energy per unit mass in the octave (E_n) divided by the typical time scale of the transfer, the eddy turnover time:

$$\prod_n \approx \frac{E_n}{\tau_n} \tag{2.48}$$

Now assume that the cascade is local, so that the dominant contribution to E_n comes from the velocity gradient at the same scale, i.e. v_n. This implies $E_n \sim {v_n}^2$ (recall that due to incompressibility all energies are taken per unit mass) and that $\tau_{vis,n} >> \tau_n$ so that there is no energy dissipation in this wavenumber band. If the energy injection rate ε (e.g. by stirring) at large scale is balanced by viscous dissipation at small scale then it is possible that the system is stationary (statistically invariant under translations in time) then $\prod_n \sim$ constant, i.e. there are no viscous losses and no sinks or sources. This is assumed to be a quasi-steady state: energy flows through the nth octave at a rate ε which is on average equal to the large-scale injection rate and to the small-scale dissipation (as we will see, such statistical stationarity is quite compatible with violent fluctuations):

$$\varepsilon = \prod_n \sim \frac{E_n}{\tau_n} \sim \frac{{v_n}^2}{\left(\frac{l_n}{v_n}\right)} \sim \frac{{v_n}^3}{l_n} \sim \text{constant} \tag{2.49}$$

(assuming that the injection rate is constant). $\prod_n$ is therefore a scale-invariant quantity (it is independent of n). This yields Kolmogorov's law (1941):

$$v_n \sim \varepsilon^{1/3} {l_n}^{1/3} \tag{2.50}$$

Since the fluctuation v_n is a scaling power law function of size l_n, we expect that the spectrum will also be a power law (see Box 2.2 for more details on Tauberian theorems that relate real space and Fourier space scaling). For wavenumber p, we therefore seek the spectral exponent β:

$$E(p) \sim p^{-\beta} \tag{2.51}$$

corresponding to the real space exponent 1/3 in Eqn. 2.50. Assuming $\beta > 1$ we get the following expression for the total variance due to all the low wavenumbers in the nth band:

$$v_n^2 \approx l_n^2 \int_{k_n/\sqrt{2}}^{\sqrt{2}k_n} dp\, p^2\, E(p) \tag{2.52}$$

(since the variance in a spherical shell between p and $p + \mathrm{d}p$ is $4\pi\, p^2 \mathrm{d}p$, and we ignore the constant factor). We thus obtain:

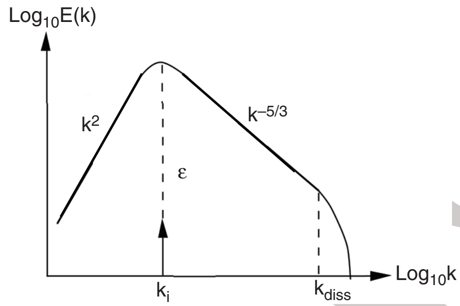

Fig. 2.5 Schematic diagram of 3D energy cascade showing the equipartition ("equilibrium") range at low wavenumbers, the energy flux injection wavenumber k_i, the "inertial" $k^{-5/3}$ range and the dissipation range $k > k_{diss}$ dominated by viscous β dissipation.

$$v_n^2 \approx l_n^2\, k_n^{3-\beta} \sim l_n^{2-3+\beta} \tag{2.53}$$

(since $l_n \sim k_n^{-1}$). Comparing this with Eqn. (2.50), we obtain $2 - 3 + \beta = 2/3$, or:

$$\beta = \frac{5}{3} \tag{2.54}$$

The Kolmogorov–Obukhov spectrum is thus derived:

$$E(k) \sim \varepsilon^{2/3} {k_n}^{-5/3} \tag{2.55}$$

A schematic diagram of the 3D cascade is shown in Fig. 2.5. The slope of the spectrum on the low-frequency side of the injection wavenumber is of the form $E(k) \sim k^2$. This follows since using statistical mechanical arguments, one expects that there is a low-frequency "equilibrium" range where each mode has roughly the same energy (equipartition). The spectral form $E(k) \sim k^2$ then follows, since there are $k^2 \mathrm{d}k$ modes between wavenumbers k and $k + \mathrm{d}k$.

