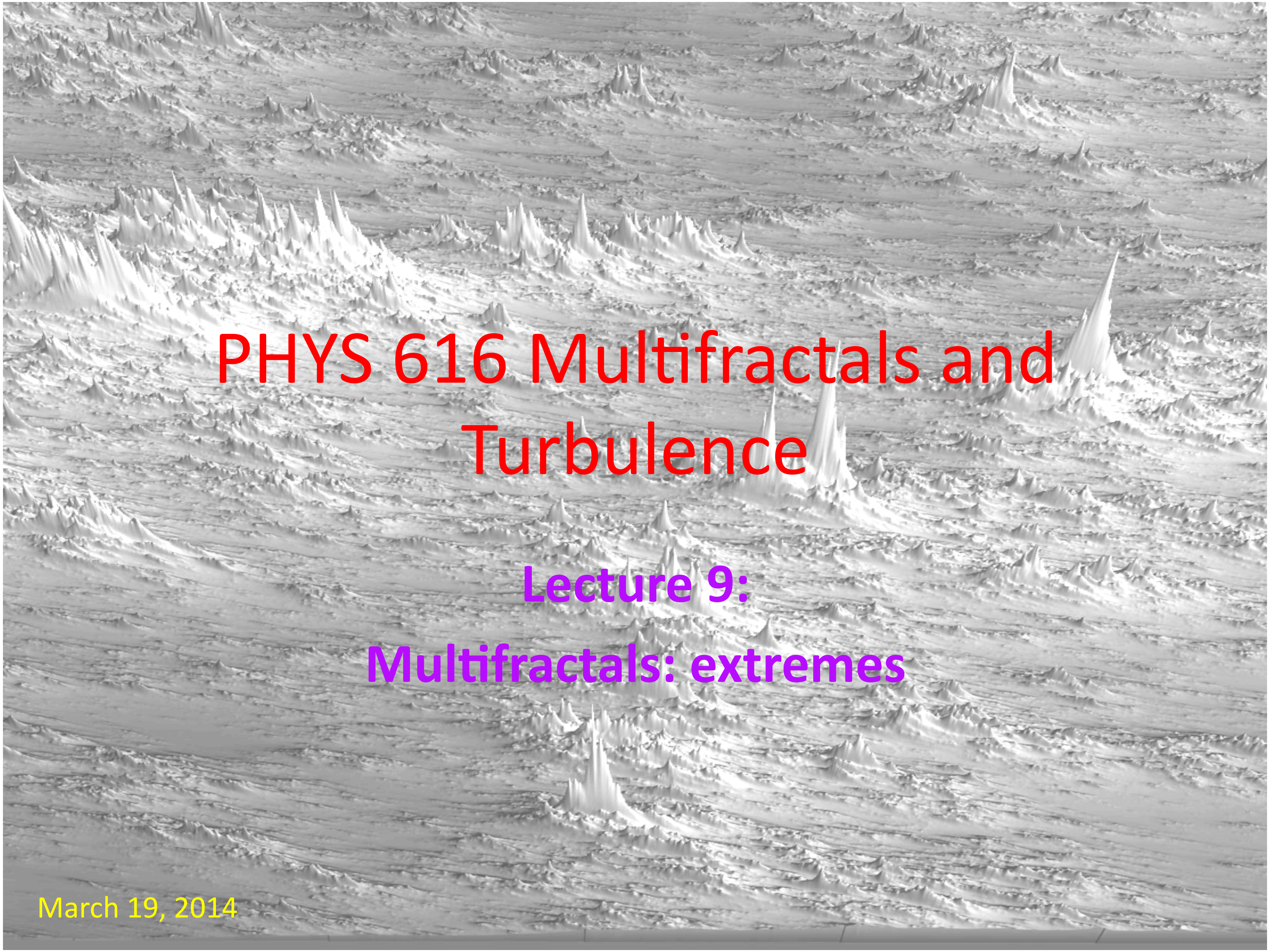


PHYS 616 Multifractals and Turbulence

Lecture 9:
Multifractals: extremes

March 19, 2014

Cascades, extremes and divergence of moments

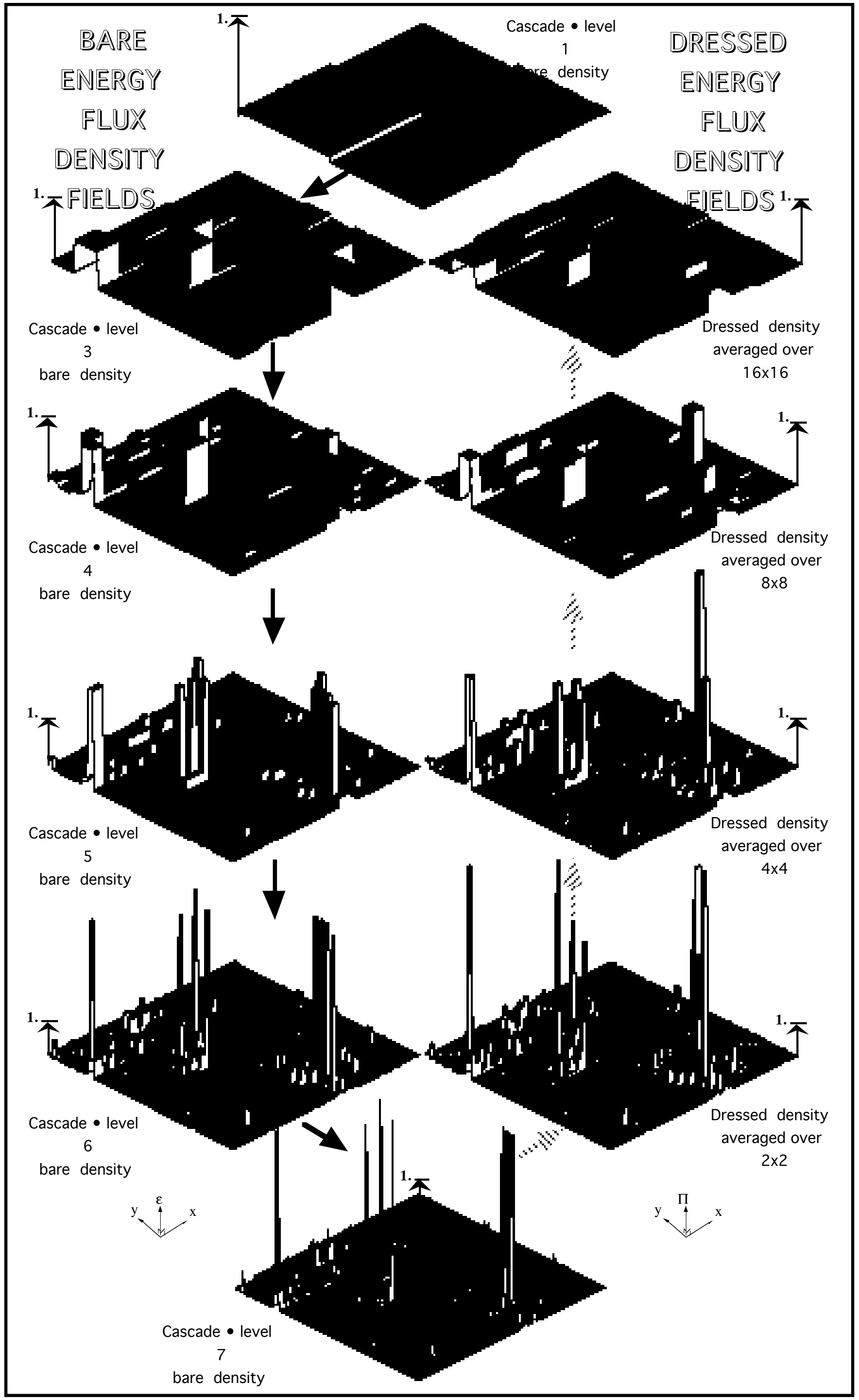
Divergence of Statistical Moments and extremes

Dressed and bare moments

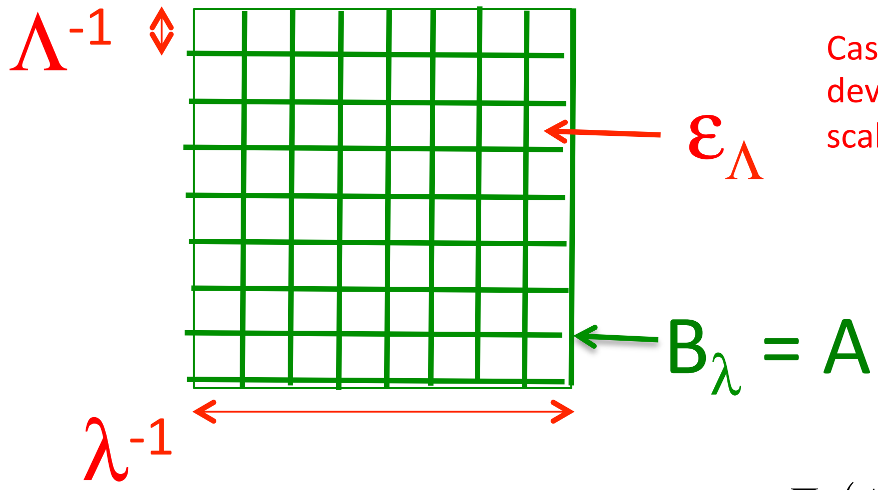
An example of an α model cascade. The left hand side shows the step by step construction of a ("bare") multifractal cascade starting with an initially uniform unit flux density. The right hand side shows the result of spatial averaging (to the same scale as the left image) of the cascade developed over the full range (a factor $\lambda = 2^7$ here, bottom centre): the "dressed" cascade discussed in the text. The vertical axis represents the density of energy ε flux to smaller scales which is conserved by the non-linear terms in the dynamical equations governing fluid turbulence. At each step the horizontal scale is divided by two, and independent random factors are chosen either <1 or >1.

"Bare" statistics – properties of cascade completed over range λ:

$$\left\langle \varepsilon_\lambda^q \right\rangle = \lambda^{K(q)}; \quad \Pr\left(\varepsilon_\lambda > \lambda^\gamma\right) \approx \lambda^{-c(\gamma)}$$



Definition of dressed flux



Cascade ε developped over the scale ratio Λ

In order to define the dressed flux, start by defining the Λ resolution flux $\Pi_\Lambda(A)$ over the set A:

$$\Pi_\Lambda(A) = \int_A \varepsilon_\Lambda d^D \underline{x}$$

Cascade ε developped over range Λ, integrated over the set A

We can now define the "partially dressed" flux density $\varepsilon_{\lambda,\Lambda(d)}$ as:

$$\varepsilon_{\lambda,\Lambda(d)} = \frac{\Pi_\Lambda(B_\lambda)}{vol(B_\lambda)}$$

Cascade ε developped over range Λ, averaged over the scale ratio λ

where $vol(B_\lambda) = \lambda^{-D}$ is the D-dimensional volume of a ball (interval, square, cube etc.) of size L/λ and the "(fully) dressed flux density" as:

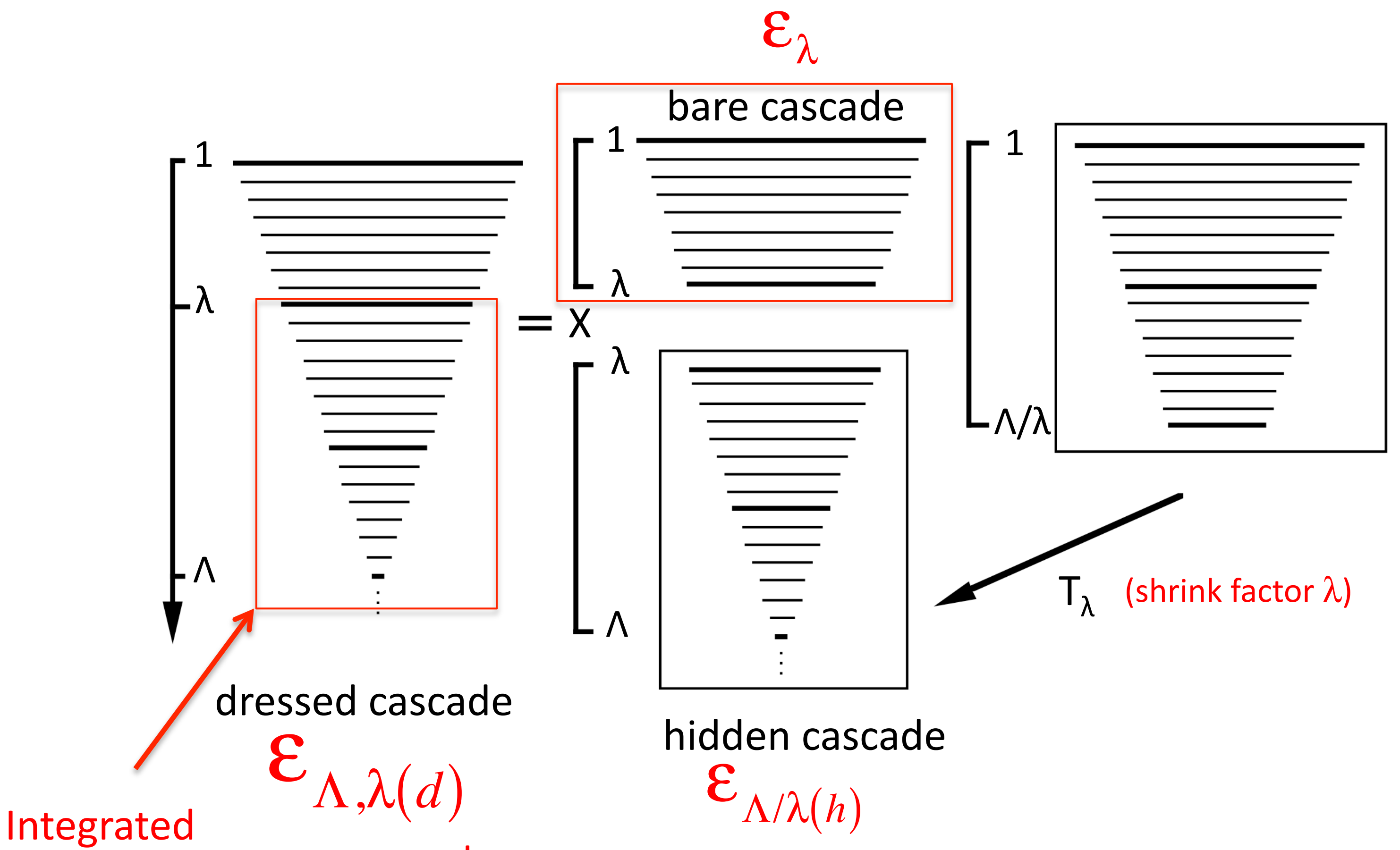
$$\varepsilon_{\lambda,(d)} = \lim_{\Lambda\to\infty} \varepsilon_{\lambda,\Lambda(d)}$$

Cascade ε developped over infinite range, averaged over the scale ratio λ

The terms "bare" and "dressed" are borrowed from renormalization jargon and are justified because the "bare" quantities neglect the small scale interactions (<L/λ), whereas the "dressed" quantities take them into account.

Factorization property of the cascade

$$\varepsilon_{\Lambda} = \varepsilon_{\lambda} T_{\lambda}\left(\varepsilon_{\Lambda/\lambda}\right)$$



$$\varepsilon_{\Lambda,\lambda(d)} = \varepsilon_{\lambda}\varepsilon_{\Lambda/\lambda(h)} \xrightarrow{\lambda \to \infty} \varepsilon_{\lambda(d)} = \varepsilon_{\lambda}\varepsilon_{\infty(h)}$$

bare (ε_{λ}); partially dressed ($\varepsilon_{\Lambda,\lambda(d)}$); Partially hidden ($\varepsilon_{\Lambda/\lambda(h)}$)

bare (ε_{λ}); Fully dressed ($\varepsilon_{\lambda(d)}$); hidden ($\varepsilon_{\infty(h)}$)

The hidden factor

Factorization shows that:

$$\varepsilon_{\lambda(d)} = \varepsilon_{\lambda}\varepsilon_{\infty(h)}$$

With the "hidden" factor given by:

$$\varepsilon_{\infty(h)} = \lim_{\Lambda\to\infty} \varepsilon_{\Lambda/\lambda(h)} = \Pi_{\infty}\left(B_1\right)$$

i.e. $\varepsilon_{\infty(h)}$ is a fully developed, fully integrated cascade and from the factorization:

$$\varepsilon_{\lambda(d)} = \varepsilon_{\lambda}\Pi_{\infty}\left(B_1\right)$$

and taking q^{th} moments:

$$\left\langle \varepsilon_{\lambda(d)}^{q} \right\rangle = \left\langle \varepsilon_{\lambda}^{q} \right\rangle \left\langle \Pi_{\infty}\left(B_1\right)^{q} \right\rangle$$

Since for any q, finite λ, $\left\langle \varepsilon_{\lambda}^{q} \right\rangle = \lambda^{K(q)}$ is always finite, the finiteness of $\left\langle \varepsilon_{\lambda(d)}^{q} \right\rangle$ depends on $\left\langle \Pi_{\infty}\left(B_1\right)^{q} \right\rangle$.

Divergence of High Order Statistical Moments

Our goal is to determine the statistics of the fully integrated, fully developped cascade: $\left\langle \Pi_\infty (B_1)^q \right\rangle$

We are interested in the statistics of the dressed and partially dressed density: $\varepsilon_{\lambda,\Lambda(d)} = \Pi_\Lambda \left(B_\lambda\right) / vol\left(B_\lambda\right)$ we will consider the mean of the q^{th} power of the flux on the set A (dimension D) of the cascade constructed down to the scale L/λ :

$$\left\langle \Pi_\lambda^q(\mathrm{A}) \right\rangle = \left\langle \left[\int_{\mathrm{A}} d^D \mathbf{x}\, \varepsilon_\lambda \right]^q \right\rangle$$

q^th order dressed moment over set A

when q is an integer ≥ 1:

$$\left\langle \left[\int_{\mathrm{A}} d^D \mathbf{x}\, \varepsilon_\lambda \right]^q \right\rangle = \left\langle \int_{\mathrm{A}} \cdots \int_{\mathrm{A}} d^D \mathbf{x}_1 \cdots d^D \mathbf{x}_q\, \varepsilon_\lambda(x_1) \cdots \varepsilon_\lambda(x_q) \right\rangle$$

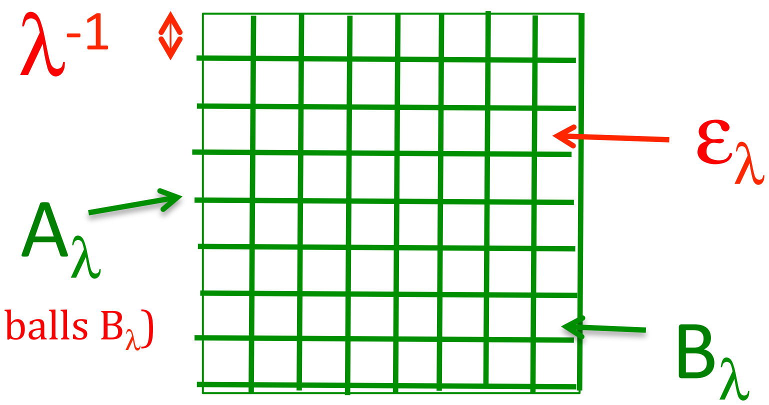
The complexity of this multiple integral suggests the introduction of "trace moments" which are obtained by integrating over the subset of the integral obtained by taking $x_1 = x_2 = x_3 = \ldots$;

$$Tr_A\, (\varepsilon_\lambda)^q = \int_A d^{qD}\mathbf{x} \left\langle \varepsilon_\lambda^q \right\rangle$$

q^th order resolution λ trace moments

where A_λ is the set A at resolution λ (*i.e.*, obtained by a disjoint covering of A with balls B_λ, ε_λ is the usual (bare) flux density at resolution λ).

Properties of Trace moments



A at resolution λ
(= cover of A with disjoint balls B_λ)

Usual moments: $$\left\langle (\Pi_\lambda(A))^q \right\rangle = \left\langle \left(\underbrace{\sum_A (\Pi_\lambda(B_\lambda))}_{\Pi_\lambda(A)} \right)^q \right\rangle$$

$$\Pi_\lambda(A) = \int_A \varepsilon_\lambda d^D \underline{x}$$

$$\Pi_\lambda(B_\lambda) = \lambda^{-D}\varepsilon_\lambda$$

Trace moments: $$Tr_A\ (\varepsilon_\lambda)^q = \int_A d^{qD}\mathbf{x}\left\langle \varepsilon_\lambda^q \right\rangle$$

$$Tr_A \varepsilon_\lambda^q = \sum_A \left\langle (\Pi_\lambda(B_\lambda))^q \right\rangle$$

Using the fact that for any positive x_i $\left(\sum_i x_i^q\right)^{1/q}$ is a decreasing function of q, (a Jensen inequality) we have:

$$(\Pi_\lambda(B_\lambda))^q = \left(\sum_A (\Pi_\lambda(B_\lambda))\right)^q \begin{cases} > \sum_A (\Pi_\lambda(B_\lambda))^q; & q > 1 \\ = \sum_A (\Pi_\lambda(B_\lambda))^q; & q = 1 \\ < \sum_A (\Pi_\lambda(B_\lambda))^q; & q < 1 \end{cases}$$

Properties of Trace moments

Taking ensemble averages we obtain:

$$\left\langle \Pi_\lambda{}^q(A) \right\rangle \geq \mathrm{Tr}_A\, \varepsilon_\lambda{}^q \qquad (q > 1)$$

$$\leq \mathrm{Tr}_A\, \varepsilon_\lambda{}^q \qquad (q < 1)$$

The use of trace moments rather than the usual moments has a number of advantages. First, it is defined for all q (whereas the usual moments can only be expanded as multiple integrals for positive integer q). Second, trace moments are Hausdorff measures since we can use the scaling of $\langle \varepsilon_\lambda{}^q \rangle$ to obtain a Hausdorff measure over a higher dimensional space (for convenience we have left out the inf, *etc.*). We anticipate that in the limit $\lambda \to \infty$ they will either diverge to ∞ or converge to 0; in fact, they will have two transitions!

Degenerate cascades

$$Tr_A\ (\varepsilon_\lambda)^q = \int_A d^{qD}\mathbf{x}\langle\varepsilon_\lambda^q\rangle \sim \sum_{A_\lambda}\langle\varepsilon_\lambda^q\rangle\lambda^{-qD} = \sum_{A_\lambda}\lambda^{K(q)}\lambda^{-qD}$$

From previous

Now use box counting in the sum ; there will be terms, each of value $\langle\varepsilon_\lambda{}^q\rangle\,\lambda^{-qD}$

$$\mathrm{Tr}_{A_\lambda}\,\varepsilon_\lambda{}^q = \lambda^D\cdot\lambda^{K(q)}\cdot\lambda^{-qD} = \lambda^{K(q)-(q-1)D} = \lambda^{(q-1)(C(q)-D)}$$

Take $\lambda\to\infty$

Where we have used the dual codimension function $C(q) = K(q)/(q\text{-}1)$.

The case $C(q)>D$ for $q<1$:

Due to the monotonicity of $C(q)$ this is equivalent to $C_1>D$. In this case,

$$\lim_{\lambda\to\infty}\mathrm{Tr}_{A_\lambda}\,\varepsilon_\lambda{}^q \to 0 \Rightarrow \left\langle\Pi_\infty{}^q(\mathrm{A})\right\rangle = 0$$

The case: $C(q)>D$ for *$q<1$* (i.e. $C_1>D$)

Recall: $\left\langle(\Pi_\lambda(A))^q\right\rangle \le Tr_A\varepsilon_\lambda^q;\quad q<1$

for all $q<1$, hence the process is *degenerate* on the space. This implies that when $C_1>D$, then the mean of the bare process is too sparse to be observed in the space D; in fact, the above shows that if $C_1>D$ it is impossible to normalize the process so that the dressed mean $\langle\Pi_\infty(A)\rangle$ is finite.

Nondegenerate cascades

The case: $C(q)<D$ for $q>1$ (i.e. $C_1<D$)

$$Tr_{A_\lambda} \varepsilon_\lambda{}^q = \lambda^D \cdot \lambda^{K(q)} \cdot \lambda^{-qD} = \lambda^{K(q)-(q-1)D} = \lambda^{(q-1)(C(q)-D)}$$

Take $\lambda \to \infty$

In this case, the trace moments diverge for $q<1$, but this does not affect the convergence of the dressed moments (the trace moments are upper bounds here). On the other hand, for $q>1$, we find:

$$\lim_{\lambda\to\infty} Tr_{A_\lambda} \varepsilon_\lambda{}^q \to \infty \Rightarrow \langle \Pi_\infty(A) \rangle \to \infty$$

q>1, $C(q)>D$ recall:

$$\langle \Pi_\lambda{}^q(A) \rangle \geq Tr_A \varepsilon_\lambda{}^q \qquad (q>1)$$

for all $C(q)>D$. Using the implicit definition of q_D: $C(q_D) = D$, we thus obtain:

$$\left\langle \left(\Pi_\infty(A)\right)^q \right\rangle \to \infty; \quad q > q_D$$

The case: $C(q)<D$ for $q>1$

i.e., in this case, divergence of the trace moments implies divergence of the corresponding dressed moments.

Multifractal Butterfly effect

Full cascade averaged at scale λ^{-1} → $\varepsilon_{\lambda(d)} = \varepsilon_\lambda \varepsilon_{\infty(h)}$

large scales (scale range λ)

small scales: integration over a fully developped cascade: $\varepsilon_{\infty(h)} = \Pi_\infty(B_1)$

The hidden moments diverge:

$$\left\langle \varepsilon^q_{\infty,(h)} \right\rangle \approx \begin{matrix} O(1); & q < q_D \\ \infty; & q \geq q_D \end{matrix}$$

Divergence due to small scales: the multifractal butterfly effect

q_D is the solution to the implicit equation $K(q_D) = D(q-1)$

Divergence of dressed moments:

$$\left\langle \varepsilon^q_{\lambda(d)} \right\rangle = \lambda^{K_d(q)}$$

where:

$$K_d(q) = \begin{cases} K(q); & q < q_D \\ \infty; & q \geq q_D \end{cases}$$

Discontinuity in first derivative = first order multifractal phase transition

Long range dependencies place this outside the framework of Extreme Value Theory

Probability distributions

$$\langle \varepsilon^q_{\lambda(d)} \rangle = \infty,\ q \geq q_D \quad \Leftrightarrow \quad \Pr(\varepsilon_{\lambda(d)} > s) \sim s^{-q_D},\ \ s >> 1$$

Mandelbrot 1974, S+L 1987

Proof (RHS->RHS): $\Pr\left(\varepsilon'_\lambda > \varepsilon_\lambda\right) \approx \varepsilon_\lambda^{-q_D}$ hence $p(\varepsilon_\lambda) \approx \frac{d\Pr}{d\varepsilon_\lambda} \approx \varepsilon_\lambda^{-q_D-1}$

and $\left\langle \varepsilon^q_\lambda \right\rangle = \int p(\varepsilon_\lambda)\varepsilon^q_\lambda d\varepsilon_\lambda = \int \varepsilon_\lambda^{-q_D-1}\varepsilon^q_\lambda d\varepsilon_\lambda = \varepsilon_\lambda^{-q_D+q}$ Which diverges at large ε_λ when $q>q_D$

The dressed codimension function $c_d(\gamma)$

To calculate the corresponding dressed codimension $c_d(\gamma)$, we can use the Legendre transform of $K_d(q)$ to obtain:

$$c_d(\gamma) = c(\gamma), \quad \gamma \le \gamma_D$$
$$c_d(\gamma) = q_D(\gamma - \gamma_D) + c(\gamma_D), \quad \gamma > \gamma_D$$

$$c_d(\gamma) = \max_q \left(q\gamma - K_d(q)\right)$$

For all $\gamma > \gamma_D$, the max is at $q = q_D$

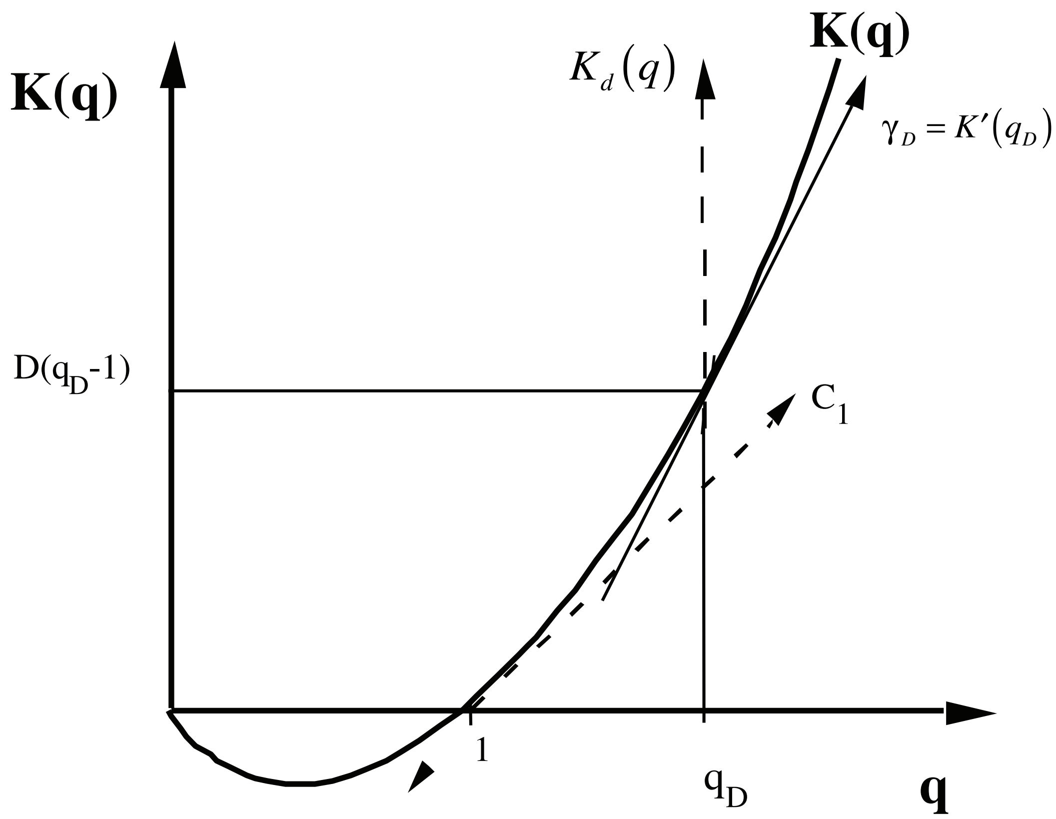
where $\gamma_D = K'(q_D)$ is the critical singularity corresponding to the critical q_D. This transition from convex "bare" behaviour to linear "dressed" behaviour represents a discontinuity in the second derivative of $c(\gamma)$; hence a "second order multifractal phase transition" for c (for K, see below).

Linear $c_d(\gamma)$ and power law ε_λ

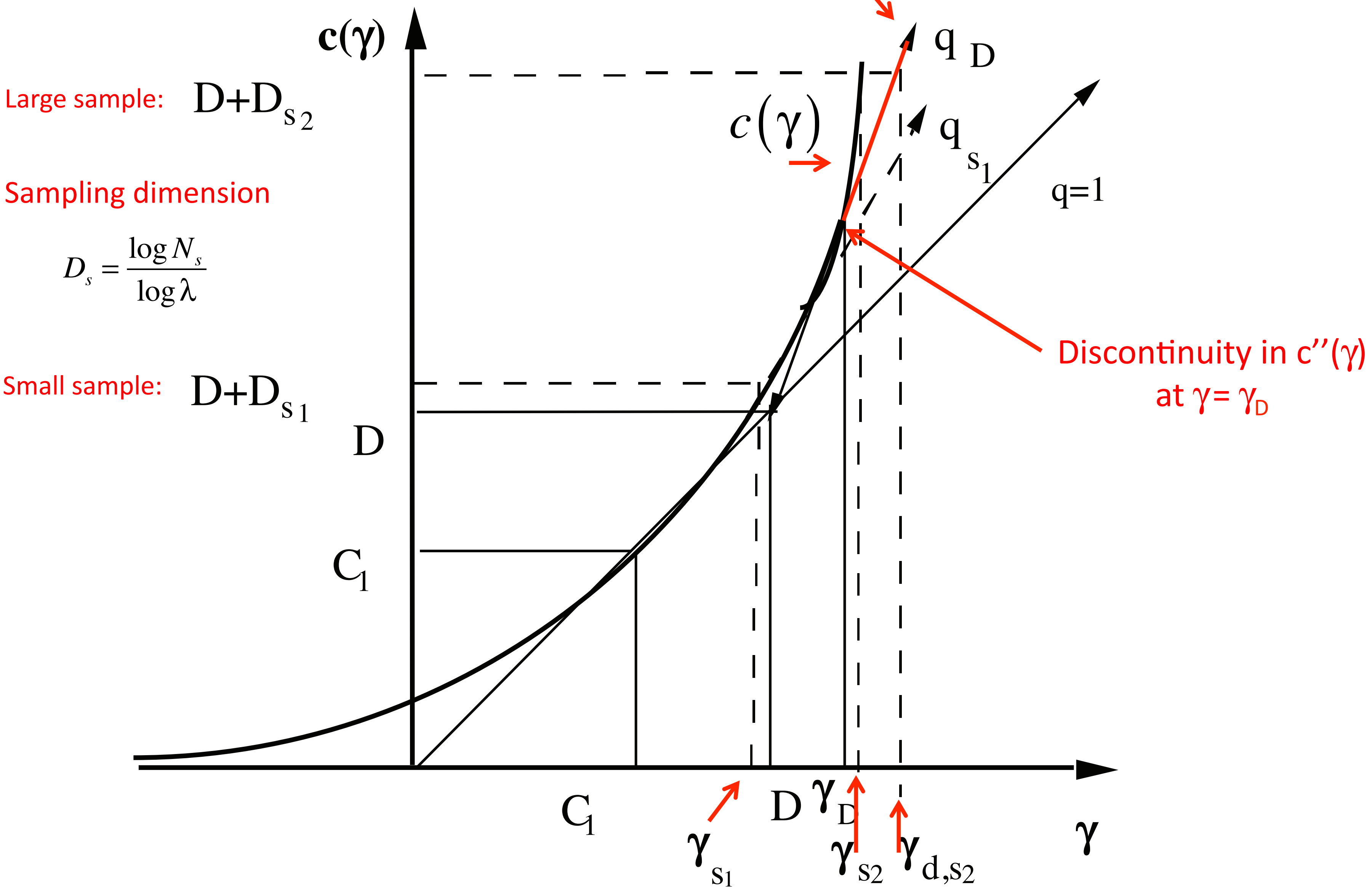
$$\Pr\left(\varepsilon'_\lambda > \varepsilon_\lambda\right) \approx \lambda^{-c_d(\gamma)} \qquad \gamma = \frac{\log \varepsilon_\lambda}{\log \lambda}$$

$$c_d(\gamma) = q_D\gamma + \log_\lambda A; \quad \log_\lambda A = -q_D\gamma_D - c(\gamma_D); \quad \gamma > \gamma_D$$

$$\Pr\left(\varepsilon'_\lambda > \varepsilon_\lambda\right) \approx A\lambda^{-q_D\gamma} = Ae^{-q_D\left((\log\varepsilon_\lambda)/\log\lambda\right)\log\lambda} = A\varepsilon_\lambda^{-q_D}; \quad \varepsilon_\lambda > \lambda^{\gamma_D}$$



Multifractal phase transitions:
Effect of finite sampling on codimensions



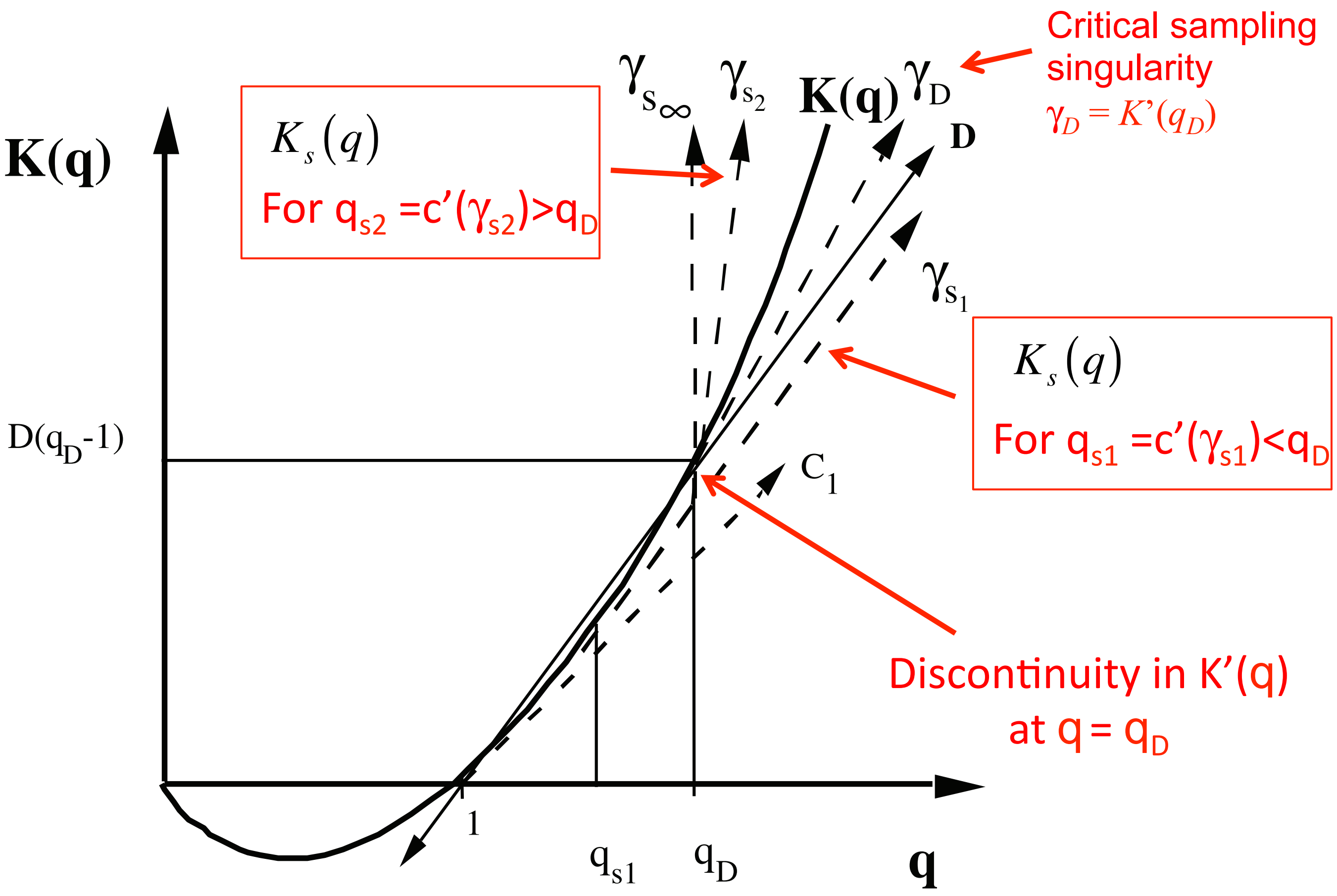
Schematic diagram of c(γ), $c_d(\gamma)$ indicating two sampling dimensions D_{S1}, D_{S2} and their corresponding $\gamma_{S1} < \gamma_D < \gamma_{S2} < \gamma_{d,S2}$; the critical tangent (slope q_D) contains the point (D, D).

First order Multifractal phase transitions: Effect of finite sampling on K(q)

$$K_s(q) = \max_{\gamma<\gamma_s}\left(q\gamma - c_d(\gamma)\right)$$

Equation for q_D:

$$K(q_D) = D(q-1)$$

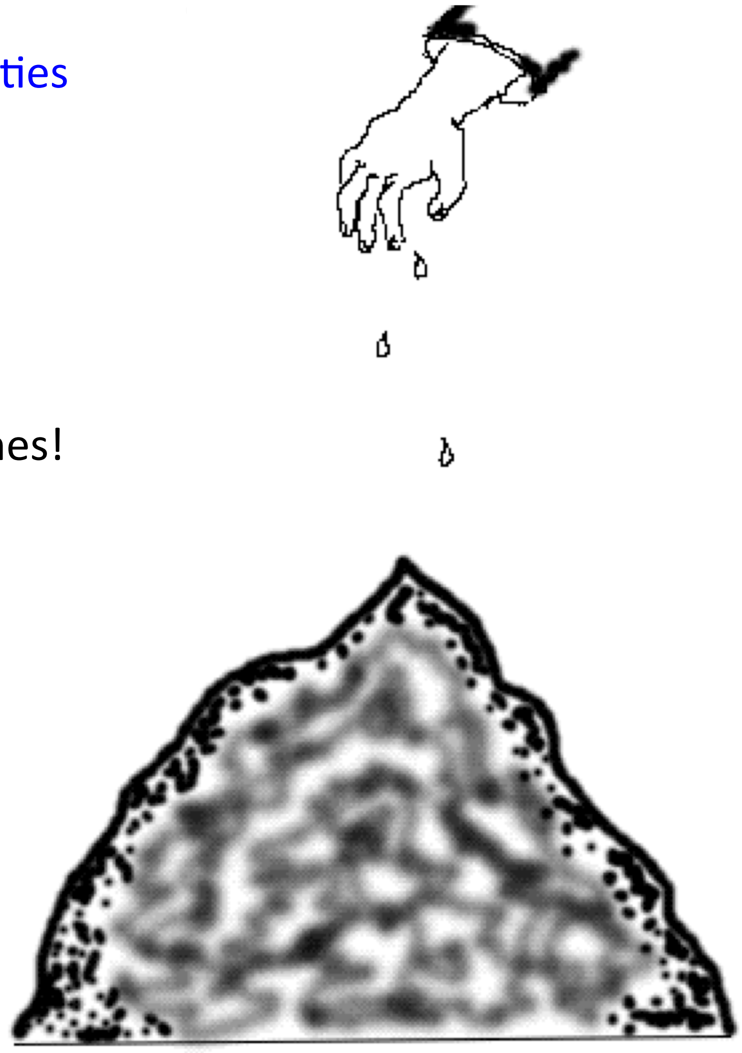


Self-Organized criticality (SOC)

Operational definition of SOC:
Spatial scaling and Power law probabilities

Sandpile "mean shape"
= result of extreme
avalanches

The mean field results from catastrophes!



Classical SOC: zero flux limit
Nonclassical multifractal SOC: quasi constant flux

$$\Pr(\varepsilon_{\lambda(d)} > s) \sim s^{-q_D}, \;\; s >> 1$$

Divergence of moments in Laboratory turbulence

Let's test the prediction:

$$\Pr\left(\varepsilon > s\right) \approx s^{-q_{D,\varepsilon}} \qquad s >> 1$$

Dissipation Range:

$$\varepsilon \approx \nu \underline{v} \cdot \nabla^2 \underline{v} \approx \nu \frac{\Delta v^2}{\Delta x^2}$$

$$\Pr(\varepsilon > s) = \Pr\left(\frac{\nu \Delta v^2}{\Delta x^2} > s\right) = \Pr\left(\Delta v > \left(\frac{\Delta x}{\nu^{1/2}}\right) s^{1/2}\right) \qquad q_{D,\varepsilon} = q_{D,v(diss)} / 2$$

Inertial Range:

$$\varepsilon \approx \frac{\Delta v^3}{\Delta x}$$

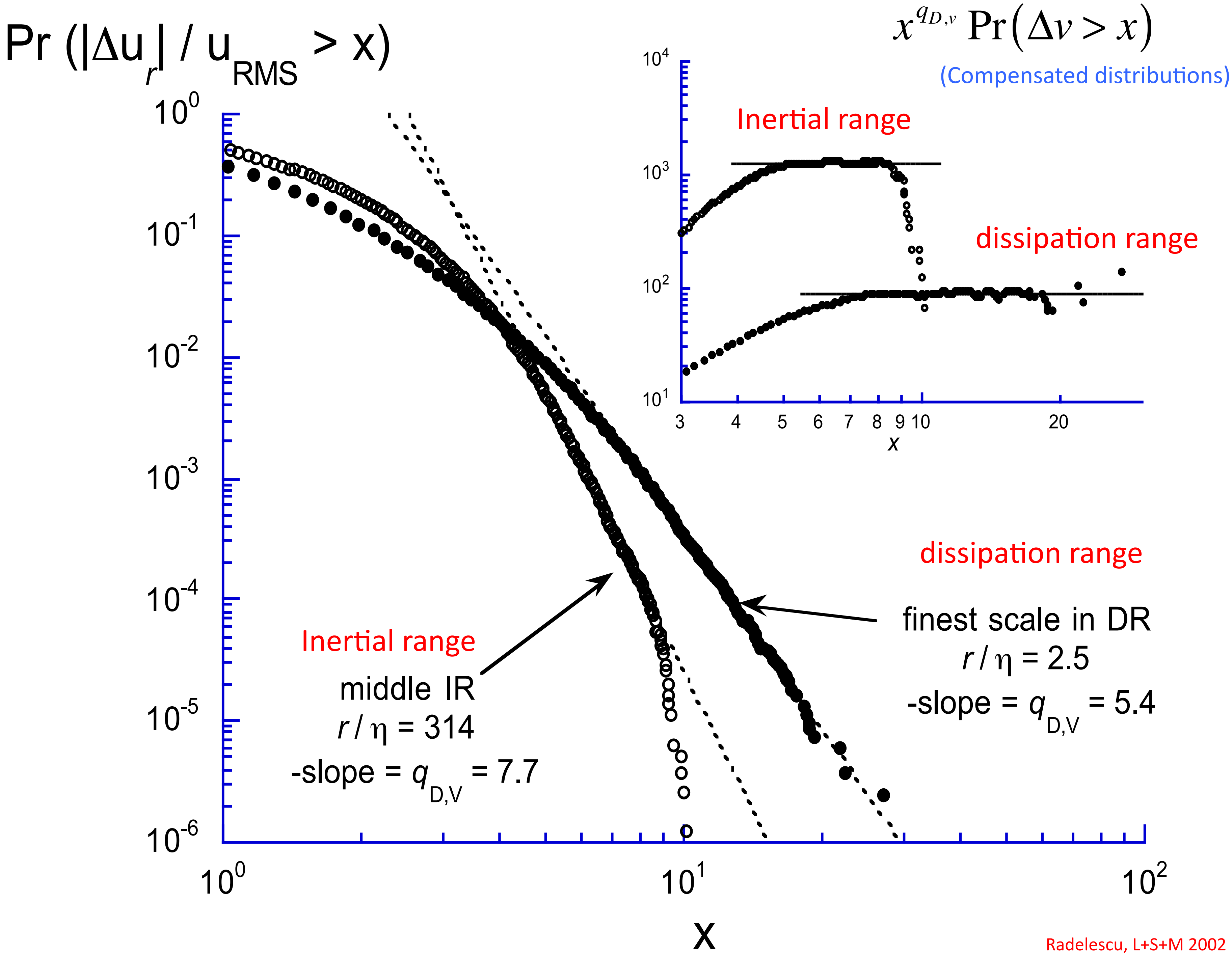
$$\Pr(\varepsilon > s) = \Pr\left(\frac{\Delta v^3}{\Delta x} > s\right) = \Pr\left(\Delta v > \Delta x s^{1/3}\right) \qquad q_{D,\varepsilon} = q_{D,v(inertial)} / 3$$

Laboratory Data:

Dissipation range estimate: $q_{D,v(diss)} \approx 5.4;$ $q_{D,\varepsilon} \approx 2.7$

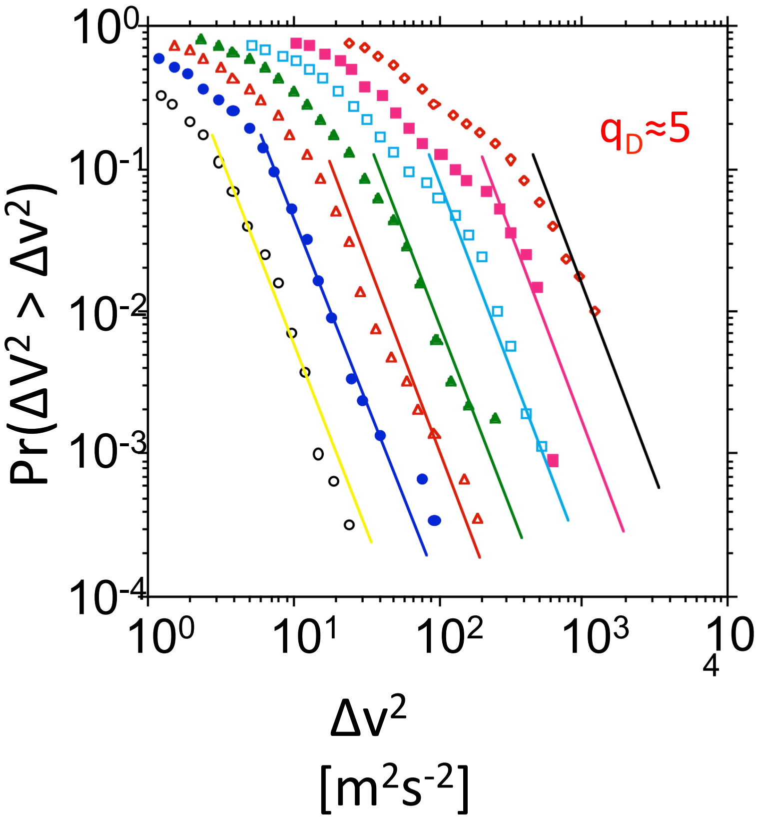
Inertial range estimate: $q_{D,v(inertial)} \approx 7.7;$ $q_{D,\varepsilon} \approx 2.6$

Radelescu, L+S+M 2002



Pr (|Δu_r| / u_RMS > x)
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
10^0
10^1
10^2
x
Inertial range
middle IR
r / η = 314
-slope = q_D,V = 7.7
dissipation range
finest scale in DR
r / η = 2.5
-slope = q_D,V = 5.4
x^{q_D,v} Pr(Δv > x)
(Compensated distributions)
Inertial range
dissipation range
10^4
10^3
10^2
10^1
3
4
5
6
7
8
9
10
20
x
Radelescu, L+S+M 2002

Divergence of moments in the horizontal wind field

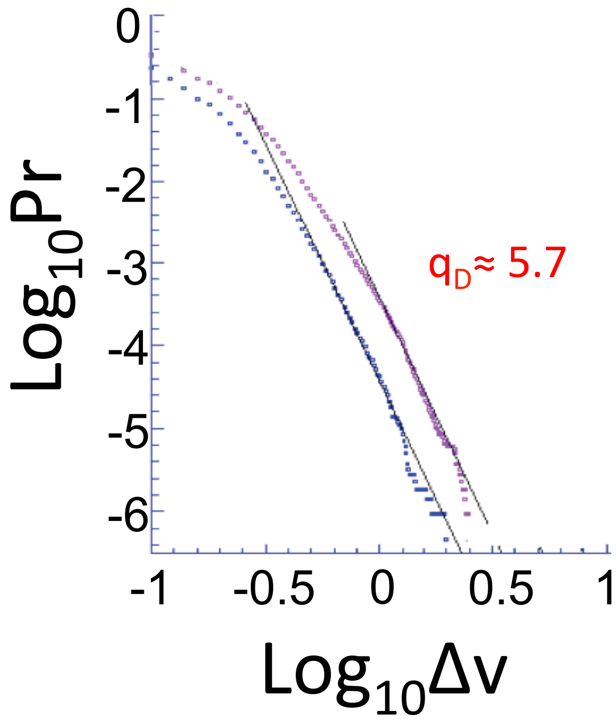


Across vertical layers

radiosondes

Layers 50,100, 200, 400,...3200m

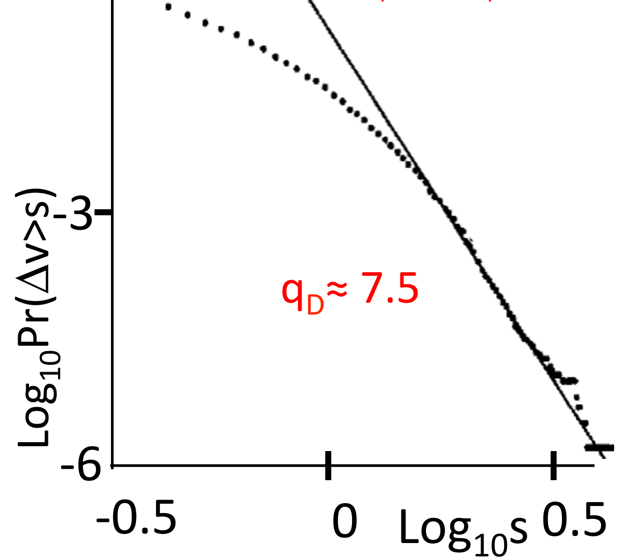
(S+L1985)



In the horizontal

aircraft: 40, 80 m

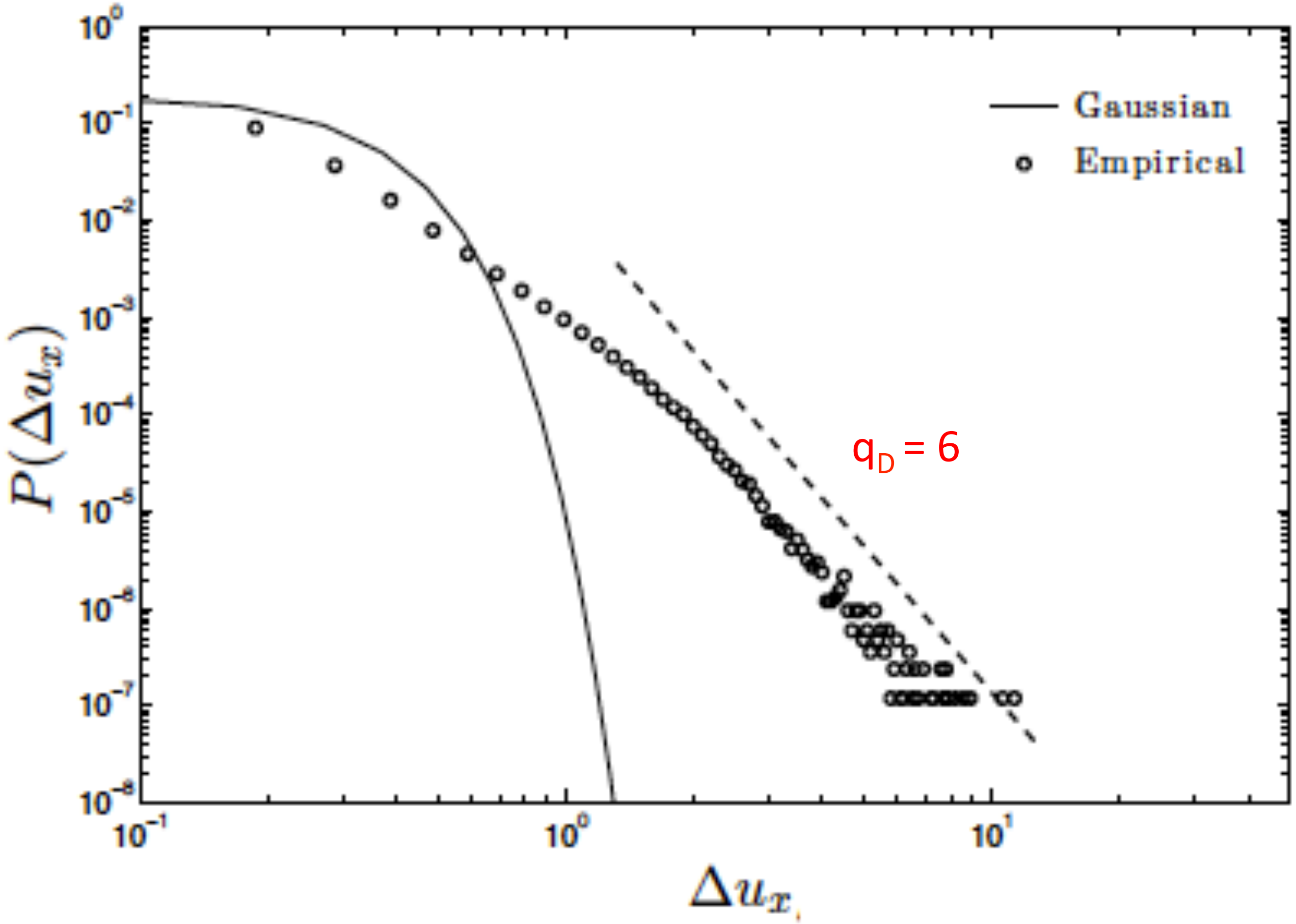
(L+S 2007)



In time

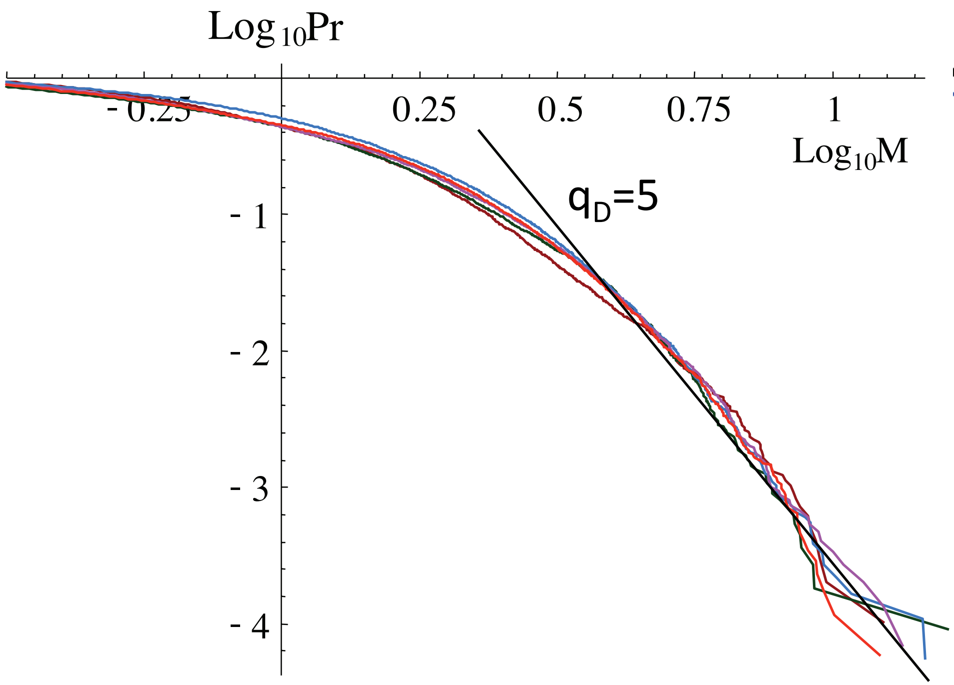
sonic probe, 10 Hz

(Schmitt, S+L 1994)

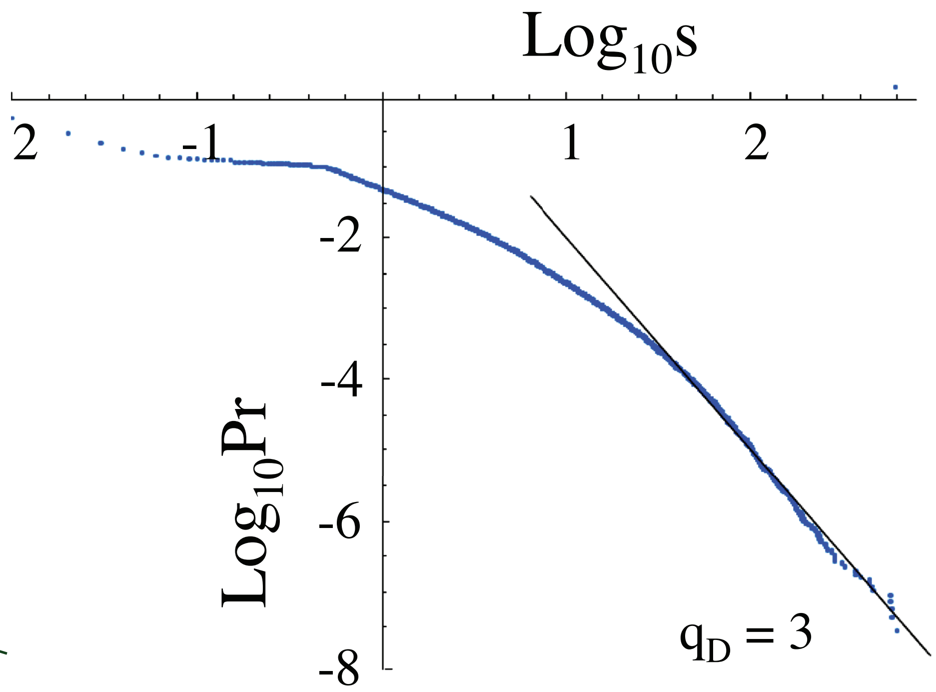


Corsica horizontal wind data at 20s resolution (Fitton et al 2012)

Precipitation



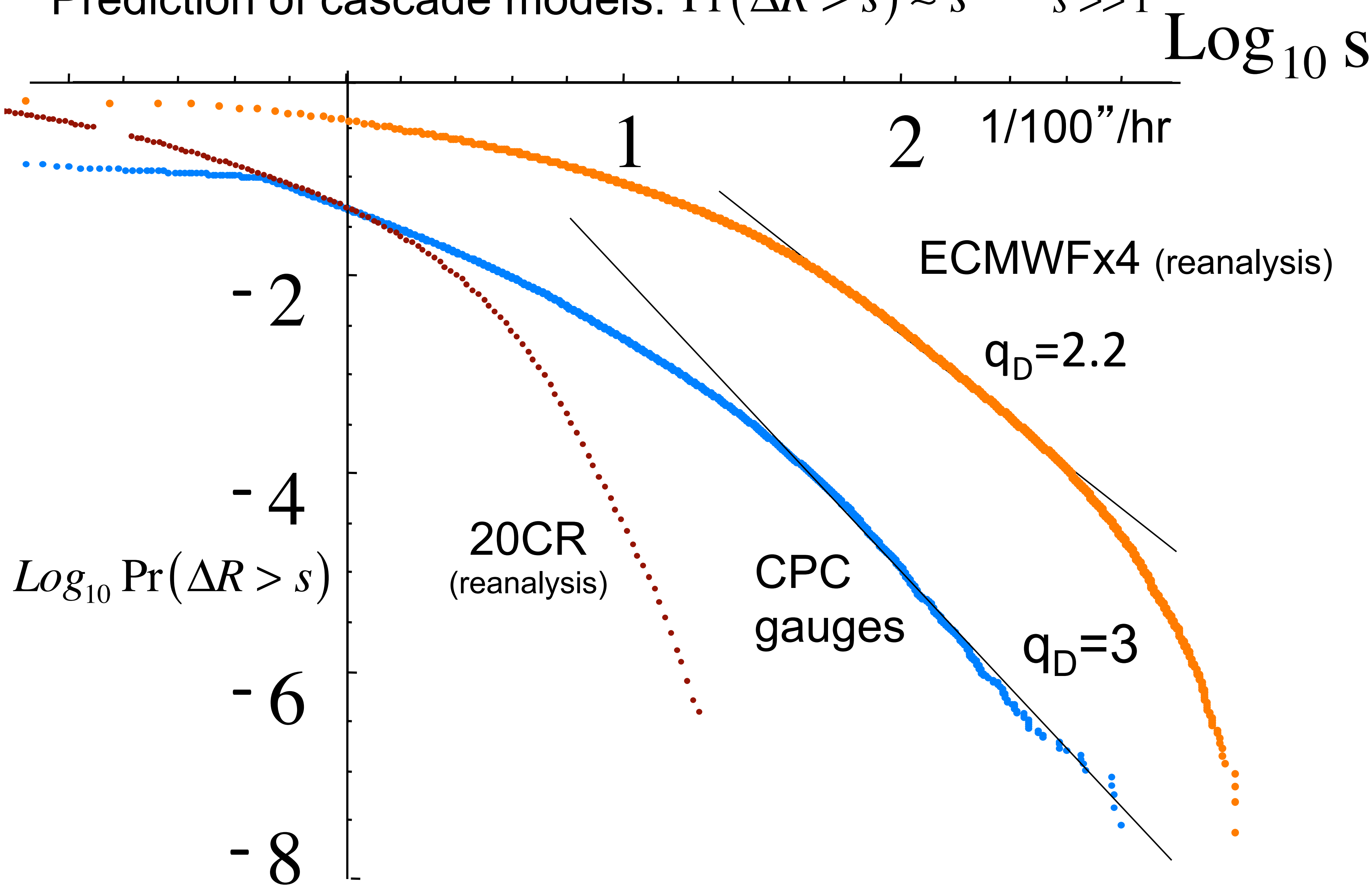
Probability distributions of rain water volumes in 10x10x10cm cubes from stereophotography of raindops.



Probability distributions of Rain rate from rain gauges

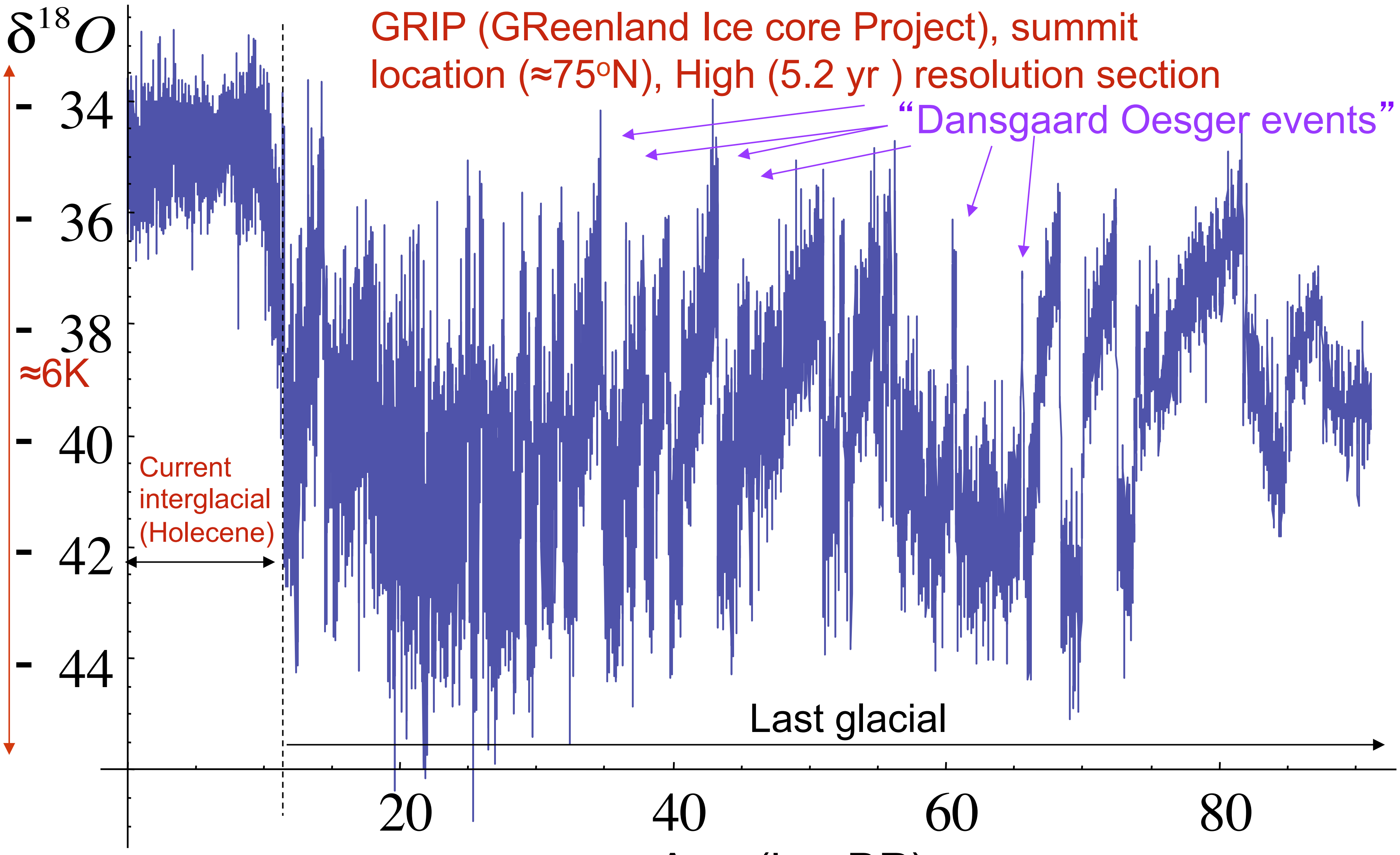
Extremes: data versus models

Prediction of cascade models: $\Pr(\Delta R > s) \approx s^{-q_D} \quad s >> 1$



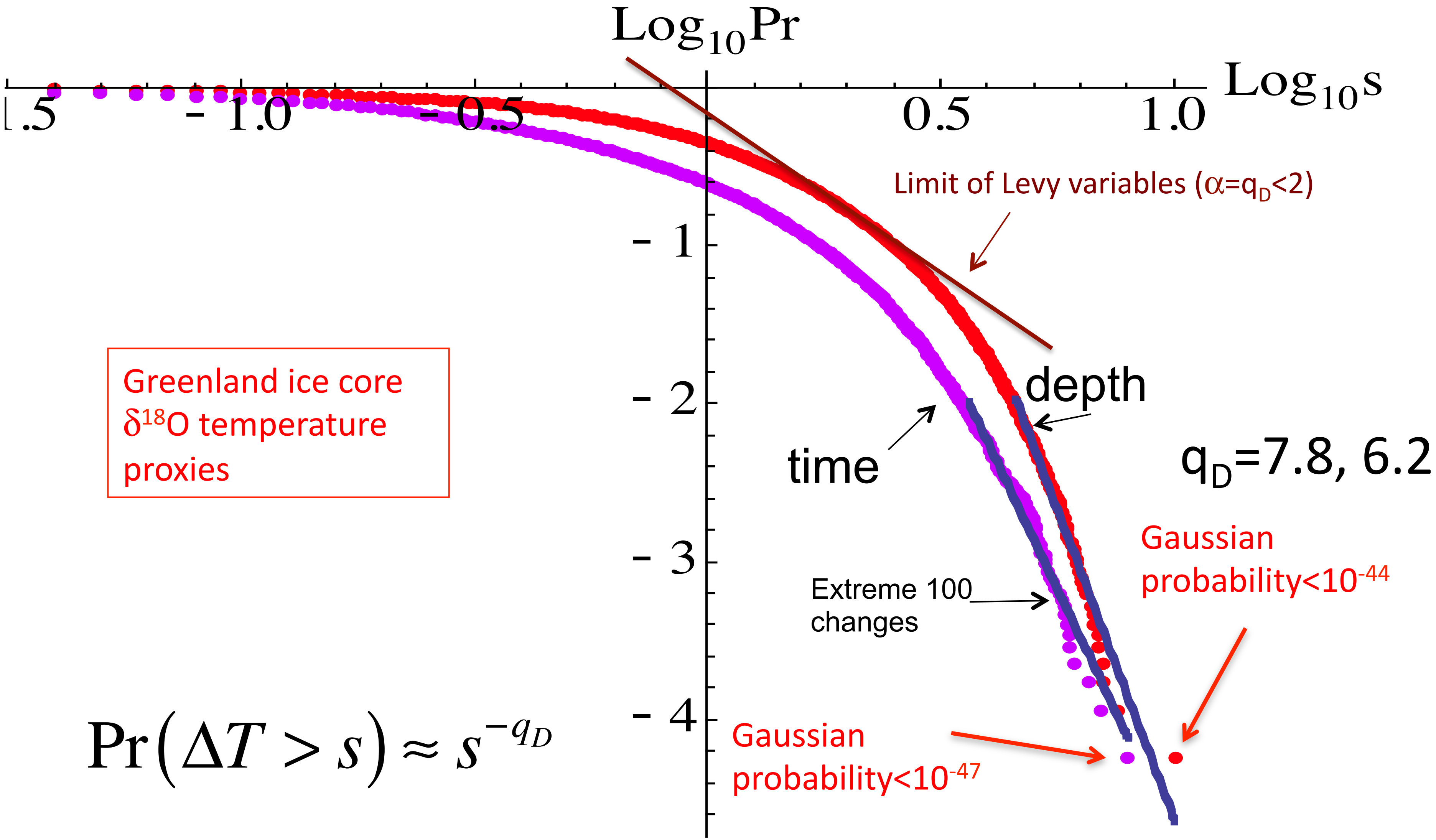
Abrupt events, extreme changes

Abrupt events, extreme changes



Data: thanks to P. Ditlevsen

GRIP Probabilities of extreme changes



Greenland ice core $\delta^{18}O$ temperature proxies

$$\Pr(\Delta T > s) \approx s^{-q_D}$$

q_D estimates for various geophysical fields

Table 5.1a A summary of various estimates of the critical order of divergence of moments (q_D) for various atmospheric fields.

Field	Data source	Type	q_D	Reference
Horizontal wind	Sonic	10Hz, time	7.5	Schmitt *et al.*, 1994
	Sonic	10 Hz	7.3	Finn *et al.*, 2001
	Hot wire probe	Inertial range	7.7	Fig. 5.22, Radulescu *et al.*, 2002
	Hot wire probe	Dissipation range	5.4	Fig. 5.22, Radulescu *et al.*, 2002
	Anemometer	15 minutes	7	Tchiguirinskaia *et al.*, 2006
	Anemometer	Daily	7	Tchiguirinskaia *et al.*, 2006
	Aircraft, stratosphere	Horizontal, 40 m	5.7	Lovejoy and Schertzer, 2007
	Aircraft, troposphere	Horizontal, 280 m – 36 km	≈ 5	Fig. 5.10
	Aircraft, troposphere	Horizontal, 40 m – 20 km	≈ 7 ± 1	Chigirinskaya *et al.*, 1994
	Aircraft, troposphere	Horizontal, 100 m	≈ 5	Schertzer and Lovejoy, 1985
	Radiosonde	Vertical, 50 m	5	Schertzer and Lovejoy, 1985, Lazarev *et al.*, 1994
	Scaling gyroscopes cascade (SGC) model (Box 3.4)	Time	6.9 ± 0.2	Chigirinskaya and Schertzer, 1996
Potential temperature	Radiosonde	Vertical, 50 m	3.3	Schertzer and Lovejoy, 1985
Humidity	Aircraft, troposphere	Horizontal, 280 m – 36 km	≈ 5	Fig. 5.10
Temperature	Aircraft, troposphere	Horizontal, 280 m – 36 km	≈ 5	Fig. 5.10
	Hemispheric, global	Annual, monthly	≈ 5, 5	Lovejoy and Schertzer, 1986, and unpublished analysis respectively
	Daily, stations	Average over 53 stations in France, daily single station (Macon)	4.5, 4.5	Ladoy *et al.*, 1991
Paleotemperatures	Ice cores	350 years (time), 0.55 m, 1 m (depth)	5, 5	Lovejoy and Schertzer, 1986, Fig. 5.21 respectively
Geopotential anomalies	Reanalyses	500 mb, daily	2.7	Sardeshmukh and Sura, 2009
Vorticity anomalies	Reanalyses	300 mb, daily	1.7	Sardeshmukh and Sura, 2009
Visible radiances (ocean surface)	Remote sensing	7 m resolution MIES data	3.6	Lovejoy *et al.*, 2001
Passive scalar (SF_6)	Fast response SF_6 analyzer	1 Hz	4.7	Finn *et al.*, 2001
Vertical CO_2 flux (above a field)	Aircraft new ground	Horizontal ≈ 1 km resolution	5.3	Austin *et al.*, 1991
Seveso pollution	Ground concentrations	In-situ measurements	2.2	Salvadori *et al.*, 1993
Chernobyl fallout	Ground concentrations	In-situ measurements	1.7	Chigirinskaya *et al.*, 1998; Salvadori *et al.*, 1993
Density of meteorological stations	WMO surface network	Geographic location of stations	3.7 ± 0.1	Tessier *et al.*, 1994

Most exponents: range 3-5

L+S 2013

q_D estimates for various hydrological fields

Table 5.1b A summary of various estimates of the critical order of divergence of moments (q_D) for various hydrological fields.

Field	Data source	Type	q_D	Reference
Radar reflectivity of rain	Radar reflectivity factor	1 km^3 resolution	1.1	Schertzer and Lovejoy, 1987
Rain rate	Gauges	Daily, Nimes	2.6	Ladoy *et al.*, 1991
	Gauges	Daily, time, France	≈ 3	Ladoy *et al.*, 1993
	Gauges	Daily, USA	1.7–3	Georgakakos *et al.*, 1994
	High-resolution gauges	8 minutes	≈ 2	Olsson, 1995
	High-resolution gauges	15 s	2.8–8.5	Harris *et al.*, 1996
	Gauges	Daily, time	3.6 ± 0.07	Tessier *et al.*, 1996
	Gauges	1–8 days	3.5	De Lima, 1998
	Gauges	Hourly, time	4.0	Kiely and Ivanova, 1999
	Gauges	Daily, four series from 18th century	3.78 ± 0.46	Hubert *et al.*, 2001
	Gauges	Hourly, time	≈ 3	Fig. 5.10c; Schertzer *et al.*, 2010
	Gauges	Hourly, time	≈ 3	Fig. 5.20b; Lovejoy *et al.*, 2012
	High-resolution gauges	15 s, averaged to 30 minutes	2.23	Verrier, 2011
Raindrop volumes	Stereophotography	10 m^3 sampling volume	5	Lovejoy and Schertzer, 2008
Liquid water at turbulent scales	Stereophotography	Total water in 40 cm cubes	3	Lovejoy and Schertzer, 2006b
Stream flow	River gauges (France)	Daily	3.2 ± 0.07	Tessier *et al.*, 1996
	River gauges (USA)	Daily	3.2 ± 0.07	Pandey *et al.*, 1998; Tessier *et al.*, 1996
	River gauges (France)	Daily	2.5–10	Schertzer *et al.*, 2006

Most exponents: ≈ 3

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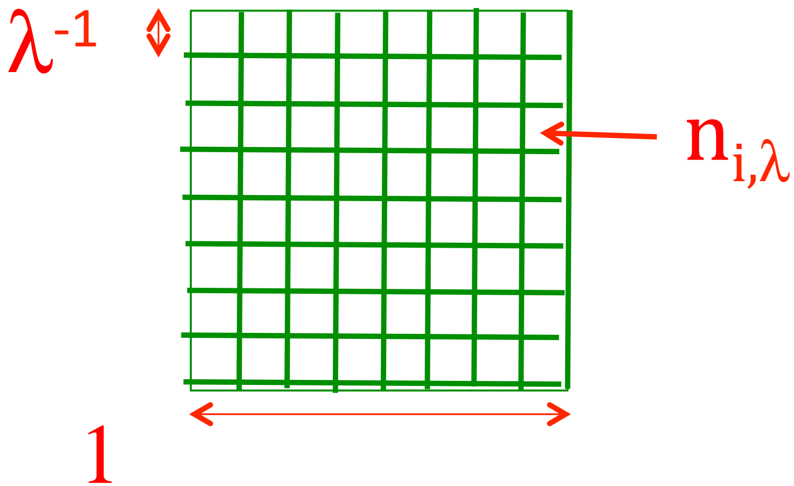
Multifractal analysis of sets of points: Codimension versus dimension multifractal formalism

Box, information, and correlation dimensions (1)

We introduced both the box (D_{box}) and correlation (D_{cor}) dimensions of a set of points: the first is the exponent of the average number of disjoint boxes size L/l needed to cover the set, while the second is the exponent of the number of point pairs separated by a distance ≤ L/l. Since both dimensions are in common use (D_{cor} particularly for characterizing strange "chaotic"/"strange" attractors) such as the Mandelbrot set, let us now consider the relation between the two. First suppose that the set of interest (denoted A) can be embedded in a d-dimensional "cube" of size L; and cover the cube with a grid of λ^d disjoint boxes each of size $l = L/\lambda$. Denote the number of points in the i^{th} l-sized grid box by $n_{i,l}$ so that the total number of points is:

$$N = \sum_{i=1}^{\lambda^d} n_{i,\lambda}$$



$l = \lambda^{-1}$

Box, information, and correlation dimensions (2)

If the points are from a strange attractor (such as the Lorenz attractor), then the space is the system's phase space and (with an ergodic hypothesis) we can interpret $P_{i,l} = n_{i,l}/N$ is an empirical frequency that approximates the probability of finding the system in the i^{th} box at phase space resolution l = L/l, this would be its asymptotic limit for an infinite resolution. In order to characterize the scale by scale statistics of the attractor, similarly to estimating the "trace moments" we can use a "partition function" approach to introduce the following family of measures indexed by q (Hentschel and Procaccia, 1983), (Grassberger, 1983), (Halsey et al., 1986):

$$\mu_q(\lambda) = \sum_{i=1}^{\lambda^d} P_{i,\lambda}^q$$

$P_{i,l} = n_{i,l}/N$

and with the corresponding scaling exponents

$$\mu_q(\lambda) \propto l^{\tau(q)} \propto \lambda^{-\tau(q)}$$

Box, information, and correlation dimensions (3)

q = 0: adopt the convention that for any x, $x^0 = 1$ if $x>0$, and $x^0 = 0$ if $x = 0$. In this case, μ_0 is simply the number of boxes needed to cover the set and $\tau(0) = -D_{box}$.

q = 2: in each box, the number of points which are within a distance l of each other is equal to the number of pairs in the box: (for large n and ignoring constant factors). However we have so that we see that μ_2 is proportional to the number of point pairs within a distance l, and hence $\tau(2) = D_{cor}$ (the correlation dimension).

The above suggests the definition:

Renyi dimension

$$D(q) = \frac{\tau(q)}{q-1} = \frac{1}{q-1} \lim_{l \to 0} \left[\frac{\log \mu_q}{\log l} \right]; \quad l = L/\lambda$$

Box, information, and correlation dimensions (4)

What about the value $q = 1$? In this case, since the sum of the probabilities is unity, we have $\mu_1 = \sum P_{i,\lambda} = 1$ so that we must use l'Hopital's rule to evaluate the limit q->1. We find:

$$D(1) = \lim_{l\to 0}\left[\frac{\sum_{i=1}^{\lambda^d} p_{i,\lambda}\log p_{i,\lambda}}{\log l}\right]; \quad l = L/\lambda$$

$D(1)$ is thus the exponent of the information I_l: $\quad I_\lambda = \sum_{i=1}^{\lambda^d} p_{i,\lambda}\log p_{i,\lambda}$

so that $I_l \approx l^{DI}$ where the information dimension $D_I = D(1)$.

Since we show that $\tau(q) = D(q\text{-}1) - K(q)$ so that the convexity of $K(q)$ implies the concavity of $\tau(q)$ so that $D(q)$ is a monotonically decreasing function of q; we therefore have the hierarchy: $D_{box} \leq D_I \leq D_{cor}$.

Codimension and dimension multifractal formalisms

Codimension (stochastic) — Dimension (deterministic)

Singularities

$$\varepsilon_\lambda = \lambda^\gamma$$

density

$$l = \lambda^{-1} \qquad vol(B_\lambda) = \lambda^{-d}$$

$$\Pi_\lambda = \int_{B_\lambda} \varepsilon_\lambda d^d \underline{x} = \lambda^{-\alpha_d}$$

integral

$$\Pi_\lambda = P_\lambda \qquad \varepsilon_\lambda = p_\lambda$$

$$\Pi_\lambda = \varepsilon_\lambda vol(B_\lambda) = \lambda^{\gamma - d}$$

$$\alpha_d = d - \gamma$$

Probabilities

$$\Pr(\varepsilon_\lambda = \lambda^\gamma) \approx \lambda^{-c(\gamma)}$$

$$Number(\Pi_\lambda = \lambda^{-\alpha_d}) = \lambda^{-f_d(\alpha_d)}$$

$$Number = \lambda^d \Pr$$

$$f_d(\alpha_d) = d - c(\gamma)$$

Statistical Moments

$$\langle \varepsilon_\lambda^q \rangle = \lambda^{K(q)}$$

$$\sum_{i=1}^{\lambda^d} \Pi_{\lambda,i}^q = \lambda^{-\tau_d(q)}$$

$$\sum_{i=1}^{\lambda^d} \Pi_{\lambda,i}^q = \left\langle \sum_{i=1}^{\lambda^d} (\lambda^{-d}\varepsilon_\lambda)^q \right\rangle = \lambda^{d(q-1)} \langle \varepsilon_\lambda^q \rangle = \lambda^{K(q)-d(q-1)}$$

$$\tau_d(\alpha_d) = d(q-1) - K(q)$$

$$c(\gamma) \overset{L.T.}{\leftrightarrow} K(q); \quad f_d(\alpha_d) \overset{L.T.}{\leftrightarrow} \tau(q)$$