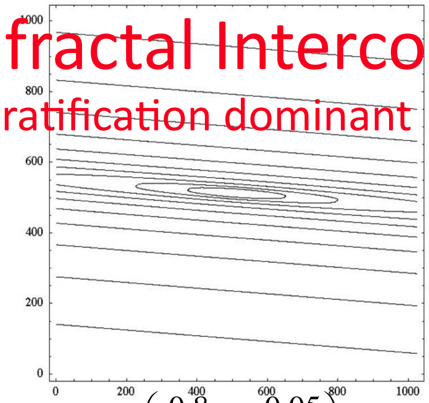
Mono, multifractal Intercomparison,

stratification dominant

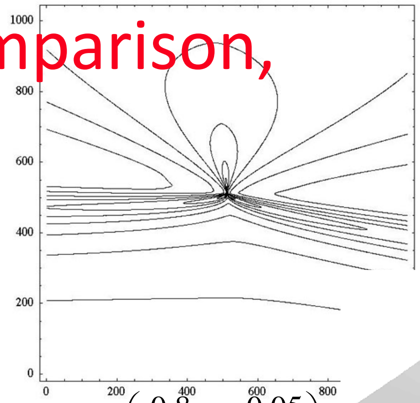
Contours of the s
functions



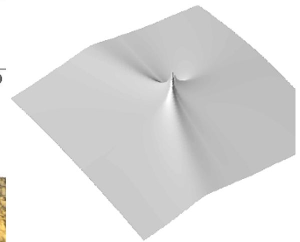
$$G=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$



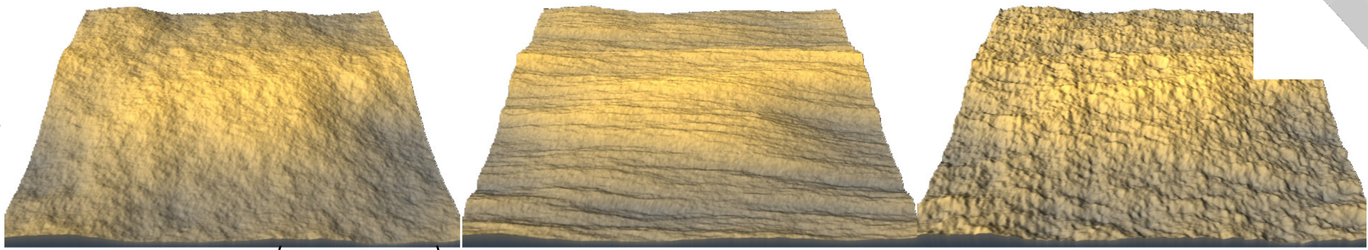
$$G=\begin{pmatrix}0.8 & -0.05\\ 0.05 & 1.2\end{pmatrix}$$



$$G=\begin{pmatrix}0.8 & -0.05\\ 0.05 & 1.2\end{pmatrix}$$

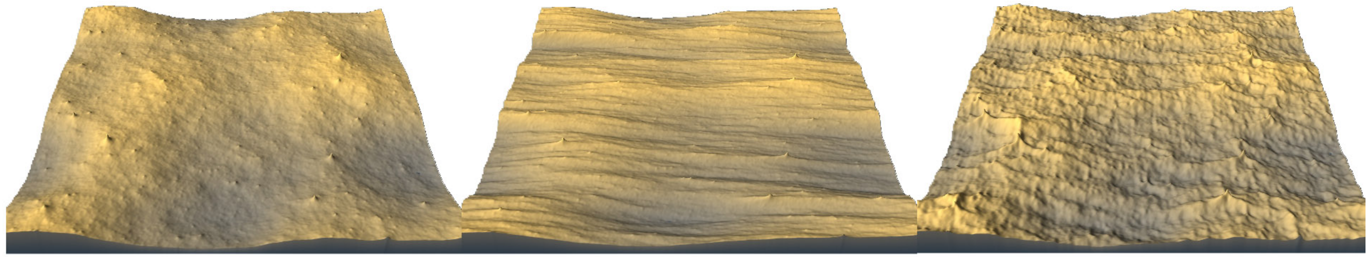


Fractional
Brownian motion,
H=0.7



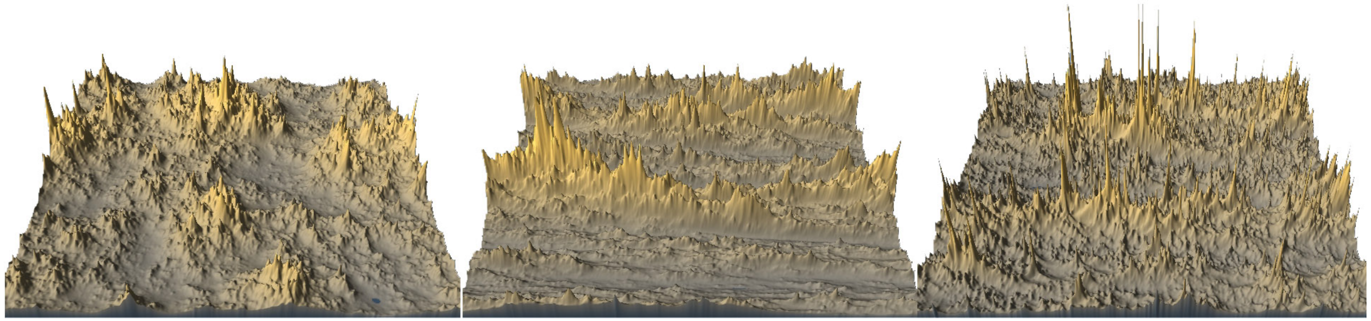
$$\left\langle \Delta h(\Delta r)^{q}\right\rangle \approx \Delta r^{qH-K(q)}\qquad K(q)=0$$

Fractional Levy
motion,
H=0.7, α =1.8



Multifractal FIF
H=0.7, α =1.8,
C_1=0.12

$$K(q)=\frac{C_1}{\alpha-1}\left(q^{\alpha}-q\right)$$

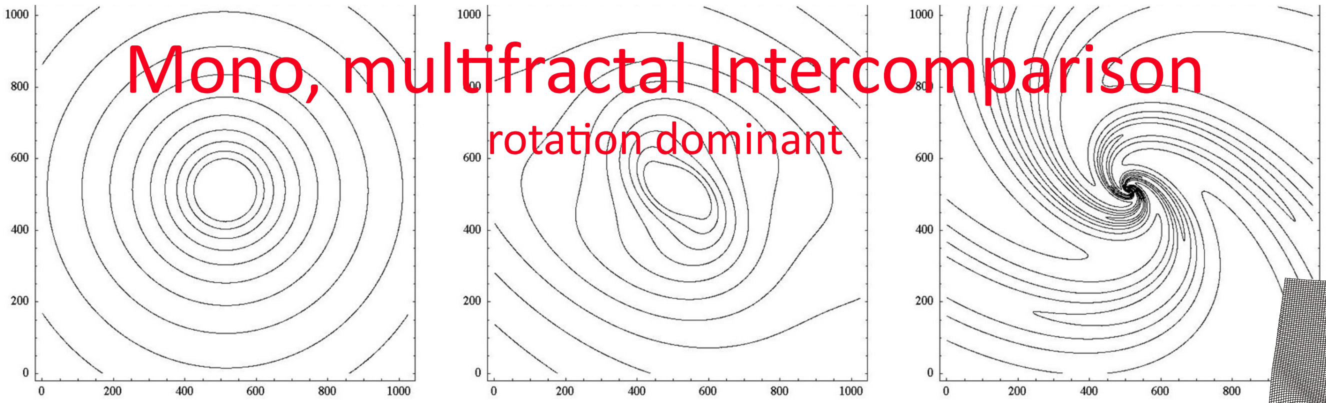


isotropic | Anisotropic no trivial anisotropy | Anisotropic with trivial anisotropy

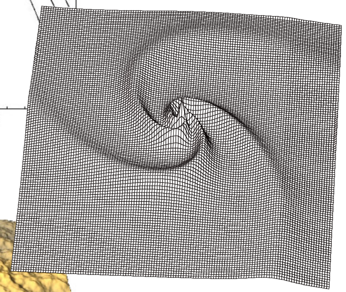
Mono, multifractal Intercomparison

rotation dominant

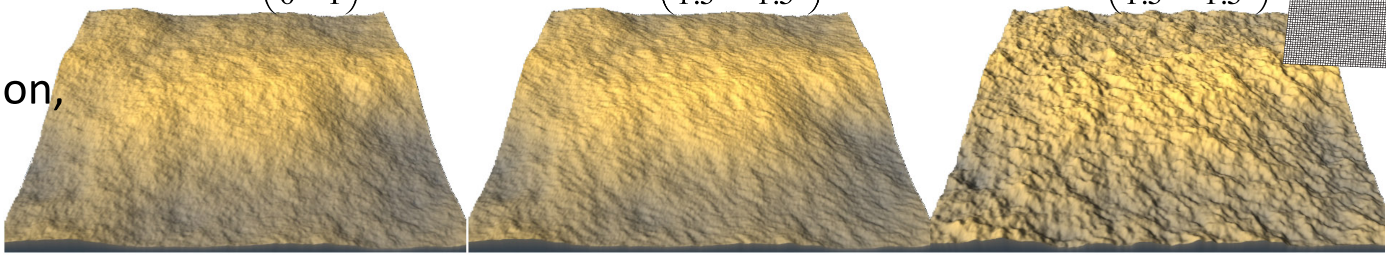
Contours of the scale functions



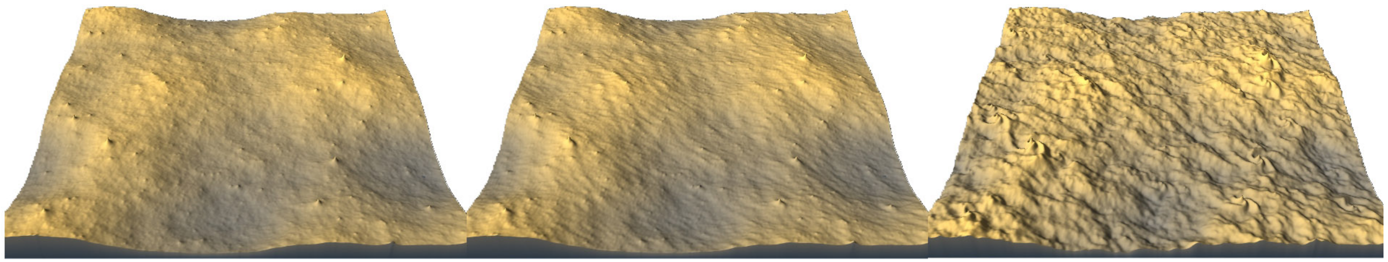
$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $G = \begin{pmatrix} 0.5 & -1.5 \\ 1.5 & 1.5 \end{pmatrix}$ $G = \begin{pmatrix} 0.5 & -1.5 \\ 1.5 & 1.5 \end{pmatrix}$



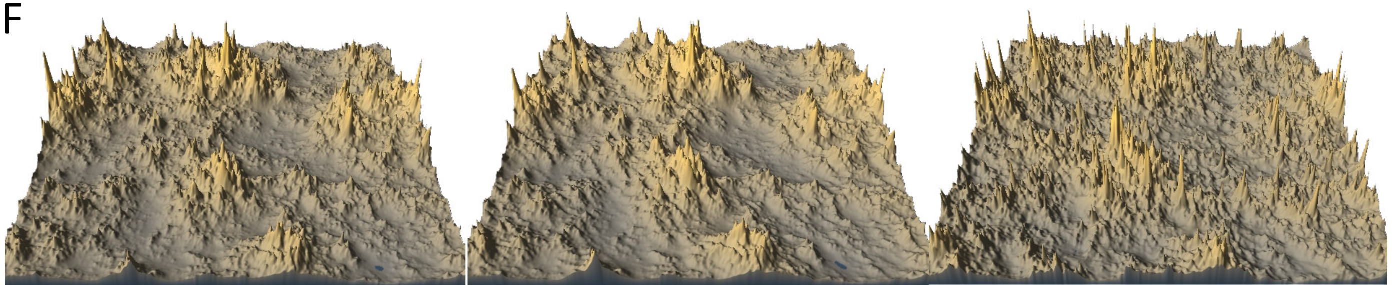
Fractional Brownian motion, H=0.7



Fractional Levy motion, H=0.7, α=1.8



Multifractal, FIF H=0.7, α =1.8, C_1=0.12



isotropic Anisotropic no trivial anisotropy Anisotropic with trivial anisotropy

Effect of varying C_1, H (self-affine, l_s=1)

C_1=0.05

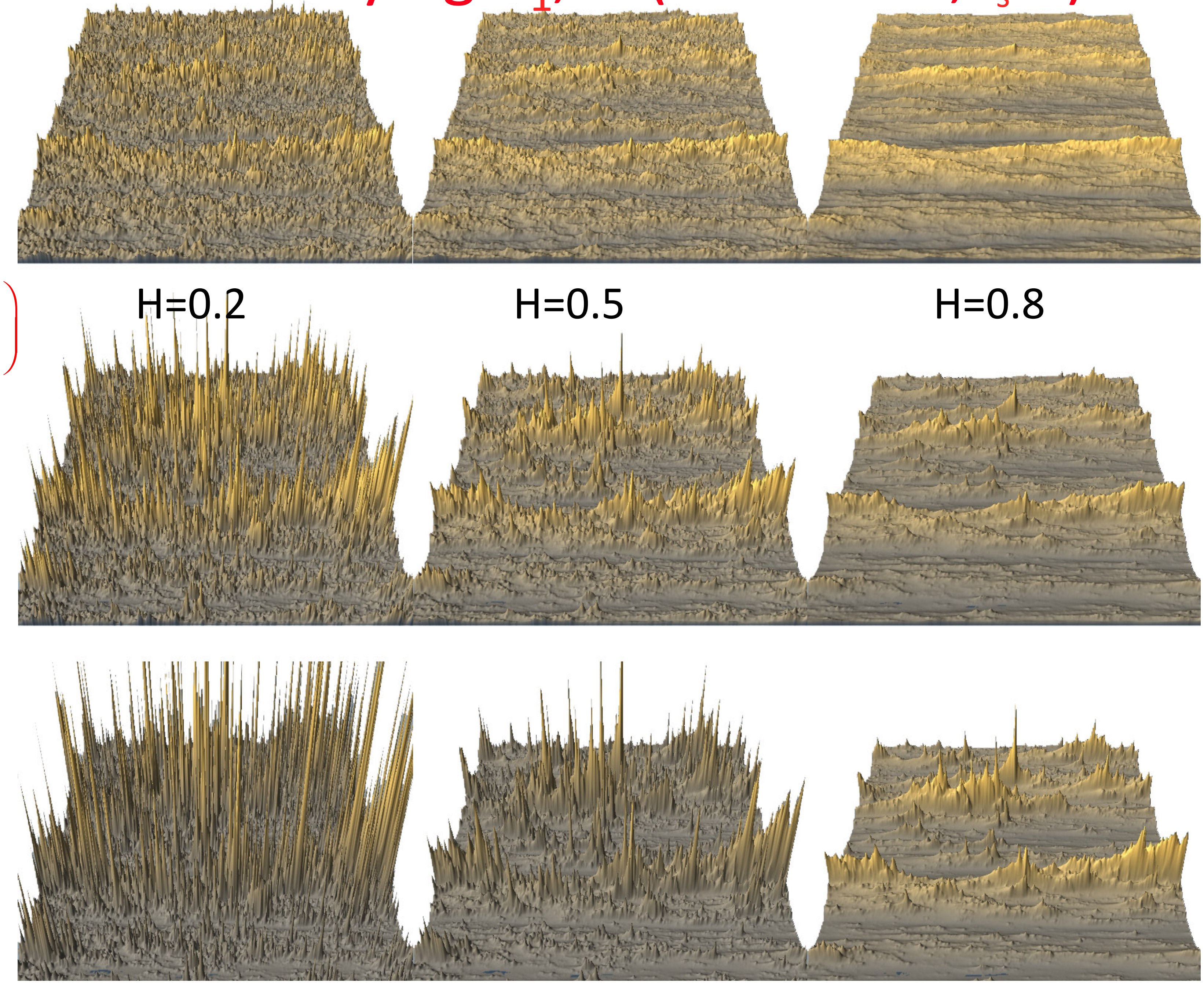
All:

α=1.8

$$G = \begin{pmatrix} 0.8 & 0 \\ 0 & 1.2 \end{pmatrix}$$

C_1=0.15

C_1=0.25



Effect of varying C_1, H (self-affine, $l_s=64$)

$C_1=0.05$

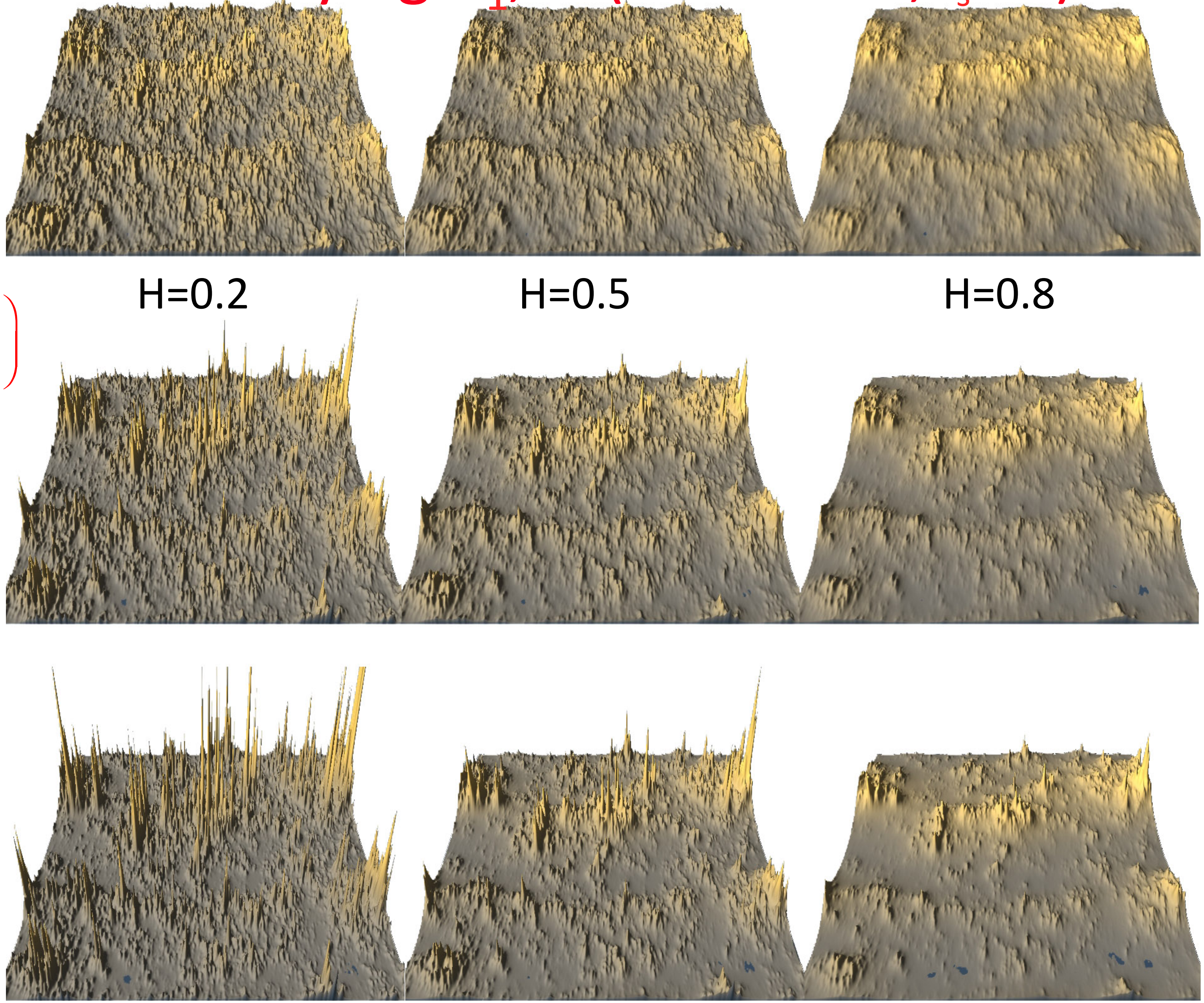
All:

$\alpha=1.8$

$$G=\begin{pmatrix} 0.8 & 0 \\ 0 & 1.2 \end{pmatrix}$$

$C_1=0.15$

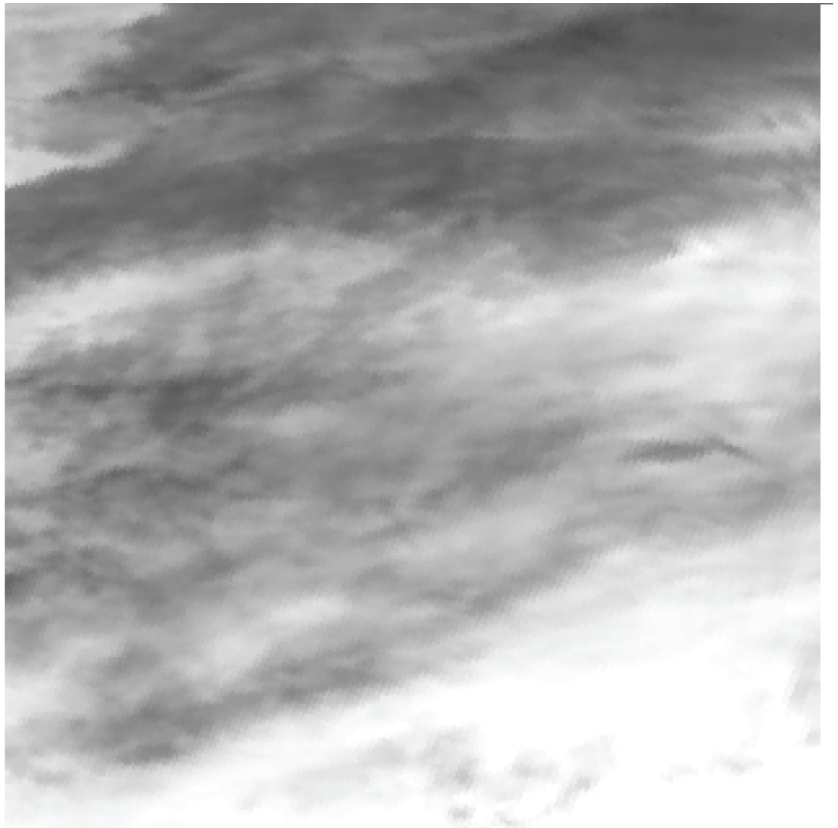
$C_1=0.25$



H=0.2 H=0.5 H=0.8

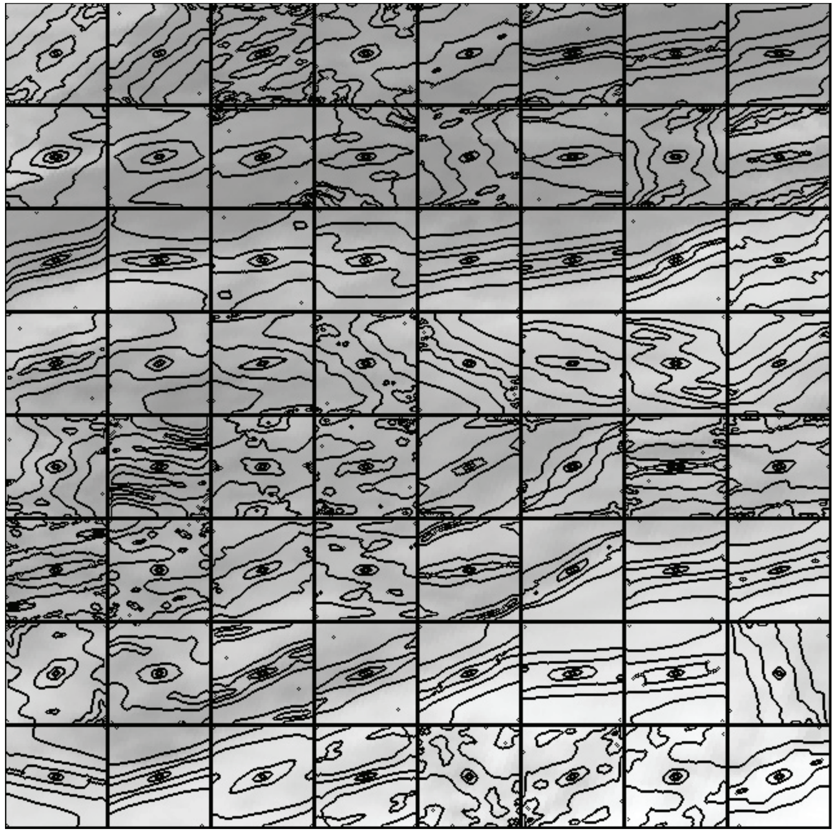
Nonlinear GSI

IR satellite picture



An infra red satellite image from a NOAA AVHRR satellite at 1.1 *km* resolution, 512x512 pixels

2D structure function for small sections



Contours of $S_2(\underline{\Delta r})$ estimated for each 64x64 pixel box from the image at left.

The generator of the infinitesimal scale change $g(x)$ and Nonlinear GSI

To go beyond linear GSI whose generator G is a fixed matrix, one first considers infinitesimal scale transformations; we will consider reductions of scale by a finite $\Delta\lambda$ and then take the small scale limit.

Consider the vector r_λ obtained by reducing the unit vector by a scale ratio λ:

$$\underline{r}_\lambda = \lambda^{-G}\underline{r}_1$$

In order to change the scale of the vector $\underline{r}_\lambda$ by $\Delta\lambda$, we need to reduce it by a scale ratio $1+\Delta\lambda/\lambda$:

$$\underline{r}_\lambda + \underline{\Delta r}_\lambda = \left(1+\frac{\Delta\lambda}{\lambda}\right)^{-G}\underline{r}_\lambda$$

hence dropping the indices and taking the limit $\Delta\lambda \to d\lambda$ we obtain:

$$d\underline{r} = -\frac{d\lambda}{\lambda}G\cdot\underline{r}$$

The nonlinear generalization of this is obtained by introducing the infinitesimal (generally nonlinear) generator $\underline{g}(\underline{r})$:

$$d\underline{r} = -\frac{d\lambda}{\lambda}\underline{g}(\underline{r})$$

Relation between linear and nonlinear GSI

Linear GSI is the special case where $\underline{g}(\underline{r})$ is linear and G is therefore the (fixed) Jacobian matrix of $\underline{g}$:

$$G_{ij} = \frac{\partial g_i}{\partial x_j}$$

where as usual, $\underline{r} = (x_1, x_2, x_3)$. To keep closer links to the linear case, this can be written in terms of the infinitesimal operator G_{op} defined as:

$$G_{op}\underline{r} = \underline{g}(\underline{r})$$

So that:

$$d\underline{r} = -\frac{d\lambda}{\lambda} G_{op}\underline{r}$$

This can (at least formally) be integrated to obtain:

$$\underline{r}_\lambda = \lambda^{-G_{op}} \underline{r}_1$$

($\underline{r}_1$ is a unit vector, $\underline{r}_\lambda$ is a unit vector reduced by a factor λ). In this way we can keep the power law notation for the scale change operator T_λ:

$$T_\lambda = \lambda^{-G_{op}}$$

The scale function equation

For any vector, T_λ increases scale by a factor λ, therefore as usual, the scale function has the basic property:

$$\left\| T_\lambda \underline{r} \right\| = \lambda^{-1} \left\| \underline{r} \right\|$$

We can now obtain the basic equation for the scale function. Consider the scale of a vector reduced from scale λ to scale λ+Δλ, as above by the reduction factor (1+Δλ/λ). The basic scale function equation $\left\| T_\lambda \underline{r} \right\| = \lambda^{-1} \left\| \underline{r} \right\|$ becomes:

$$\left\| \left(1+\frac{\Delta\lambda}{\lambda}\right)^{-G_{op}} \underline{r} \right\| = \left(1+\frac{\Delta\lambda}{\lambda}\right)^{-1} \left\| \underline{r} \right\|$$

If we now perform Taylor series expansions and take the limit Δλ->0, and using $G_{op} = \underline{g}(\underline{r})$ we obtain the basic equation for the scale function:

$$g_i \frac{\partial}{\partial x_i} \left\| \underline{r} \right\| = \left\| \underline{r} \right\|$$

summing over the indices *i*, or in vector form:

$$\left(\underline{g}(\underline{r}) \cdot \nabla \right) \left\| \underline{r} \right\| = \left\| \underline{r} \right\|$$

The solution of the scale function equation

In the special case of linear GSI this yields:

$$\underline{r}^T \cdot G^T \cdot \nabla \|\underline{r}\| = \|\underline{r}\|$$

As expected, to solve this partial differential equation for the scale function, we can use the same series of transformations of variables as used to solve the scale function equation previously:

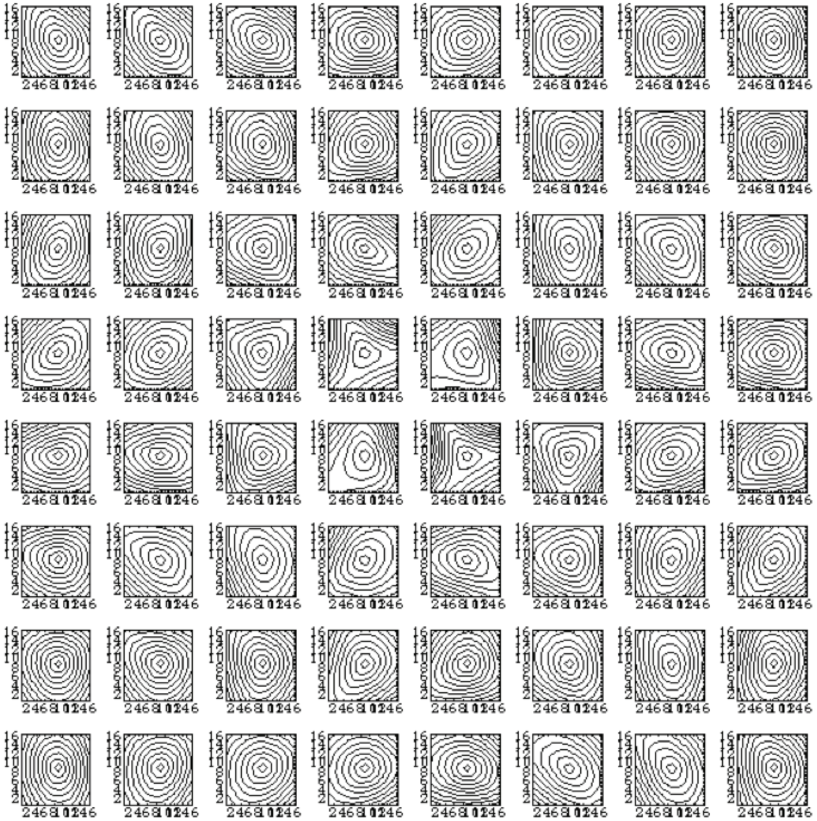
$$\frac{\partial}{\partial \log R^{(2)}} \log \|\underline{r}\| = 1$$

whose general solution is:

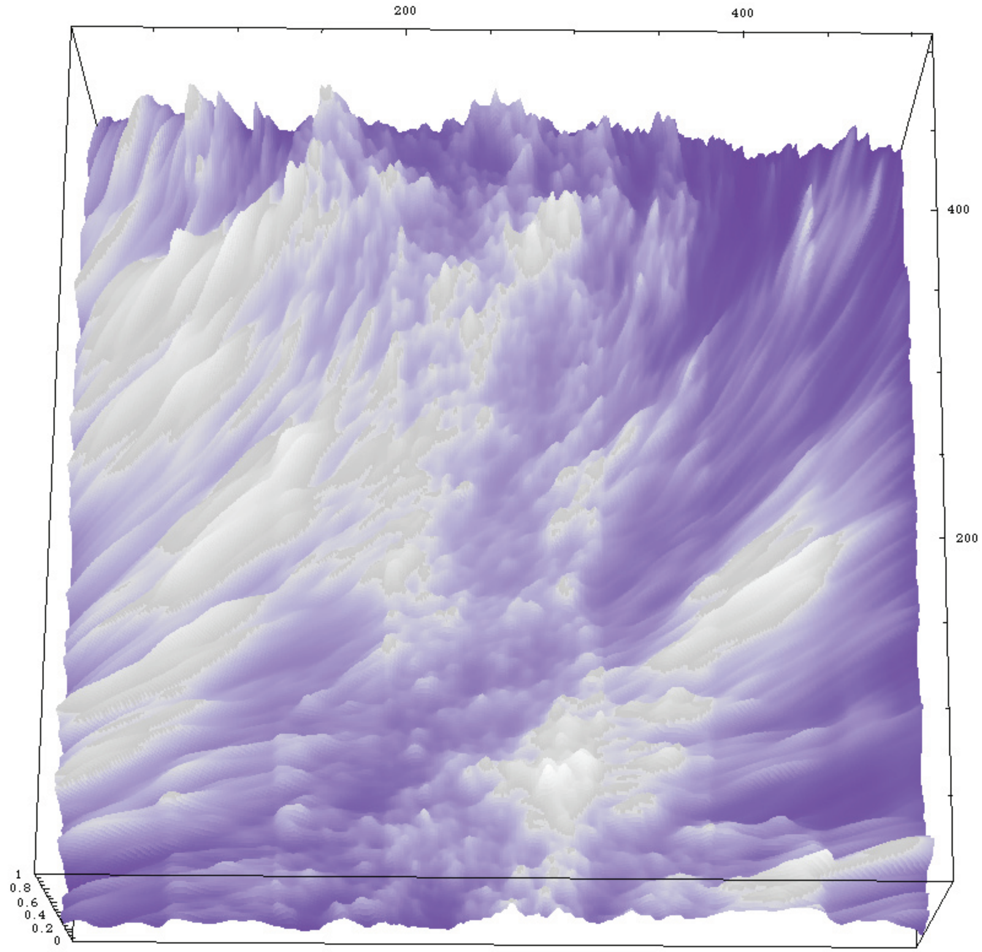
$$\|\underline{r}\| = R^{(2)} \Theta\left(\theta^{(2)}\right)$$

where $R^{(2)}$ is the polar coordinate representation of $(x^{(2)}, y^{(2)})$ and where Θ (an arbitrary function of angle) here appears as a function of integration.

Example of nonlinear GSI scale functions



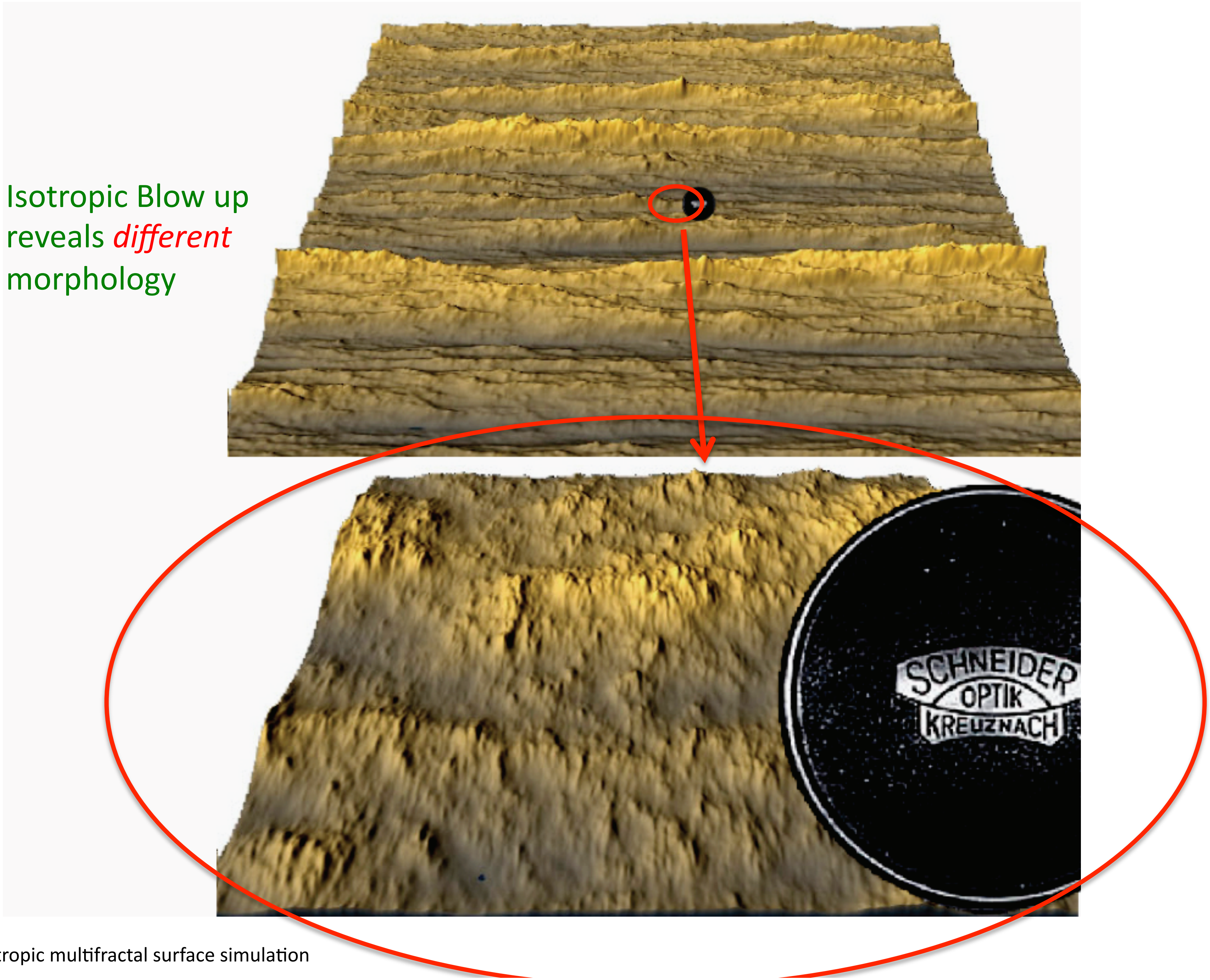
Nonlinear GSI Simulation



Phenomenological Fallacy

1) Morphology not dynamics is taken as fundamental

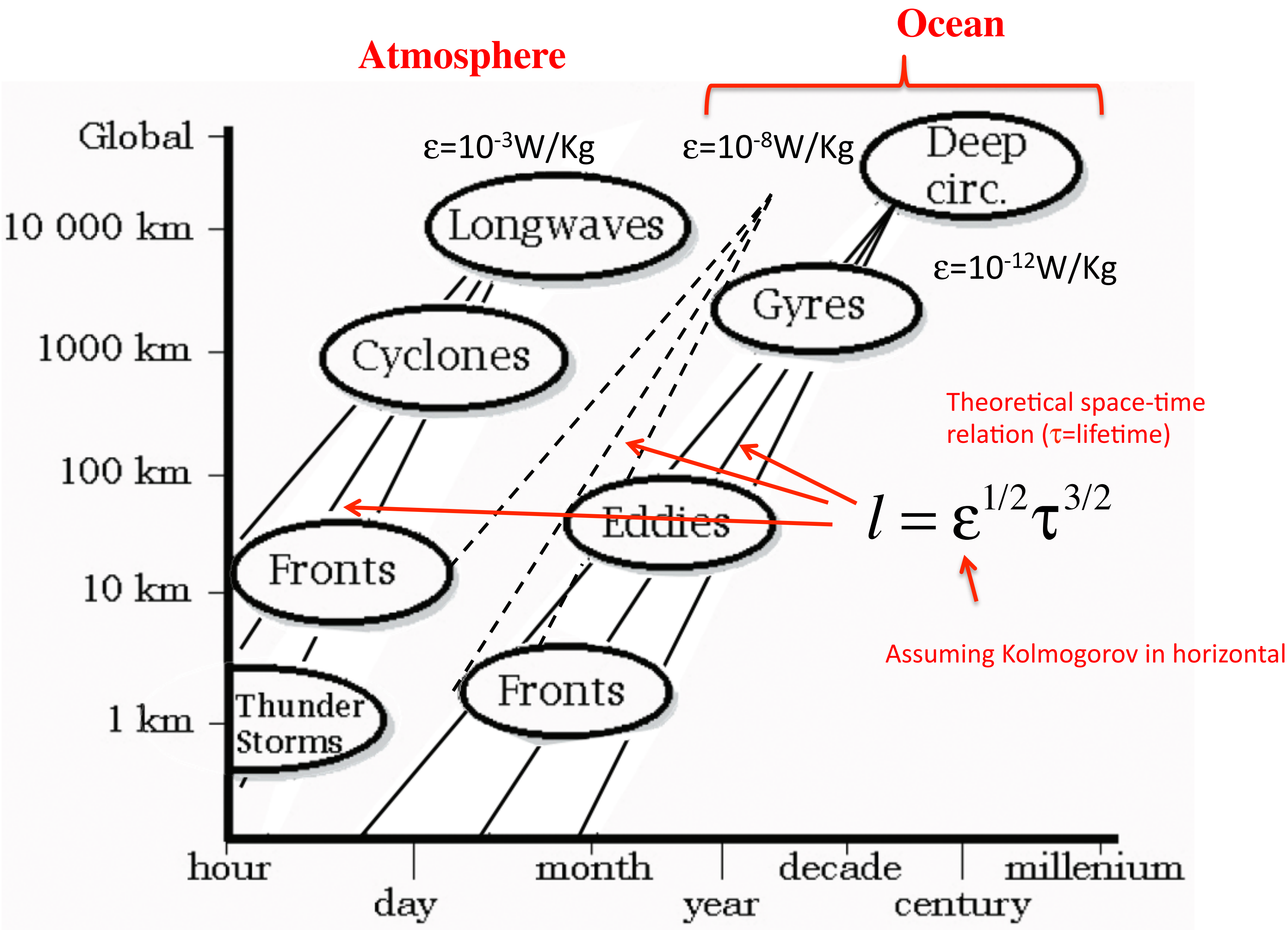
2) Scaling is reduced to the isotropic (self-similar) special case

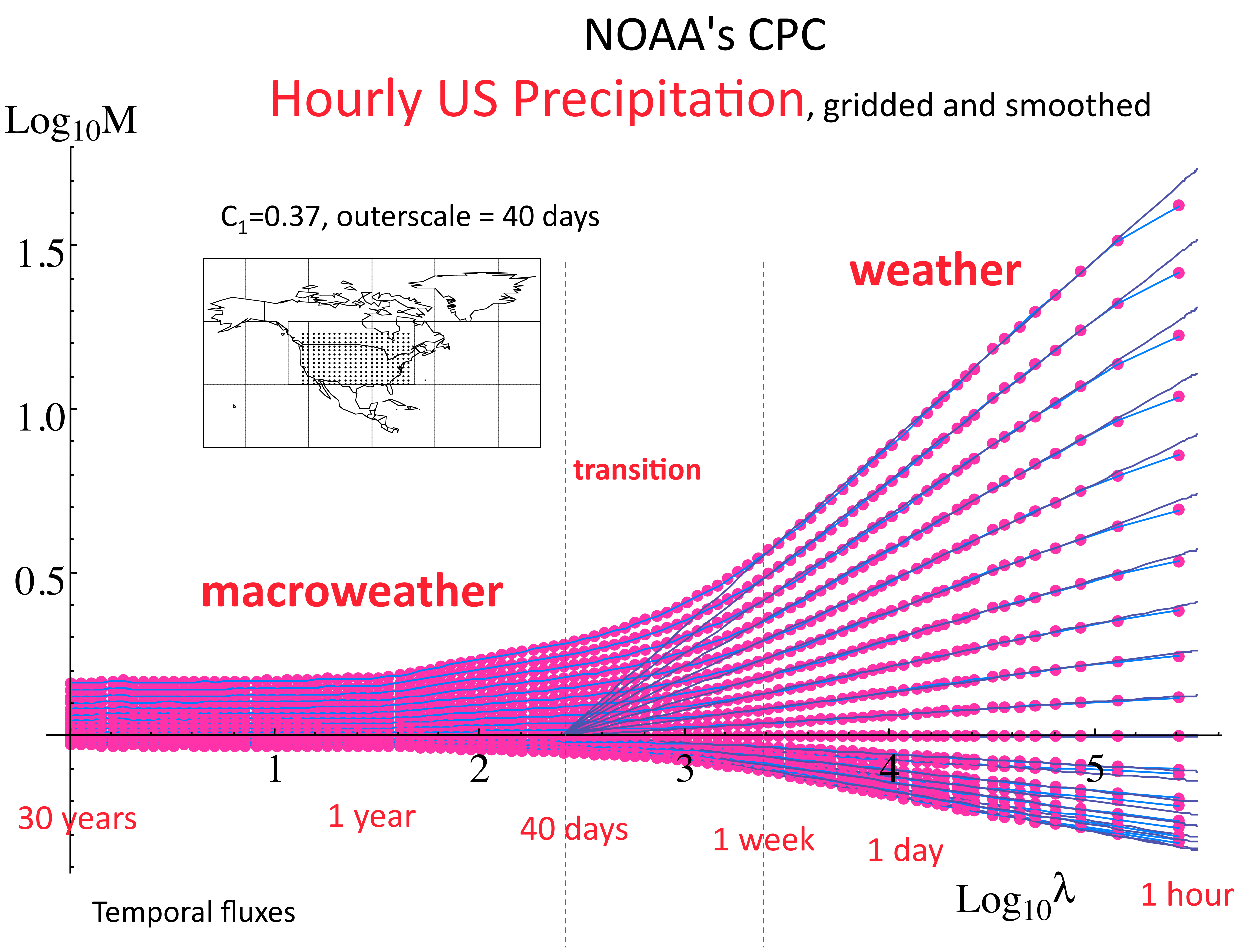


Anisotropic multifractal surface simulation

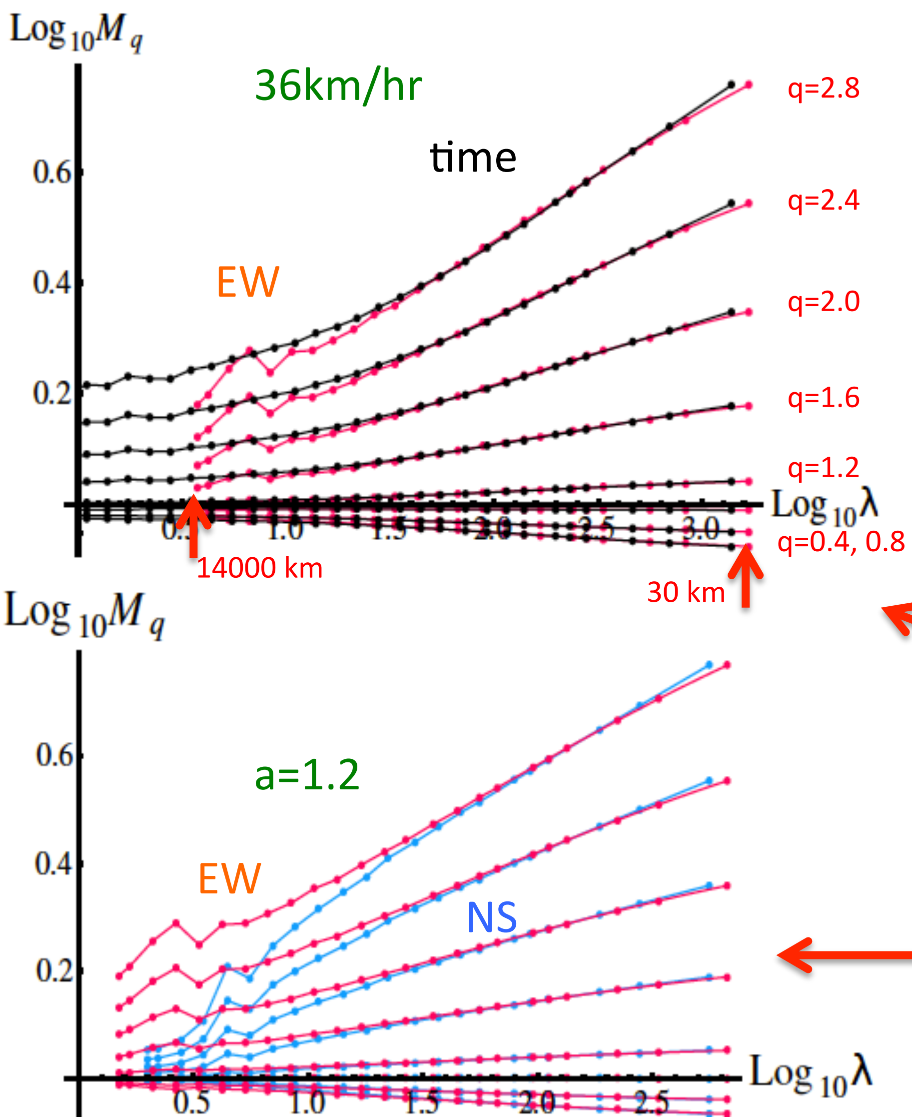
Extension from space to space-time (including waves)

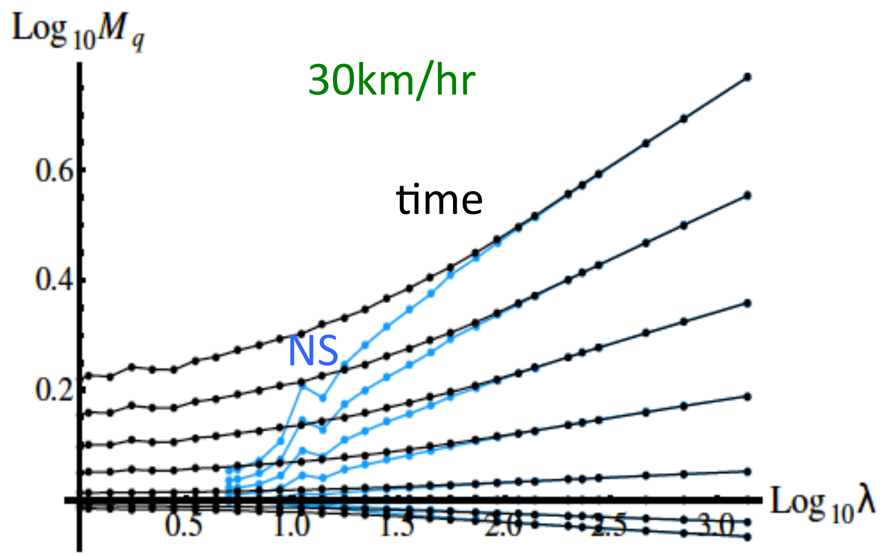
Space-Time ("Stommel") diagramme




NOAA's CPC
Hourly US Precipitation, gridded and smoothed
$Log_{10}M$
C_1=0.37, outerscale = 40 days
1.5
1.0
0.5
weather
transition
macroweather
1
2
3
4
5
30 years
1 year
40 days
1 week
1 day
$Log_{10}\lambda$
1 hour
Temporal fluxes

1400 MTSAT IR images 30°S - 40°N, Pacific
(Spectrum, 1-D subspaces)





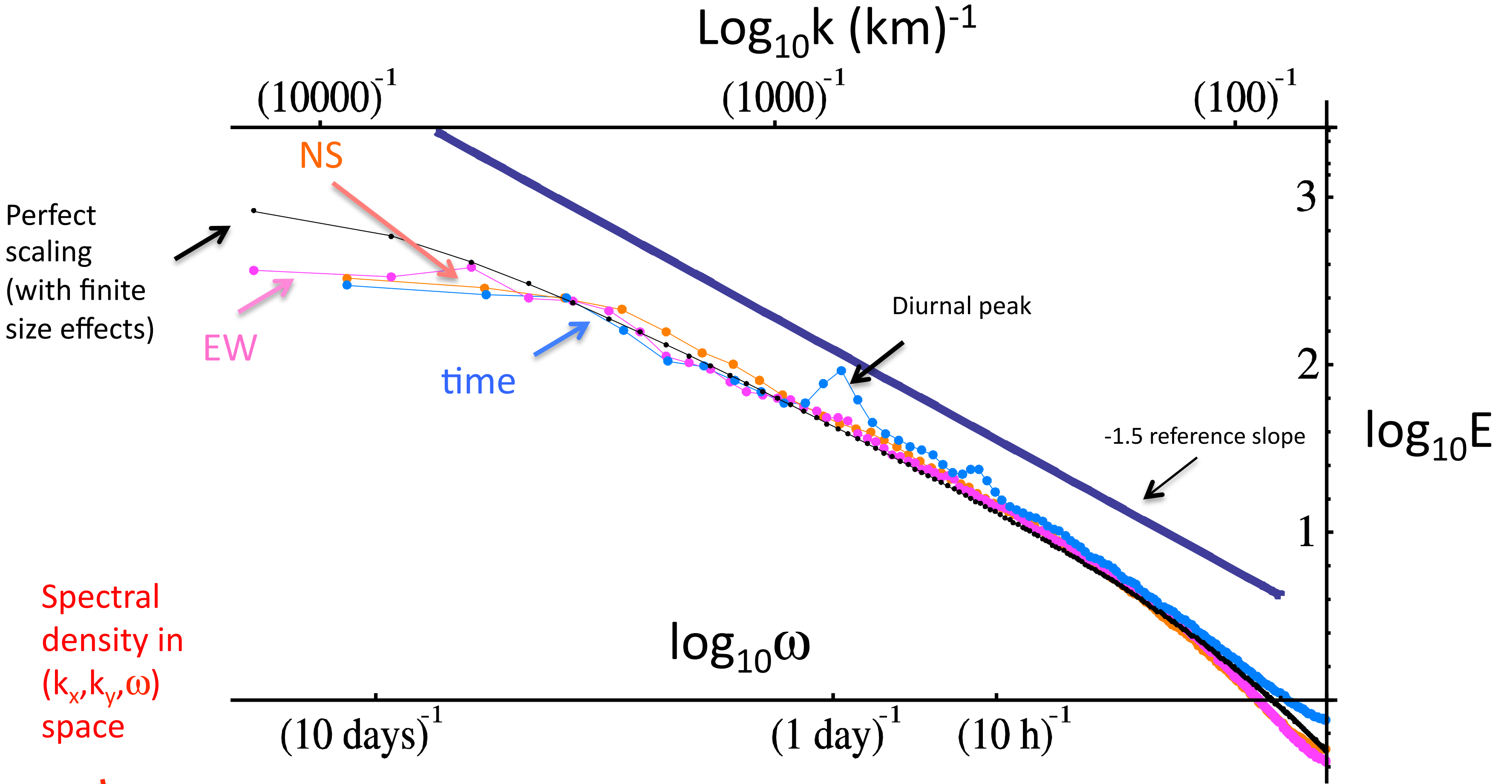
Superposition of temporal and spatial trace moments

Superposition of EW and NS spatial trace moments

1400 MTSAT IR images

30°S - 40°N, Pacific (Spectrum, 1-D subspaces)

1 hour, 30 km resolution



Spectral density in (k_x,k_y,ω) space

Space-time scaling is accurately respected:

$$P\left(\lambda^{-1}\left(\underline{k},\omega\right)\right)=\lambda^{s}P\left(\left(\underline{k},\omega\right)\right) \quad \longleftrightarrow \quad E\left(k_x\right)\approx k_x^{-\beta};\quad E\left(k_y\right)\approx k_y^{-\beta};\quad E\left(\omega\right)\approx\omega^{-\beta}$$

$\beta = s-2$

$\underline{k}=(k_x,k_y)$

Causal and acausal impulse response functions and fractional derivatives

Consider the H^{th} order fractional derivative equation for the impulse response function $g(t)$ (the "Green's function"):

$$\frac{d^H g}{dt^H} = \delta(t)$$

Fractional differential equation for the Green's function

where $\delta(t)$ is the usual Dirac delta function. Fourier transforming both sides of the equation, we obtain:

$$(i\omega)^H \tilde{g}(\omega) = 1 \qquad \text{hence} \qquad \tilde{g}(\omega) = (i\omega)^{-H}$$

where we have used the fact that the Fourier transform of the δ function =1, that the Fourier transform of d/dt is $-i\omega$, and have indicated the Fourier transform by the tilde. This g can be used to solve the general inhomogeneous fractional differential equation:

$$\frac{d^H h}{dt^H} = f(t); \quad h = I^H f$$

Fractional differential equation for general forcing function

where $h(t)$ is the response to the forcing $f(t)$. We have written the equation both in differential and in the equivalent integral form where I^H is the H^{th} order integral operator, the inverse of d^H/dt^H. The solution of the above is thus:

$$\tilde{h} = \tilde{g}\tilde{f} = (i\omega)^{-H}\tilde{f} \overset{F.T.}{\leftrightarrow} h = g * f$$

Causal, acausal fractional integrations

We see that:

$$h = I^H f = g * f$$

"*" indicates convolution and we have used the fact that multiplication in Fourier space corresponds to convolution in real space.

where:

$$\tilde{g}(\omega) = (i\omega)^{-H} \overset{F.T.}{\leftrightarrow} \quad g(t) = \frac{\Theta_{Heavi}(t)\, t^{-(1-H)}}{\Gamma(H)}; \qquad \Theta_{Heavi}(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

$\Theta_{Heavi}(t)$ is the Heaviside function. Writing the final solution explicitly, we obtain:

$$h(t) = I_L^H f(t) = \frac{1}{\Gamma(H)} \int_{-\infty}^{t} (t-t')^{(H-1)} f(t')\, dt'$$

Value at t depends only on the past

Liouville (causal) fractional integration

The Riemann-Liouville ("RL") fractional integration:

$$I_{RL}^H f(t) = \frac{1}{\Gamma(H)} \int_{-\infty}^{\infty} |t-t'|^{H-1} f(t')\, dt'$$

Riemann-Liouville (symmetric, acausal) fractional integration

Note: the integrals only converge for 0<H<1. To fractionally integrate outside this range, you can combine usual differentiation/ integration with fractional differentiation/ integration

is based on the Green's function:

$$\tilde{g}(\omega) = |\omega|^{-H} \sqrt{\frac{2}{\pi}} \sin\frac{\pi}{2}(1-H) \overset{F.T.}{\leftrightarrow} \quad g(t) = \frac{|t|^{H-1}}{\Gamma(H)}$$

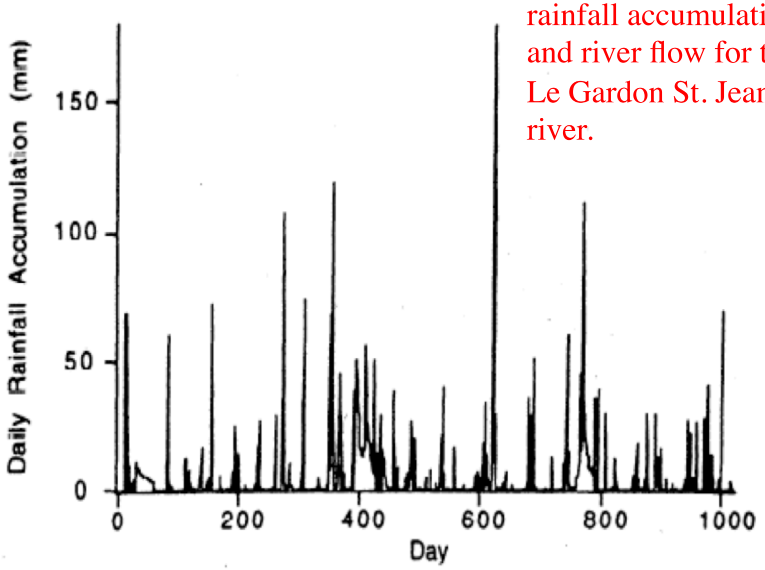
Example: rivers

The relation between rainfall ($R(t)$) in small river basins could be considered as the forcing of the corresponding river flow $Q(t)$ in the fractional differential equation:

$$\frac{d^H Q}{dt^H} = R(t); \quad R(t) = \frac{1}{\Gamma(H)} \int_{-\infty}^{t} (t - t')^{H-1} Q(t') dt'$$

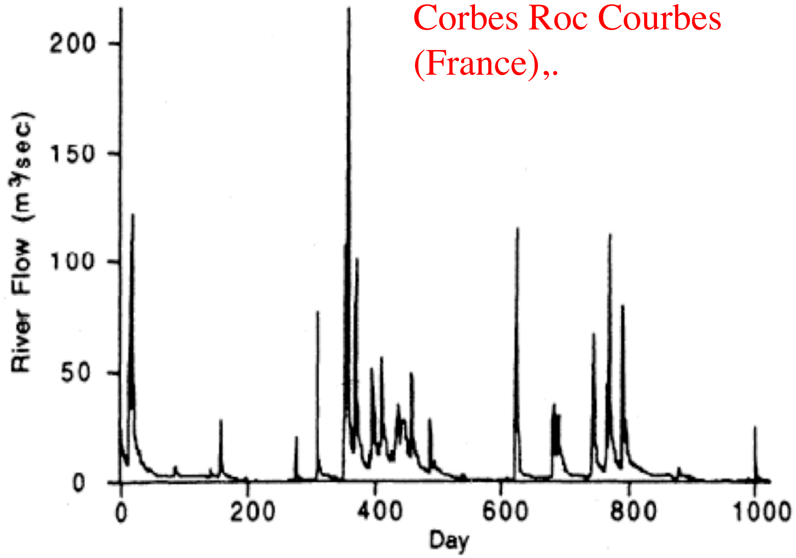
where empirically it was found that $H = 0.3$. In this hydrology context, the Green's function g corresponding to the Liouville fractional integral is called a "transfer function". Physically this convolution corresponds to a specific power law (scaling) "storage" model for the runoff and ground water processes which are thus assumed to be scaling over a wide range.

R(t)



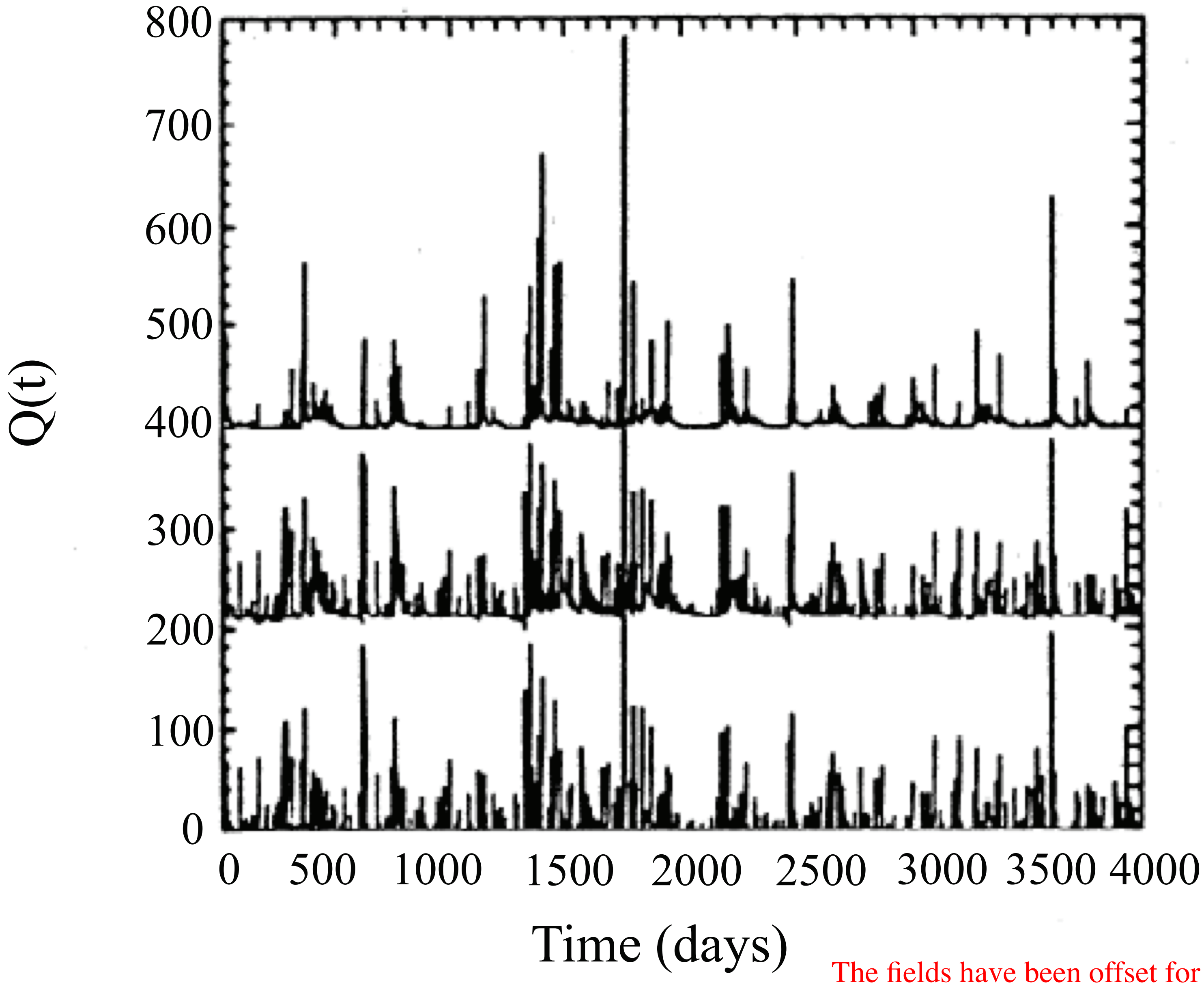
Simultaneous daily rainfall accumulation and river flow for the Le Gardon St. Jean river.

Q(t)



The nearby station of Corbes Roc Courbes (France),.

Rivers



River flow data

Rainfall data fractionally integrated

Rainfall data

The fields have been offset for clarity

Causality in Space-time: propagators

If we extend the above discussion to space-time, then the corresponding Green's function/impulse response function is called a "propagator". Let us consider as an example the propagator for the classical wave equation:

$$\left(\nabla^2 - \frac{1}{V^2}\frac{\partial^2}{\partial t^2}\right) g(\underline{r},t) = \delta(\underline{r},t)$$

Classical wave equation

where V is the wave speed. Taking the space-time fourier transform of both sides, we find:

$$\tilde{g}(\underline{k},\omega) = \left(\omega^2 / V^2 - |\underline{k}|^2\right)^{-1}$$

Classical wave propagator

Due to the negative sign, the character of this propagator is totally different from those obtained with a positive sign (relevant to space-time localized strucutres).

Its behaviour is totally dominated by the waves satisfying the relation $\omega^2 / V^2 = |\underline{k}|^2$ which makes the propagator singular, this is indeed the significance of this "dispersion" relation.

Dispersion relation

Turbulence and (fractional) propagators

Example: The classical wave equation with forcing

$$\left(\nabla^2 - \frac{1}{V^2}\frac{\partial^2}{\partial t^2}\right) I(\underline{r},t) = f(\underline{r},t)$$

Solution by Fourier transforms

propagator

$$\tilde{I}(\underline{k},\omega) = \tilde{g}(\underline{k},\omega)\tilde{f}(\underline{k},\omega) \qquad \tilde{g}(\underline{k},\omega) = \left(\omega^2/V^2 - \left|\underline{k}\right|^2\right)^{-1}$$

$$\tilde{g}\left(\lambda^{-1}\left(\underline{k},\omega\right)\right) = \lambda^H \tilde{g}\left(\left(\underline{k},\omega\right)\right); \quad H = 2$$

Isotropic scale change symmetry

Fractional wave equation

$$\left(\nabla^2 - \frac{1}{V^2}\frac{\partial^2}{\partial t^2}\right)^{H/2} I(\underline{r},t) = f(\underline{r},t) \quad ; \quad \tilde{g}(\underline{k},\omega) = \left(\omega^2/V^2 - \left|\underline{k}\right|^2\right)^{-H/2}$$

Note: the dispersion relation is independent of H (>0)

Spatial turbulence

Isotropic

Turbulent law (space)

$$\Delta I\left(\underline{\Delta r}\right) = \varphi\left|\underline{\Delta r}\right|^{H}$$

FIF model

Turbulent law (Fourier space)

$$\tilde{I}\left(\underline{k}\right) = \tilde{g}_{tur}\left(\underline{k}\right)\tilde{\varphi}\left(\underline{k}\right)$$

Kolmogorov values

$$\varphi = \varepsilon^{1/3}; \quad H = 1/3$$

$$\tilde{g}_{tur}\left(\underline{k}\right) = \left|\underline{k}\right|^{-H}$$

Anisotropic extension

$$\left|\underline{k}\right| \to \left\|\underline{k}\right\|$$

Fourier scale function

Scaling equation

$$\left\|\lambda^{G}\underline{k}\right\| = \lambda\left\|\underline{k}\right\|$$

$$G = \begin{pmatrix} 1 & . & . \\ . & 1 & . \\ . & . & H_z \end{pmatrix}$$

Generator of the anisotropy

Canonical scale function
(vertical stratification)

$$\left\|\underline{k}\right\| = l_s^{-1}\left(\left(k_x l_s\right)^2 + \left(k_y l_s\right)^2 + \left(k_z l_s\right)^{2/H_z}\right)^{1/2}$$

Sphero-scale

Turbulence in Space-time (horizontal)

Theory (assuming largest eddies "sweep" smaller ones)

Observable

Turbulent flux forcing

$$g^{-1}(\underline{r},t) * I(\underline{r},t) = \varphi(\underline{r},t)$$

$$g(\underline{r},t) \overset{F.T.}{\leftrightarrow} \tilde{g}(\underline{k},\omega)$$

propagator

$$\tilde{I}(\underline{k},\omega) = \tilde{g}(\underline{k},\omega)\tilde{\varphi}(\underline{k},\omega)$$

$$\tilde{g}(\underline{k},\omega) = \left(-i\omega' + \left\|\underline{k}\right\|\right)^{-H_{tur}}$$

$$\omega' = \left(\omega + \underline{k}\cdot\underline{\mu}\right)\sigma^{-1} \qquad \left\|\underline{k}\right\| = \left(k_x^2 + k_y^2 / a^2\right)^{1/2}$$

$$\sigma = \left(1 - \left(\mu_x^2 + a^2\mu_y^2\right)\right)^{1/2}$$

$$\underline{\mu} = \left(\overline{v_x}, \overline{v_y}\right) / V_w \qquad V_w = \varepsilon_{L_e} L_e^{1/3}$$

EW/NS aspect ratio = a

mean horizontal wind= $\left(\overline{v_x}, \overline{v_y}\right)$

Mean planetary scale energy flux ε_{L_e}

Planet size: L_e = 20000 km

Turbulence and waves

Turbulence forcing

Turbulent flux

$$\tilde{I}(\underline{k},\omega)=\tilde{g}_I(\underline{k},\omega)\tilde{\varphi}(\underline{k},\omega)$$

Turbulence-waves

$$\tilde{g}_I(\underline{k},\omega)=\underbrace{\tilde{g}_{wav}(\underline{k},\omega)\tilde{g}_{tur}(\underline{k},\omega)}_{\text{Wheeler Kiladis 1999 factorization}}$$

Propagator symmetry constraints

Reality

$$\tilde{g}\left(\underline{k},\omega\right)=\tilde{g}^*\left(-\underline{k},-\omega\right)$$

Space-time scaling

$$\tilde{g}\left(\lambda^{-1}\left(\underline{k},\omega\right)\right)=\lambda^H\tilde{g}\left(\left(\underline{k},\omega\right)\right)$$

Causality

Poles of g in ω plane are below real axis:

$\omega=-i|\underline{k}|\Phi\left(\hat{k}\right)$ causal if $\mathrm{Re}\left(\Phi\left(\hat{k}\right)\right)>0$

$$\|\underline{k}\|=|\underline{k}|\Phi\left(\hat{k}\right);\quad \hat{k}=\frac{\underline{k}}{|\underline{k}|}$$

General form

$$\tilde{g}(\underline{K})=[\![\underline{K}]\!]^{-H}\qquad [\![\underline{K}]\!]=\left(-i\omega+\|\underline{k}\|\right)F\left(\widehat{K}\right)$$

$$\underline{K}=(\underline{k},\omega);\quad \underline{k}=\left(k_x,k_y\right)$$

Simple wave ansatz

Simple scaling wave propagator

Fractional (and anisotropic) wave equation propagator

$$\tilde{g}_{wav}(\underline{k},\omega)=\left(\omega'^2/v_{wav}^2-\left\|\underline{k}\right\|^2\right)^{-H_{wav}/2}\qquad H=H_{tur}+H_{wav}\qquad \begin{aligned}\omega'&=\left(\omega+\underline{k}\cdot\underline{\mu}\right)\sigma^{-1}\\ \left\|\underline{k}\right\|&=\left(k_x^2+k_y^2/a^2\right)^{1/2}\end{aligned}$$

Dispersion relation

$$\omega=-\underline{k}\cdot\underline{\mu}\pm\sigma v_{wav}\left\|\underline{k}\right\| \longleftarrow \omega'=\pm v_{wav}\left\|\underline{k}\right\|$$

Spectral density

Turbulent part Wave part

$$P_I(\underline{k},\omega)=P_\varphi(\underline{k},\omega)\left|\widetilde{g_I}\right|^2$$

$$\left|\widetilde{g_I}\right|^2=\left|\widetilde{g_{tub}}\right|^2\left|\widetilde{g_{wav}}\right|^2=\left(\omega'^2+\left\|\underline{k}\right\|^2\right)^{-H_{tur}}\left(\omega'^2/v_{wav}^2-\left\|\underline{k}\right\|^2\right)^{-H_{wav}/2}$$

$$P_\varphi(\underline{k},\omega)=P_0\left(\omega'^2+\left\|\underline{k}\right\|^2\right)^{-s_\varphi/2}$$

Spectrum of turbulence forcing

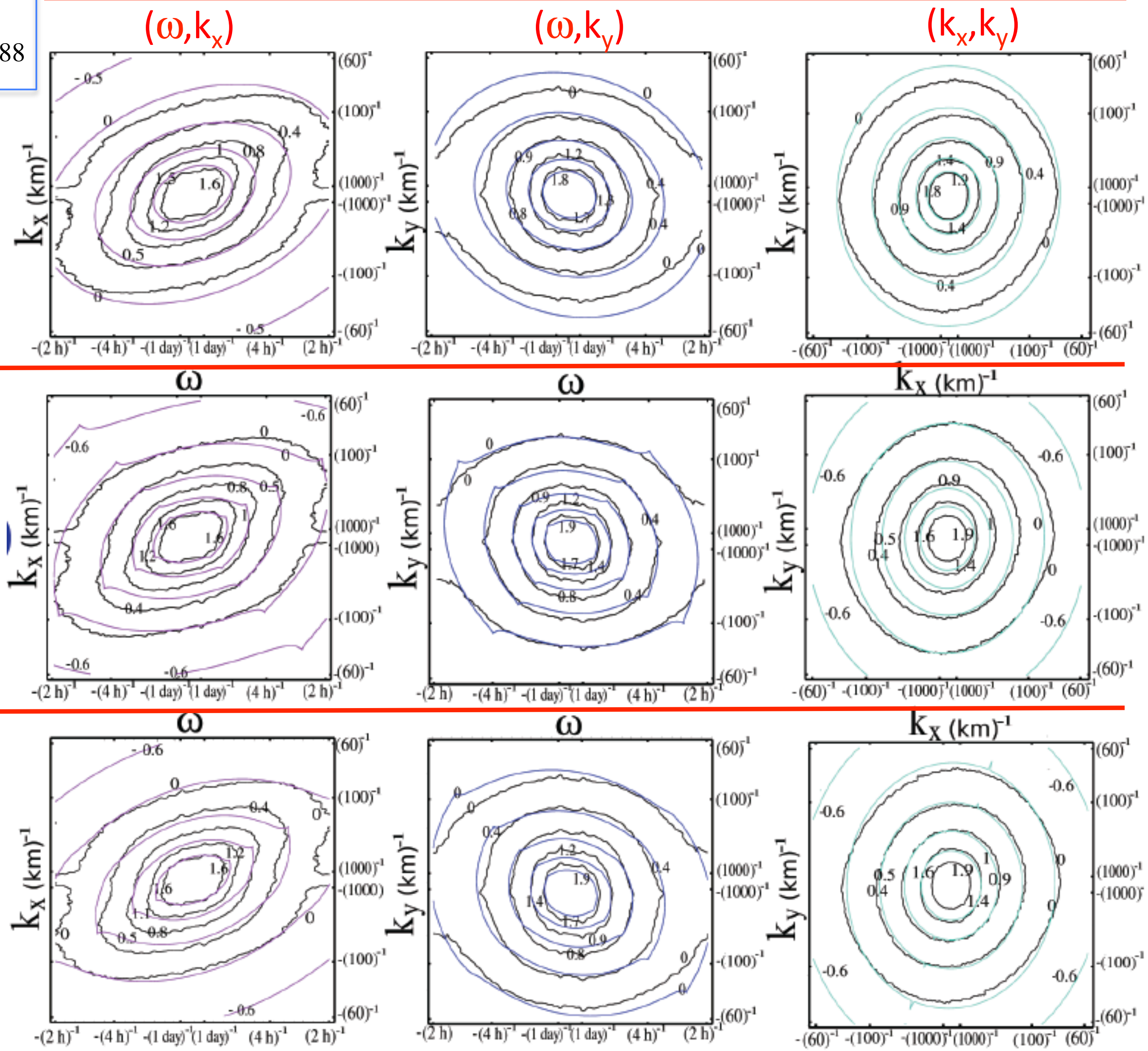
Spectrum, 2-D subspaces

Spectral density:

$$P_I(\underline{k},\omega) \propto P_\varphi(\underline{k},\omega)\left|\tilde{g}_{tur}\right|^2\left|\tilde{g}_{wav}\right|^2$$

$$P_\varphi = \left(\omega'^2 + \left\|\underline{k}\right\|^2\right)^{-s_\varphi} \qquad \tilde{g}_{tur} = \left(-i\omega' + \left\|\underline{k}\right\|\right)^{-H_{tur}}$$

$$H = H_{tur} + H_{wav} \qquad H = 0.26 \qquad s_\varphi = 2.88$$



$$\tilde{g}_{wav} = \left(\omega'^2 / v_{wav}^2 - \left\|\underline{k}\right\|^2\right)^{-H_{wav}/2}$$

$H_{wav} = 0$

no waves

$$\tilde{g}_{wav} = \left(\omega'^2 / v_{wav}^2 - \left\|\underline{k}\right\|^2\right)^{-H_{wav}/2}$$

$H_{wav} = 0.17$

EW symmetric waves

(Classical wave equation: $H_{wav}=2$)

$$\tilde{g}_{wav} = \left[i\left(\omega' / v_{wav} + \left\|\underline{k}\right\| sign\left(\underline{k}\cdot\underline{\mu}\right)\right)\right]^{-H_{wav}}$$

$$\underline{\mu} = (-0.3, 0.1)$$

Kelvin-like waves

$H_{wav} = 0.08$

(k_x,ω) spectrum, wave part only

$$P_{wav}(\underline{k},\omega)=\left|\tilde{g}_{wav}(\underline{k},\omega)\right|^2=\frac{P_I(\underline{k},\omega)}{P_\varphi(\underline{k},\omega)\left|\tilde{g}_{tur}\right|^2}$$

$$P_{wav}(k_x,\omega)=\int P_{wav}(k_x,k_y,\omega)\,dk_y$$

Dispersion relation for Kelvin waves (corresponding to *h* = 12 m, red)

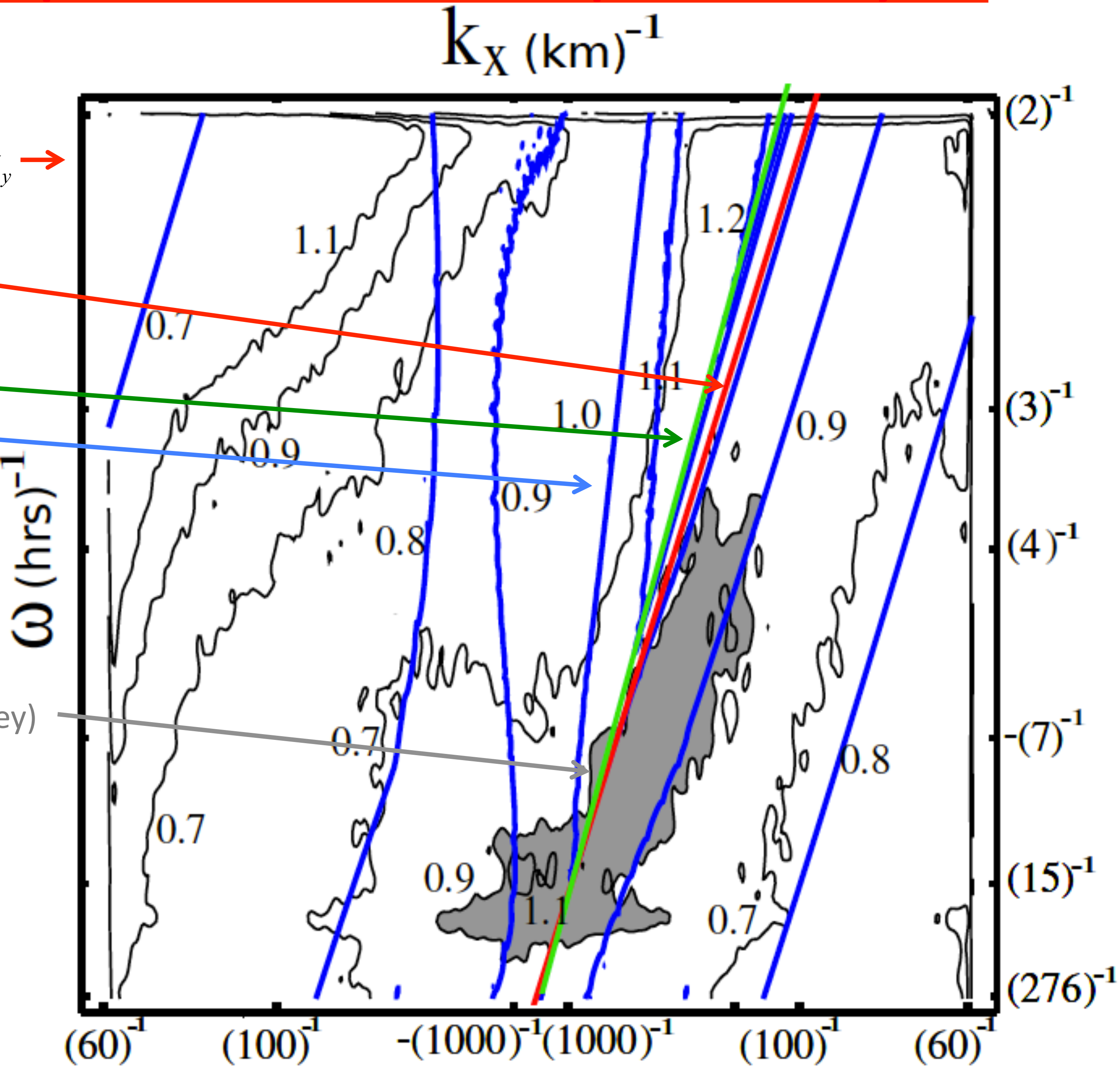
Maximal theory line in green

Theoretical contours, blue derived from:

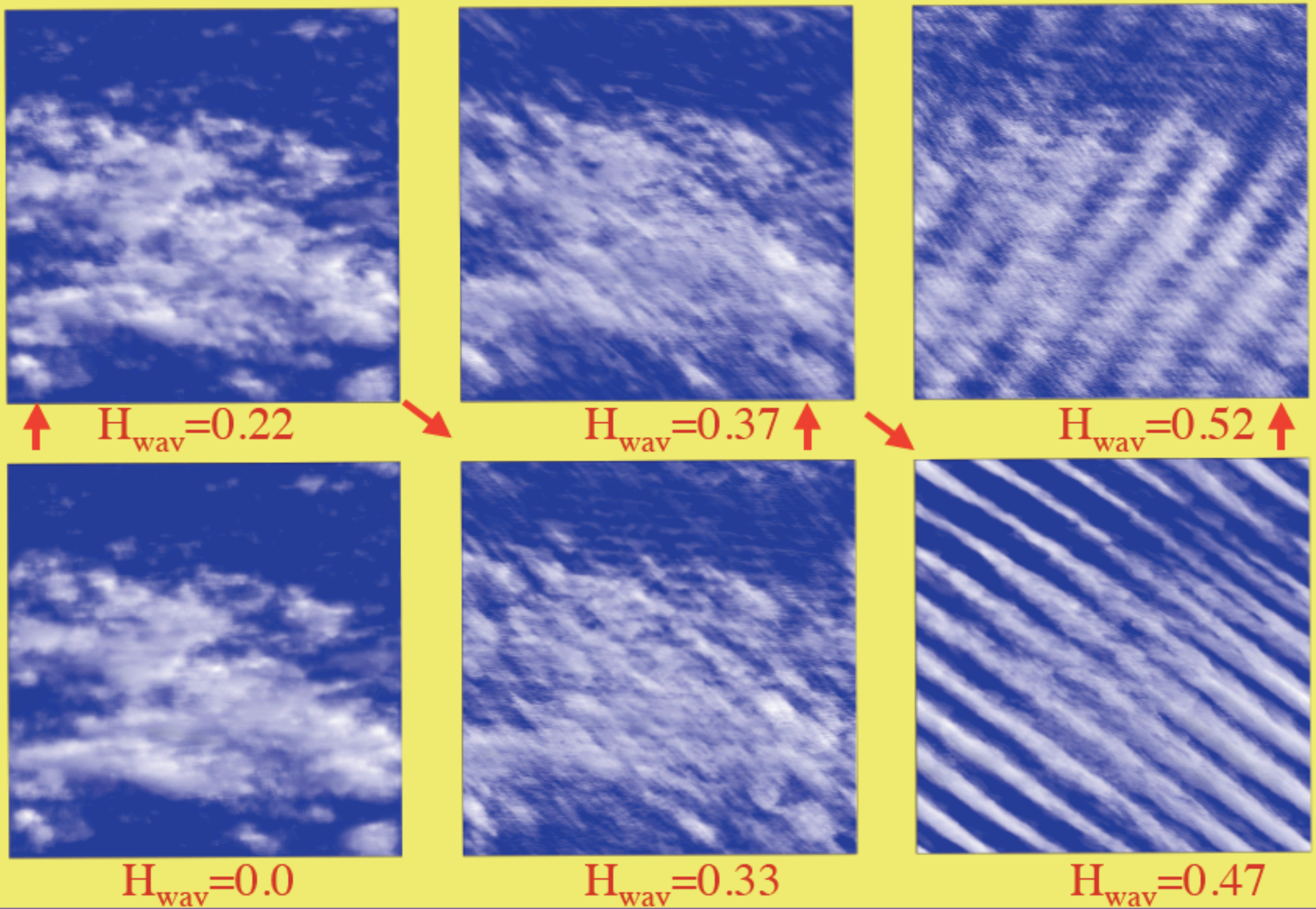
$$\left|\tilde{g}_{wav}\right|^2=\left(\omega'/v_{wav}+\left\|\underline{k}\right\| sign\left(\underline{k}\cdot\underline{\mu}\right)\right)^{-2H_{wav}}$$

$$\underline{\mu}=(-0.3,0.1)$$

Overall maximum region (grey)



Cascades from localized to increasingly unlocalized structures: $H_{wav} = 1/3 - H_{tur}$

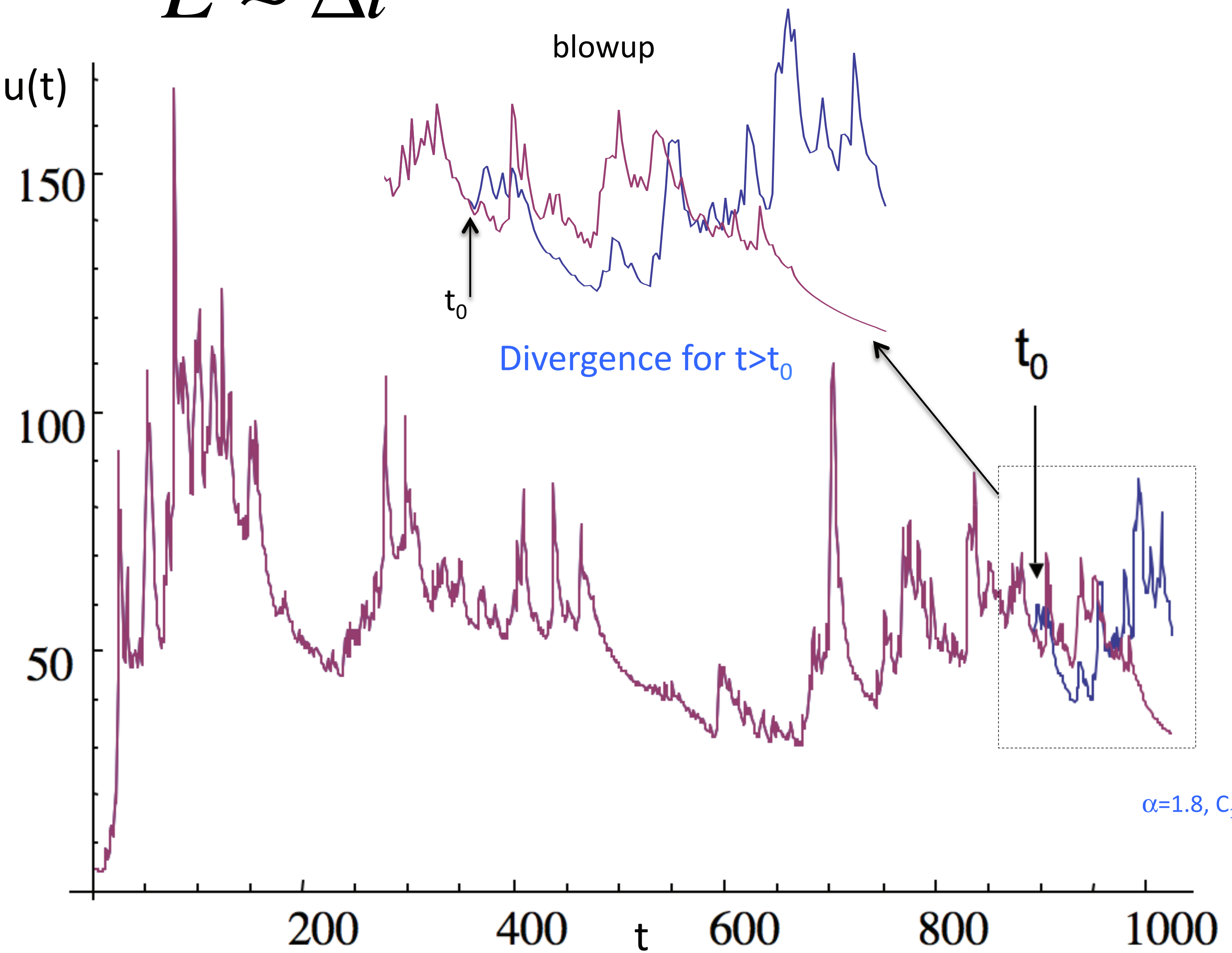


Predictability and stochastic forecasting

Predictability limits algebraic:

Prediction error: $E \approx \Delta t^{H}$

Lyapunov exponent $\rightarrow \infty$

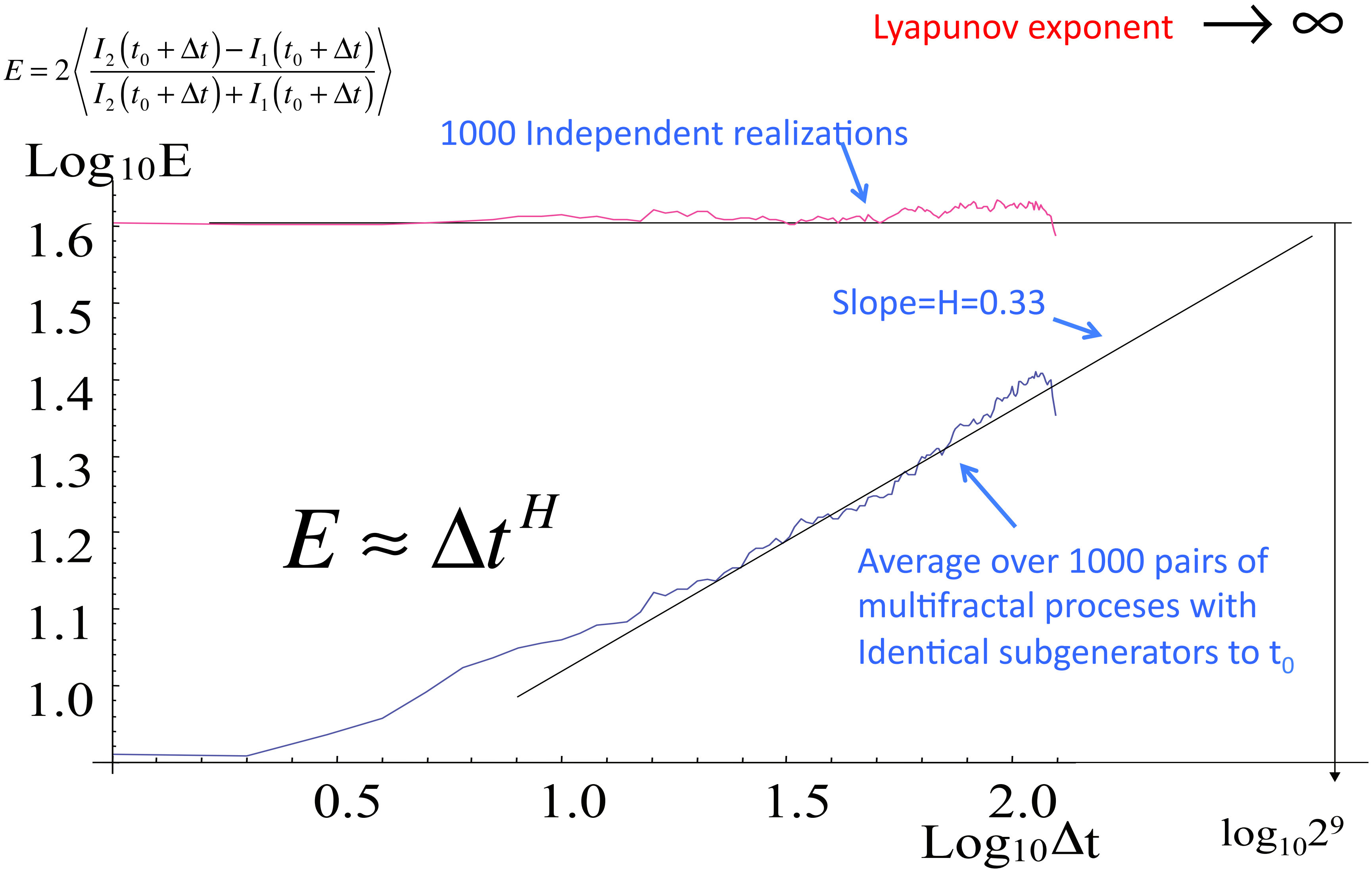


α=1.8, C_1=0.1, H=0.33

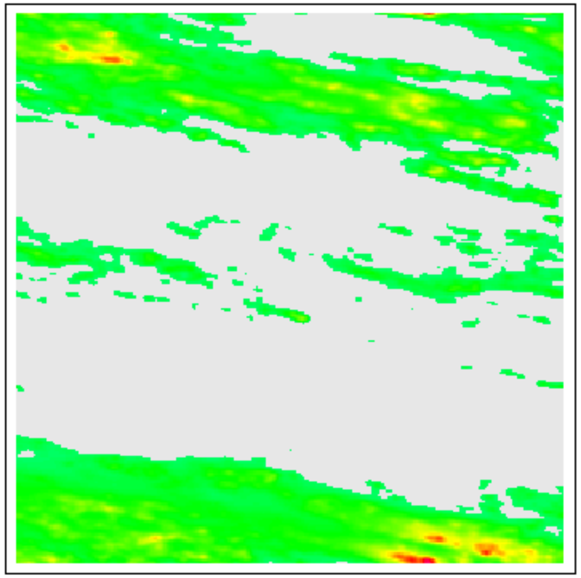
Two multifractal proceses with Identical subgenerators to t_0

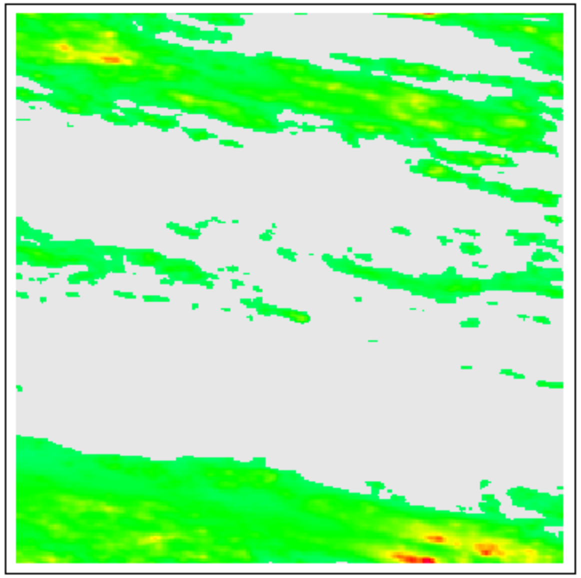
Algebraic divergence of realizations

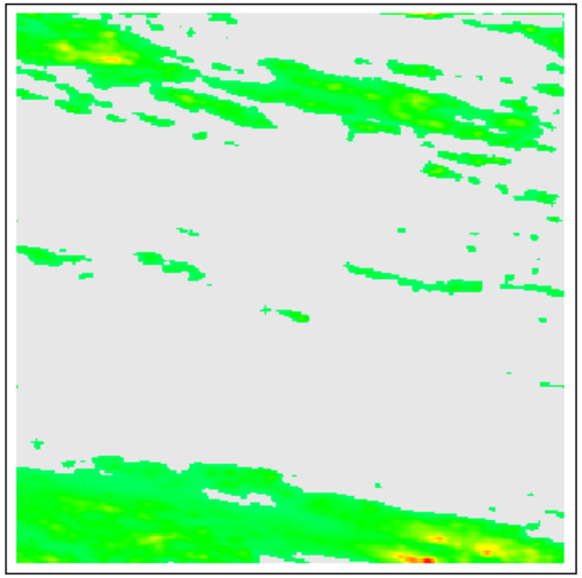
$$E = 2\left\langle \frac{I_2(t_0+\Delta t) - I_1(t_0+\Delta t)}{I_2(t_0+\Delta t) + I_1(t_0+\Delta t)} \right\rangle$$



Space-time Cascades, stochastic nowcasting (rain)







Realization A

Realization B

(all same initially)

Forecast based on first 16 time steps

Forecasting the climate

$$\Delta T(\Delta t) \approx \Delta t^{H_T}$$

Macroweather up to ≈ 100 years $H_T \approx -0.1$

$$T(t) = I^{\Delta H}\gamma = \int_{-\infty}^{t} (t-t')^{-(1-\Delta H)}\gamma(t')dt'$$

Ignore intermittency, take quasi-Gaussian model: $H_\gamma = -1/2$

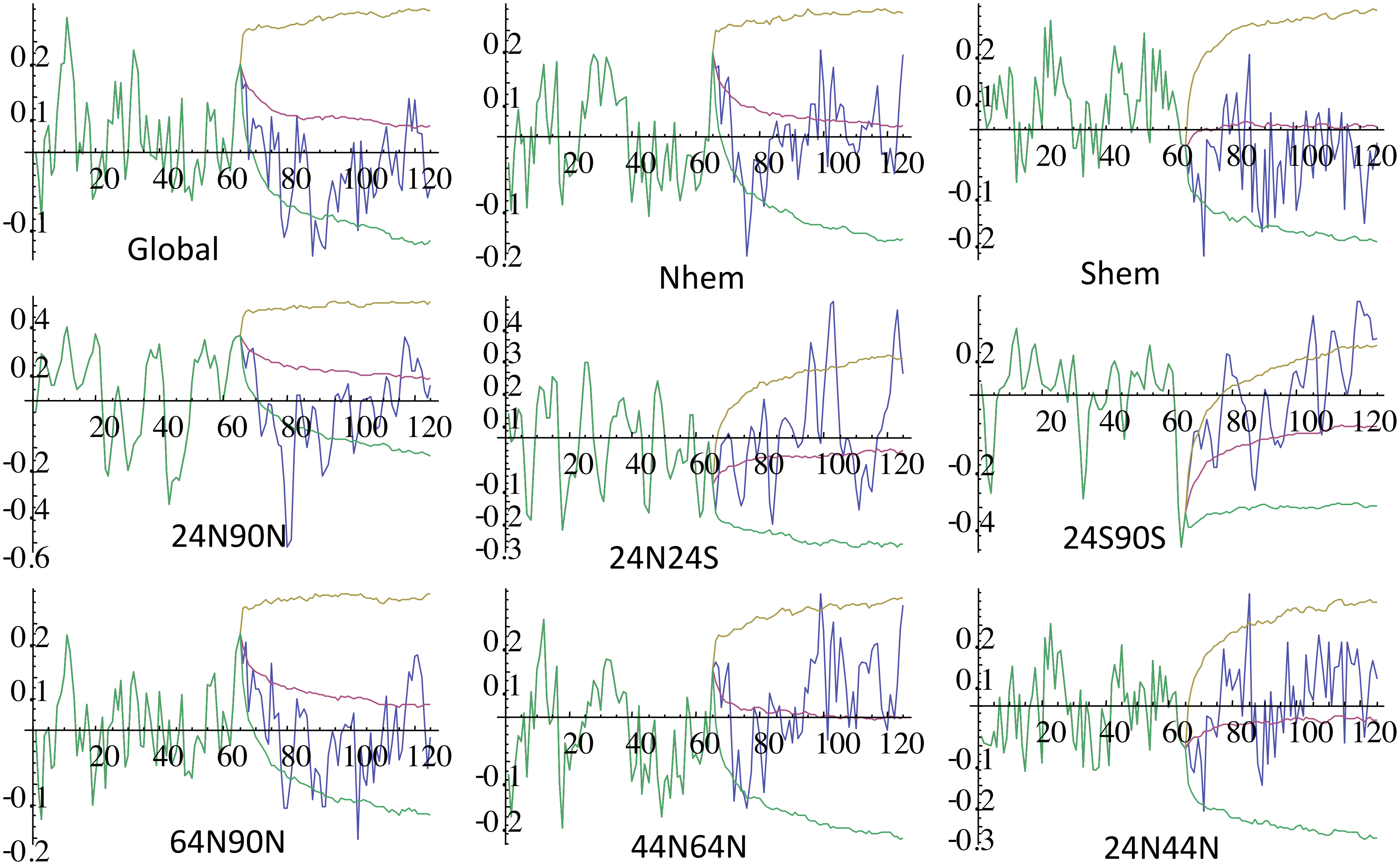
Fractional integration order ΔH

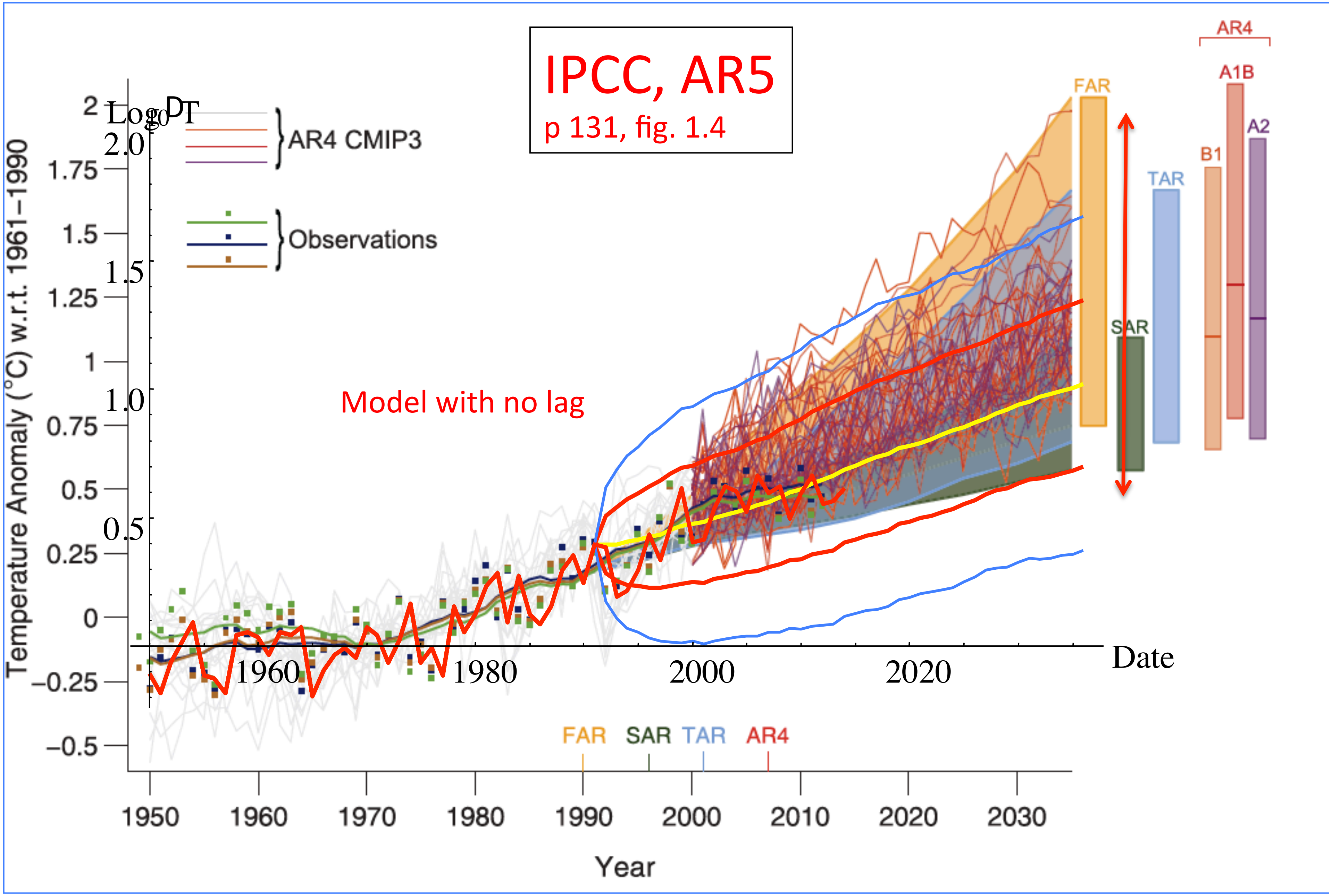
$$\Delta H = H_T - H_\gamma \approx 0.4$$

To obtain independent noises:

$$\gamma = I^{-\Delta H} T$$

9 latitude bands, $\Delta H = 0.4$, from 60 years ago





Compares favourably with GCM's