2.4.3 Vortex stretching, the break-up of eddies and the cascade direction

It is easy to identify each term in the vorticity equation (2.40): $D_{\underline{\omega}}/Dt$ is the convective (total) derivative of the vorticity (remembering that the total derivative operator is just $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{v}\cdot\nabla$, it represents the change in a quantity that moves with the flow; it is also called a Lagrangian derivative), the term $\underline{\nu}\nabla^2\underline{\omega}$ (ignored in Eqn. (2.40)) represents the molecular dissipation, the term $(\nabla\cdot\underline{v})\underline{\omega}$ is the compressibility term; we consider here the simplest incompressible case, $\nabla\cdot\underline{v} = 0$. The all-important "vortex stretching" contribution $(\underline{\omega}\cdot\nabla)\underline{v}$ is so named because its component is only positive when the gradient of $\underline{v}$ is parallel to $\underline{\omega}$, in

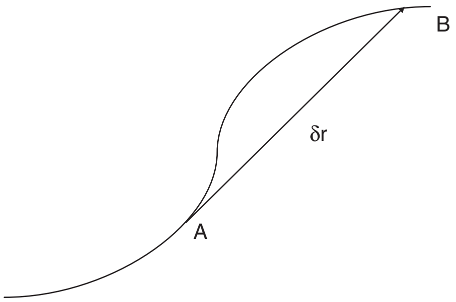

Fig. 2.6 Schematic showing an infinitesimal segment of a vortex line.

which case vortex lines are "stretched" by the velocity field (Fig. 2.6: a vortex line is like a line of electric or magnetic force – its tangent is everywhere parallel to the field lines; the analogous line for the velocity field is called a "streamline").

However, a more important property of vorticity is that – ignoring viscosity – vortex lines are material lines. To see this, let $\delta\underline{r}$ represent the vector between particles A and B (Fig. 2.6). Then the equation of evolution of $\delta\underline{r}$ is:

$$\frac{D(\delta\underline{r})}{Dt} = \frac{D(\underline{r}_A)}{Dt} - \frac{D(\underline{r}_B)}{Dt} = \underline{v}_A - \underline{v}_B = \delta\underline{v} \tag{2.56}$$

and to first order in $\delta\underline{r}$:

$$\delta\underline{v} = (\delta\underline{r}\cdot\nabla)\underline{v} \tag{2.57}$$

then:

$$\frac{D\delta\underline{r}}{Dt} = \delta\underline{v} = (\delta\underline{r}\cdot\nabla)\underline{v} \tag{2.58}$$

which is identical to the (incompressible) vorticity equation if $\delta\underline{r}$ is taken parallel to $\underline{\omega}$ (recall we are considering negligible viscosity, $\nu = 0$). This shows that if at some initial time a vortex line is composed of a given set of fluid particles then at any later time the (evolved) vortex line will still be composed of the same particles. Vortex lines are therefore material lines.

Now apply this to the evolution of vortex tubes (these are the surfaces bounded by vortex lines): the volumes enclosed by the tubes are constant, since the fluid is incompressible and vortex lines are material lines. As the system evolves, the ends of the tubes move apart on average (this is a statistical effect: in a turbulent fluid, the ends of the tube will execute a convoluted random walk; on average, they will move apart). Since the volumes of the tube are incompressible, this implies that as the lengths of the tubes increase the cross-sectional areas tend to decrease. Hence there will be "pinching" of the tube at certain regions where there is a high stretching, leading locally to extremely high gradients of $\underline{v}$. The $\nu\nabla^2\underline{v}$ term will become large and viscosity will tend to smooth the high gradients and break (smooth out) the vortex tubes. This stretching–pinching mechanism means that a fat (large) vortex tube "slims" (cross-sections become smaller) and then gets broken up, the energy flux being conserved throughout the process, except for the final viscous smoothing/dissipation at very small scales. If we now imagine a complex turbulent flow as a "spaghetti" of vortex tubes evolving in time, we can see that ends of tubes which are far apart will tend to move further apart (just as a drunkard tends to move away from his starting bar), and hence the tubes will be generally stretched and then pinched (Fig. 2.7). Since this causes tubes with initially large cross-sections to tend to evolve into tubes with small cross-sections, this gives a simple explanation for the downscale direction of energy cascades in three-dimensional turbulence, and indeed whenever vortex stretching is important.

2.4.4* The vorticity spectrum

In homogeneous isotropic turbulence $E(k)$ contains a lot (but by no means all!) of the statistical information about the turbulent flow (it is still only a second-order moment depending on only the separation $\underline{r}$ of the two points $\underline{x}$ and $\underline{x} + \underline{r}$; it is a "two-point" statistic). We now derive the relation between $E(k)$ and the spectrum of the vorticity, which will be important in considering two-dimensional turbulence. First we use the vector identity:

$$\underline{A}\cdot(\nabla \times \underline{B}) = \underline{B}\cdot(\nabla \times \underline{A}) - \nabla\cdot(\underline{A} \times \underline{B}) \tag{2.59}$$

If $\underline{A}$ and $\underline{B}$ are functions of $\underline{v}$, and if we assume that the statistical properties of $\underline{v}$ are independent of position (statistical homogeneity) then $\langle \underline{A} \times \underline{B} \rangle$ is a constant and it follows that the expectation of the last term is zero (i.e. $\nabla\cdot\text{constant} \equiv 0$). Now, using $\underline{A} = \nabla \times \underline{v} = \underline{\omega}$ and $\underline{B} = \underline{v}$, we obtain:

$$\langle \omega^2 \rangle = \langle \underline{A}\cdot(\nabla \times \underline{B}) \rangle = \langle \underline{B}\cdot(\nabla \times \underline{A}) \rangle = \langle \underline{v}\cdot(\nabla \times (\nabla \times \underline{v})) \rangle \tag{2.60}$$

Finally, using the following vector identity for incompressible flows:

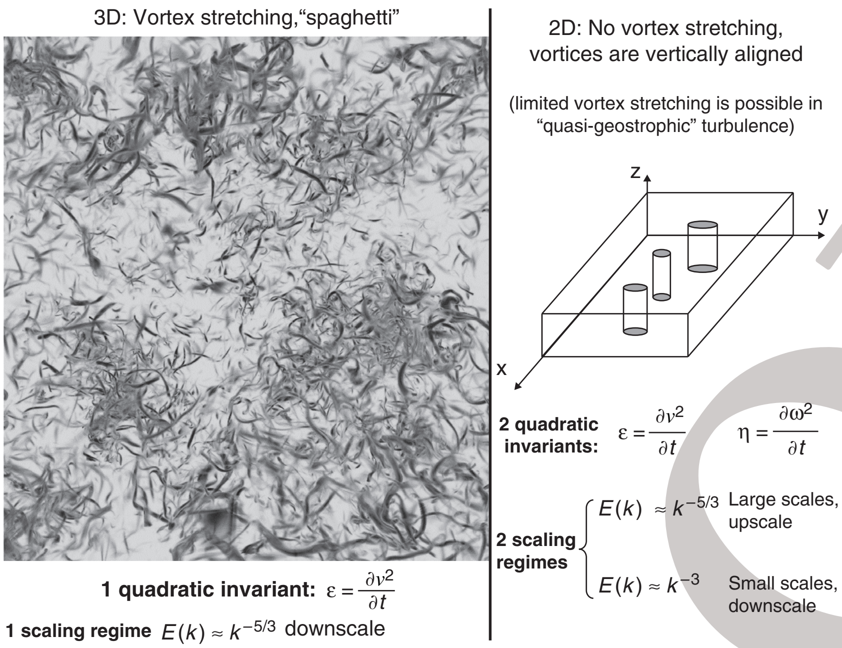

Fig. 2.7 A schematic showing the "spaghetti plate" view of vortices stretching and tangling in 3D turbulence (the left-hand side) compared with the vortex-stretching free dynamics in 2D turbulence (the right-hand side). Spaghetti of vortex tubes thanks to numerical simulations by M. Wilczek, thanks to http://www.vapor.ucar.edu/ software.

$$\nabla \times (\nabla \times \underline{v}) = -\nabla^2 \underline{v} \tag{2.61}$$

we obtain:

$$\langle \omega^2 \rangle = -\langle \underline{v} \cdot \nabla^2 \underline{v} \rangle \tag{2.62}$$

Therefore, since spectra are Fourier transforms of correlations and since the Laplacian corresponds to multiplication by $(ik)^2$ in Fourier space, we have the following relationship between the vorticity spectrum E_ω and velocity spectrum E_v:

$$E_\omega(k) = k^2 E_v(k) \tag{2.63}$$

Box 2.2 Scaling and Fourier transforms: correlation functions, structure functions and Tauberian theorems

In the following we will use both real-space and Fourier-space statistics, so it is useful to consider the general relation between real- and Fourier-space scaling. First define the Fourier transform and its inverse (note that ω in this section no longer denotes the vorticity but the angular frequency, i.e. the Fourier conjugate of the time t):

$$\tilde{v}(\omega) = F(v) = \int_{-\infty}^{\infty} dt\, e^{-i\omega t} v(t) \tag{2.64}$$

$$v(t) = F^{-1}(\tilde{v}) = \int_{-\infty}^{\infty} d\omega\, e^{i\omega t} \tilde{v}(\omega) \tag{2.65}$$

We recall two fundamental properties of Fourier transforms:

$$F\left(\frac{d^n v}{dt^n}\right) = (i\omega)^n \tilde{v}(\omega) \tag{2.66}$$

$$F(v * w) = \tilde{v}(\omega)\tilde{w}(\omega) \tag{2.67}$$

where $v * w$ is the convolution of v and w: