A final remark is in order. The bifurcation diagram provides information on the *quality* of the solutions of a model. For a complete comparison between the molecular model and the simplified equations, also the solution diagrams should be considered. Indeed, these diagrams, which report a proper norm (or seminorm) of the solution versus the parameter, together with the bifurcation diagrams give a complete insight into a model.

Acknowledgements

This work has been partly supported by MURST. Jean Gilder is gratefully acknowledged.

References

1. S.Hess, Z., *Naturforsch.* **31A**, 1034, (1976).
2. M.Doi, *J. Polym. Sci. Polym. Phys.* **19**, 229 (1981).
3. M.Doi, and S.F. Edwards, *The Theory of Polymer Dynamics*, Clarendon Press, Oxford, (1986).
4. G. Marrucci and P. L.Maffettone , *Macromolecules* **22**, 4076, (1989).
5. R. G.Larson, *Macromolecules* **23**, 3983, (1990).
6. P. L.Maffettone and S. Crescitelli, *J. Rheol.* **38**, 1559, (1994).
7. P. L.Maffettone and S. Crescitelli, *J. Non-Newtonian. Fluid Mech.* **59**, 73, (1995).
8. P. L.Maffettone and G.Marrucci, *J. Non Newtonian Fluid Mech.* **38**, 273, (1991).
9. E. J.Doedel, and Kernevez J. P., *AUTO: Software for Continuation and Bifurcation Problems in Ordinary Differential Equations*, California Institute of Technology (1986).
10. J.Guckenheimer and Holmes P., *Nonlinear oscillations, dynamical systems and bifurcation of vector fields*, Springer Verlag, (1983).

PREDICTABILITY OF MULTIFRACTAL PROCESSES: THE CASE OF TURBULENCE

D. MARSAN, D. SCHERTZER
Laboratoire de Météorologie Dynamique, Université Pierre et Marie Curie, Paris, France

S. LOVEJOY
Physics Department, McGill University, Montreal, Canada

Space-time multiplicative cascade models reproduce the scaling both in space and time of the Navier-Stokes equations. They are powerful tools in the investigation of space-time properties of turbulent flows, including their limits of predictability. We detail the use of such models in quantifying the decorrelation process associated with turbulence. It is shown that classical quantities like correlated or uncorrelated energy spectra give an incomplete description of such an intermittent dynamical process. We propose extensions of these quantities to overcome these limitations.

1 Introduction

The scaling symmetries of the Navier-Stokes equations are responsible for the intermittency in the inertial range. Most of the theoretical developments (which as the cascade models: log-normal[1], β-model[2,3], α-model[4], random β-model[5], universal multifractals[6], log-Poisson[7–9]) or applications concentrated on the spatial variability. The temporal intermittency was reduced through Taylor's frozen turbulence hypothesis to a purely spatial one, and too little attention has been paid to cascade models on space-time domains[10–12].

However, scaling in both space and time is an essential feature of turbulent dynamics. A scaling space-time framework is required for numerous applications, e.g.: sampling strategies for remote-sensing data assimilation, corrections to Taylor's hypothesis (taking into account not only the velocity at the largest scale, but indeed the velocities at all scales), or forecasting, etc. The important issue of causality in multifractal processes[12] and corresponding means for respecting it, i.e., removing the artificial temporal mirror symmetry of earlier time-space cascade models, enable us to address another important issue in the present paper: the determination of the limits of predictability of turbulent flows.

The sensitivity of nonlinear dynamics to small perturbations has been widely popularized with the help of the 'butterfly effect' metaphore in 'deterministic chaos' (few degrees of freedom). Two flows initially very close in phase-space will tend to diverge exponentially with time, becoming in a finite characteristic time (the inverse of which is the Liapounov exponent) fully uncorrelated. For fully developed turbulence (infinite number of degrees of freedom), due to scaling both in space and time, there is no characteristic time of the process, and one thus expects an algebraic decorrelation in time. The characterization of this phase-space divergence, or decorrelation process, in turbulence has been discussed mainly for atmospheric flows (Lilly [13] and Houtekamer [14] for reviews). Closure techniques for homogeneous turbulence: Quasi-Gaussian approximations [15], the Test-Field model [16], or the EDQNM model [17], lead to a characterization of the temporal evolution of the cross-correlated energy spectrum for two flows initially differing only for wavenumbers larger than an 'error cut-off wavenumber' $k_e(t=0)$. These models are intrinsically limited by strong assumptions on the statistics of the solution, thus missing the essential feature of the intermittency of the process. One may note that an approach based on shell-models has been proposed [18,19]. However, shell-models drastically lose their spatial dimensionality (see Chigirinskaya and Schertzer [20] for discussion and alternatives) and keep only a very reduced number of degrees of freedom, choosen typically around 30 for numerical purposes; therefore their relevance to turbulence predictability issues remains questionnable.

Multiplicative cascade models have a very large number of degrees of freedom, and correspond to the action of a scale-invariant generator which multiplicatively modulates, in an intermittent manner, the larger structures into smaller structures from the largest scale of the system down to the resolution scale. Their extension to space-time domains, justified by the scaling properties of the Navier-Stokes equation in both space and time, enables us to study the intermittency of the turbulent decorrelation process, commonly observed by meteorologists as long quiescent and predictable periods interrupted by short, sudden, non-predictable bursts of decorrelation.

We first recall multiplicative cascade models giving (scalar) velocity fields on space-time domains; the extension from space to space-time implies two fundamental characteristics: a scaling anisotropy between space and time (on average given by the Kolmogorov-Obukhov theory) and the respect of causality by breaking the artificial temporal mirror symmetry of earlier space-time cascade models. We then quantify the mean behavior of the decorrelation process for such fields, by considering the rather classical uncorrelated and respectively correlated energy spectra. However, it is shown that such quantities cannot describe completely or in a satisfactory manner the decorrelation process, and we propose extensions in order to explore the whole range of multifractal singularities.

2 Causal cascade models for the scalar velocity field

2.1 *Discrete space-time cascade models*

Cascade models operate through a scale-invariant generator acting from the largest scale L down to the smallest scale $l = \frac{L}{\Lambda}$ of the system, Λ being the maximum resolution, thus creating structures at all scales. For simplicity purposes, we look here at the two-dimensionnal case (2-D cut of the 3-D space). Consider a square domain of size $L \times L$, characterized by an intensity ϵ_0, and the scale invariant generator acting at any given step n by dividing the existing structures at scale $l_{n-1} = \frac{L}{\lambda_1^{n-1}}$ with intensities ϵ_{n-1} into λ_1^2 new structures at scale $l_n = \frac{l_{n-1}}{\lambda_1} = \frac{L}{\lambda_1^n}$ with intensities $\epsilon_n = \epsilon_{n-1} \times \mu\epsilon$ where λ_1 is an integer and the multiplicative increment $\mu\epsilon$ is a positive random variable with a second Laplace characteristic function $K(q)$ such that $\langle \mu\epsilon^q \rangle = \lambda_1^{K(q)}$. The iteration of this generator leads, after N steps ($\Lambda = \lambda_1^N$), to an intermittent field such that $\langle \epsilon_\Lambda^q \rangle \sim \Lambda^{K(q)}$.

A continuous version of this model (i.e. in the limit $\lambda_1 \to 1$ keeping Λ constant) would be characterized by its statistical invariance properties for any intermediate scale ratio λ, e.g. for the scaling function moment $K(q)$:

$$\forall \lambda \in (1, \Lambda): \ \langle \epsilon_\lambda^q \rangle \sim \lambda^{K(q)} \tag{1}$$

through the action of the contraction operator:

$$T_\lambda : (\underline{x}) \to \lambda^{-\mathcal{G}}(\underline{x}) \tag{2}$$

where in the simplest case shown here of a self-similar cascade the matrix $\mathcal{G}$ reduces to the identity matrice I. Linear Generalized Scale Invariance [21,22] (GSI) already involves matrices $\mathcal{G} \neq I$; diagonal matrices yield self-affine multifractals, with an associated generalized scale function, denoted $||.||$, more involved than a norm, satisfying:

$$||T_\lambda[\underline{x}]|| = \lambda^{-1}||\underline{x}|| \tag{3}$$

The structures created at all scales are interpreted as typical eddies transfering energy to smaller scales through a shearing process; such eddies possess a life-time τ_l depending on the scale l, after which they are considered to have been swept by other structures. In the framework of homogeneous turbulence [23,24], this life-time is a characteristic time for each scale, and scales like:

$$\tau_l \sim l^{\frac{2}{3}} \bar{\epsilon}^{-1/3} \tag{4}$$

with $\bar{\epsilon}$ being homogeneous in space and in scale. In the framework of inhomogeneous turbulence [25,26], the same scaling relation should hold only on the average, whereas at any given scale l, the eddy turn-over time τ_l is spatially intermittent, and depends on the non-homogeneous ϵ_l rather than on $\bar{\epsilon}$.

We thus see that a space-time cascade model for turbulence merely corresponds to a self-affine generator instead of a self-similar one as presented above, thus distinguishing the temporal and spatial coordinates. The contraction operator of Eq. 2 (now acting on a space-time domain) admits the following generator $\mathcal{G}$:

$$\mathcal{G} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \tag{5}$$

The operator is characterized by the elliptical dimension d_{el} such that the Jacobian of the transform of Eq. 2 is $\lambda^{-d_{el}}$. This dimension is the trace of $\mathcal{G}$, so that in our case $d_{el} = 5/3$; in the more general $(d+1)$-case (with d being the dimension of space, so $d+1$ being the dimension of the space-time cut), we would rather have obtained $d_{el} = d + 2/3$.

2.2 Continuous causal cascades

A major drawback of the rather pedagogical discrete cascade models is that they create artificialities due to the discrete dividing method (the fixed scale ratio λ_1 is integer), so that structures are generated only on a discrete set of scales.

In order to remove these artificialities, we now turn to continuous cascades which, contrary to the 'discrete' cascades, allow elementary cascade steps of scale ratio arbitrary close to 1. In fact, the continuous cascades may be obtained as a limit of discrete cascades by scale densification [27] and have log-infinitely divisible statistics, which recently aroused wide interest [9]. However, among these processes, strong universal multifractals [6] are the only ones which possess attractive and stable properties [28]. These processes have log-Lévy statistics; conservative universal multifractals have a moment scaling function of the form:

$$K(q) = \frac{C_1}{\alpha - 1}(q^\alpha - q) \tag{6}$$

where C_1 is the codimension of the mean singularity and measures the mean fractality of the field, whereas α is the Lévy index of the logarithm of the field ($\alpha \in (0;2]$). Self-affine spatial (i.e., non-causal) conservative multifractals ϵ_Λ are obtained by first computing their generator $\Gamma_\Lambda = log\epsilon_\Lambda$ as $\Gamma_\Lambda(\underline{x}) =$ $G_\Lambda(\underline{x}) \star \gamma(\underline{x})$, where γ is the sub-generator, i.e., an extremal Lévy white noise of Lévy index α, and G_Λ is a filter such that $G(\underline{x}) \sim ||\underline{x}||^{-\frac{d=2}{\alpha}}$, and $G_\Lambda(\underline{x}) = G(\underline{x})$ for $||\underline{x}|| \in [\lambda^{-1};1]$, with d the dimension of the space, and $||.||$ is the generalized scale function of Eq. 3. In order to generate a causal multifractal, the filter G should be defined as a retarded Green function; the simplest choice is of the form [12]:

$$\hat{G}(\underline{k},\omega) \sim \frac{1}{|\underline{k}|^{\frac{d+1}{\chi}} - (i\omega)^{\frac{3(d+1)}{2\chi}}} \tag{7}$$

where the symbol $(\hat{.})$ stands for the Fourier Transform, and χ is such that

$$\frac{1}{\alpha} + \frac{1}{\chi} = \frac{d_{el}}{2} \tag{8}$$

A more complete derivation of these results can be found in Marsan et al. [12].

2.3 The scalar velocity field

The Kolmogorov's refined similarity hypotheses [25,26] correspond to

$$\delta v_l \sim l^{\frac{1}{3}} \epsilon_l^{\frac{1}{3}} \tag{9}$$

where δv_l is the shear velocity at scale l, and ϵ_l is the spatially intermitent energy flux through this same scale. In a purely spatial domain, one can construct a field with statistical properties analoguous to the ones observed for turbulent shear velocities by performing a spatial fractional integration of order $\frac{1}{3}$ on the conservative field ϵ obtained by any of the spatial version of the models described above [29]:

$$\hat{v}_\Lambda(\underline{k}) = \hat{f}(\underline{k}) \, \widehat{\epsilon_\Lambda^{1/3}}(\underline{k}) \tag{10}$$

where the function $\hat{f}(\underline{k})$ scales like $|\underline{k}|^{-\frac{1}{3}}$. A similar method has been used [28,30–32] for the inverse problem in data analysis, i.e., in order to retrieve the energy flux ϵ from the (atmospheric horizontal wind) velocity shear δv yielding the estimates of universal parameters $C_1 = 0.15$ and $\alpha = 1.5$ of the energy flux. For a space-time domain, the analogue of Eq. 10 is

$$\hat{v}_\Lambda(\underline{k},\omega) = \hat{f}(\underline{k},\omega) \, \widehat{\epsilon_\Lambda^{1/3}}(\underline{k},\omega) \tag{11}$$

with $\hat{f}(\underline{k},\omega)$ being now a causal version of $||(\underline{k},\omega)||^{-1/3}$; the simplest choice corresponds to

$$\hat{f}(\underline{k}, \omega) \sim \frac{1}{|\underline{k}|^{1/3} - (i\omega)^{1/2}} \qquad (12)$$

3 Mean decorrelation process

For two fields $\delta v_{1\Lambda}$ and $\delta v_{2\Lambda}$ with corresponding sub-generators γ_1 and γ_2 such that $\gamma_1(x,t) = \gamma_2(x,t)$ for $t < t_0$ and $\gamma_1(x,t)$ and $\gamma_2(x,t)$ independent for $t \geq t_0$, a decorrelation process takes place as $\Delta t = t - t_0$ increases. This is due to the fact that their respective generators 'integrates' identical noises up to a scale $|\Delta t|^{3/2}$.

A very similar process can be observed, looking at only one field δv_Λ: instead of measuring the correlation between $\delta v_{1\Lambda}(.,t)$ and $\delta v_{2\Lambda}(.,t)$, we now look at the auto-correlation for an interval Δt, thus at the correlation between $\delta v_\Lambda(.,t)$ and $\delta v_\Lambda(.,t+\Delta t)$. The same phenomenology should also lead here to a decorrelation characterized by a 'cut-off' scale $|\Delta t|^{3/2}$. We will, in this third section, derive this result more carefully.

3.1 Definition of the spectra

We consider similar spectra to those introduced in homogeneous turbulence[16]:

$$E_\Lambda(k)\ \delta(k+k') = \langle \hat{v}_\Lambda(k,t)\ \hat{v}_\Lambda(k',t) \rangle \qquad (13)$$

$$E_{W_\Lambda}(k,\Delta t)\ \delta(k+k') = \langle \hat{v}_\Lambda(k,t)\ \hat{v}_\Lambda(k',t+\Delta t) \rangle \qquad (14)$$

$$E_{\Delta_\Lambda}(k,\Delta t)\ \delta(k+k') = \frac{1}{2}\langle [\hat{v}_\Lambda(k,t) - \hat{v}_\Lambda(k,t+\Delta t)]\ [\hat{v}_\Lambda(k',t) - \hat{v}_\Lambda(k',t+\Delta t)] \rangle \qquad (15)$$

One may note that the presence of the $\delta(.)$ function is merely a consequence of the statistical translation invariance of the velocity field. E_Λ is the usual spectrum of (total) energy, E_{W_Λ} and E_{Δ_Λ} are respectively the correlated energy and the uncorrelated energy spectra, for a given time lag Δt. Due to the stationnarity of E_Λ, we have indeed

$$E_{W_\Lambda}(k,\Delta t) + E_{\Delta_\Lambda}(k,\Delta t) = E_\Lambda(k) \qquad (16)$$

We define the following energy transfers as time derivatives of the corresponding spectra:

$$T_{W_\Lambda}(k,\Delta t) = \partial_t E_{W_\Lambda}(k,\Delta t) \qquad (17)$$

$$T_{\Delta_\Lambda}(k,\Delta t) = \partial_t E_{\Delta_\Lambda}(k,\Delta t) \qquad (18)$$

and because of the stationnarity of $E_\Lambda(k)$:

$$T_{W_\Lambda}(k,\Delta t) = -T_{\Delta_\Lambda}(k,\Delta t) \qquad (19)$$

hence T_{W_Λ} and $-T_{\Delta_\Lambda}$ indeed correspond to energy transfers from the correlated energy spectra into the uncorrelated energy spectra.

3.2 Determination of the spectra

For conservative multifractal fields, correlation measures in space-time domains are easily derived[12]:

$$\langle \epsilon_\Lambda^{q_1}(x,t)\epsilon_\Lambda^{q_2}(x+\Delta x, t+\Delta t)\rangle \sim \Lambda^{K(q_1)+K(q_2)}\|(\Delta x,\Delta t)\|^{K(q_1)+K(q_2)-K(q_1+q_2)} \qquad (20)$$

with the generalized scale function $\|.\|$ of Eq. 3, and $K(q)$ being the moment scaling function of the field ϵ_Λ. For example, any function of the type $\|(x,t)\| = [|x|^\xi + |t|^{\frac{3\xi}{2}}]^{\frac{1}{\xi}}$ where x and t have been nondimensionnalized by dividing them with the integral scale L and time T respectively, and ξ is positive, are scale functions (note that in the limit $\xi \to \infty$ we obtain $\|(x,t)\| = max\{|x|, |t|^{\frac{3}{2}}\}$). For $\eta = q_1 = q_2$, this leads to:

$$\langle \epsilon_\Lambda^\eta(x,t)\epsilon_\Lambda^\eta(x+\Delta x,t+\Delta t)\rangle \sim \|(\Delta x,\Delta t)\|^{-K(2,\eta)}\ \Lambda^{2K(\eta)} \qquad (21)$$

The corresponding spectrum $E_{\epsilon_\Lambda^\eta}(k\omega)$ is thus

$$E_{\epsilon_\Lambda^\eta}(k,\omega)\|(k,\omega)\|^{-d_{el}+1}\delta(k+k')\delta(\omega+\omega') = \langle \widehat{\epsilon_\Lambda^\eta}(k,\omega)\widehat{\epsilon_\Lambda^\eta}(k',\omega')\rangle \qquad (22)$$

which yields

$$E_{\epsilon_\Lambda^\eta}(k,\omega) \sim \|(k,\omega)\|^{-1+K(2,\eta)}\ \Lambda^{2K(\eta)} \qquad (23)$$

where $K(q,\eta) = K(\eta q) - qK(\eta)$ is the moment scaling function of $\epsilon_\Lambda^{(\eta)}$, the normalized η power of ϵ_Λ:

$$\epsilon_\Lambda^{(\eta)} = \frac{\epsilon_\Lambda^\eta}{\langle \epsilon_\Lambda^\eta \rangle} \qquad (24)$$

Note that the term $||(k,\omega)||^{-d_{el}+1}$ in the l.h.s. of Eq. 22 corresponds to the integration factor $|k|^{-d+1}$ in isotropic spaces, the anisotropy of the space-time domain introducing the dimension d_{el} instead of d. The spectrum E_{W_Λ} can now be written as

$$E_{W_\Lambda}(k,\omega) = |\hat{f}(k,\omega)|^2 E_{\epsilon_\Lambda^\eta}(k,\omega) \tag{25}$$

with $\hat{f}(k,\omega)$ defined in Sec. 2.3. Note that $|\hat{f}(k,\omega)|^2$, i.e., the composition of a causal and an anti-causal operators, is indeed no longer causal, since it depends on $|\omega|$ and no longer on the sign of ω; the mirror symmetry along the temporal axis is restored in this composition. This corresponds to the fact that our spectra, and merely all two-points correlation measures, possess this mirror symmetry: $E_{W_\Lambda}(k,\Delta t) = E_{W_\Lambda}(k,-\Delta t)$. We simply have $|\hat{f}(k,\omega)|^2 \sim ||(k,\omega)||^{-2/3}$.

Using Eq. 23, we get

$$E_{W_\Lambda}(k,\omega) \sim ||(k,\omega)||^{-5/3+K(2,1/3)}\ \Lambda^{2K(1/3)} \tag{26}$$

The scaling of the spectrum holds when we change the representation, though the norm involved acts on the new domain; thus

$$E_{W_\Lambda}(k,\Delta t) \sim ||(k,\Delta t)||^{-5/3+K(2,1/3)}\ \Lambda^{2K(1/3)} \tag{27}$$

The classical spectrum $E_\Lambda(k)$, scaling like $E_\Lambda(k) \sim |k|^{-5/3+K(2,1/3)}\ \Lambda^{2K(1/3)}$ (note that indeed $E_\Lambda(k) = E_{W_\Lambda}(k,\Delta t = 0)$), we eventually get

$$E_{W_\Lambda}(k,\Delta t) \sim E_\Lambda(k)[|k|^{-1}||(k,\Delta t)||]^{-5/3+K(2,1/3)} \tag{28}$$

3.3 Results

We finally obtain

$$E_\Lambda(k) \sim k^{-\frac{5}{3}+K(2,\frac{1}{3})}\ \Lambda^{2K(\frac{1}{3})} \tag{29}$$

$$E_{W_\Lambda}(k,\Delta t) \sim E_\Lambda(k)\ \phi(k,\Delta t) \tag{30}$$

$$T_{W\Lambda}(k,\Delta t) \sim k^{-\frac{2}{3}+K(2,\frac{1}{3})}\ |\Delta t|^{\frac{1}{2}}\ \dot{\phi}(k|\Delta t|^{\frac{3}{2}})\ \Lambda^{2K(\frac{1}{3})} \tag{31}$$

The ϕ-function ($\dot{\phi}$ denotes its derivative) is a cut-off function, explicitly determined by the choice of the scale function, such that:

$$\phi(k,\Delta t) = ||(1, k/k_e(\Delta t))||^{-5/3+K(2,1/3)} \tag{32}$$

with the cut-off wavenumber $k_e(\Delta t)$ scaling like $k_e(\Delta t) \sim \Delta t^{-3/2}$. As can be seen, the limit $k << k_e(\Delta t)$ leads to $E_{W_\Lambda}(k,\Delta t) \sim E_\Lambda(k)$, and the limit $k >> k_e(\Delta t)$ the spectrum $E_{W_\Lambda}(k,\Delta t)$ tends to zero as a consequence of the breaking of scaling introduced by $k_e(\Delta t)$.

The transfer T_{W_Λ} is not stationnary in the range $k << k_e(\Delta t)$. This is due to the fact that such a function is the result of a time derivative of order 1, though the physical dependence of the system is in $|\Delta t|^{\frac{3}{2}}$ (due to the scaling anisotropy between space and time). In order to obtain a stationnary transfer in this range, we can introduce the fractional transfer $T'_{W_\Lambda}(k,\Delta t) = \partial_{t^{\frac{3}{2}}} E_{W_\Lambda}(k,\Delta t)$, giving

$$T'_{\Delta_\Lambda}(k,\Delta t) \sim k^{-\frac{2}{3}+K(2,\frac{1}{3})}\ \dot{\phi}^{(\frac{3}{2})}(k|\Delta t|^{\frac{3}{2}})\ \Lambda^{2K(\frac{1}{3})} \tag{33}$$

with $\dot{\phi}^{(\frac{3}{2})}(x)$ the fractionnal derivative of order $\frac{3}{2}$ of $\phi(x)$.

Figure 1 displays the uncorrelated energy spectra $E_{\Delta_\Lambda}(k,\Delta t)$ for the first 128 time steps (a time step corresponding to the mean life-time of the structures at the smallest scale) of a simulated continuous causal multifractal scalar velocity field with the universal parameters $\alpha = 1.5$ and $C_1 = 0.15$. The spectra are averaged on 100 realizations, and are computed for 256x256 2D-cuts. As expected, the uncorrelated energy spectrum develops from large wave-numbers to smaller wave numbers; the transition wave-number $k_e(\Delta t)$ between the large wave-numbers Kolmogorov regime and the low-wave numbers range where the decorrelation process is occuring, scales as $k_e(\Delta t) \sim \Delta t^{-\frac{3}{2}}$, as shown by Figure 2 (the cut-off wavenumber $k_e(\Delta t)$ was estimated as being the smallest k such that $E_{\Delta_\Lambda}(k,\Delta t) > 0.99 E_\Lambda(k)$).

4 Limits of the (second order) correlation measures and beyond

Eq. 20 shows that the correlation with any order (q_1,q_2) decays algebraically with the generalized scale, whereas spectra considered in the previous section correspond to second order moment and therefore yield information only for $q_1 = q_2 = 1$. As any given order of moment of $\langle\epsilon_\Lambda^q\rangle$ of a multifractal field ϵ_Λ corresponds to a given singularity $\gamma_q = \dot{K}(q)$ using the Legendre transform [33], this corresponds to characterizing only one singularity of the decorrelation process. Therefore, we consider the following η^{th} order correlation structure functions (in fact η^{th} order correlation measures, with $q_1 = q_2 = \eta$):

$$C_\Lambda(\epsilon,\eta,\Delta x,\Delta t) = \frac{\langle\epsilon_\Lambda^\eta(x,t)\ \epsilon_\Lambda^\eta(x+\Delta x, t+\Delta t)\rangle}{\langle\epsilon_\Lambda^\eta(x,t)\rangle^2} \tag{34}$$

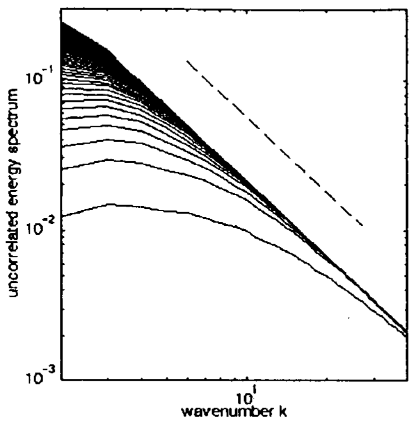

Figure 1: Uncorrelated energy spectra E_{Δ_Λ} for the first 128 time steps, from bottom to top (a time step beingthe mean life-time of smallest scale structures), for 100 realizations of a causal cascade which scale resolution $\Lambda = 256$. As expected, a scaling range close to Kolmogorov $-\frac{5}{3}$ scaling (displayed by the dashed line) develops from large wavenumbers to small wavenumbers.

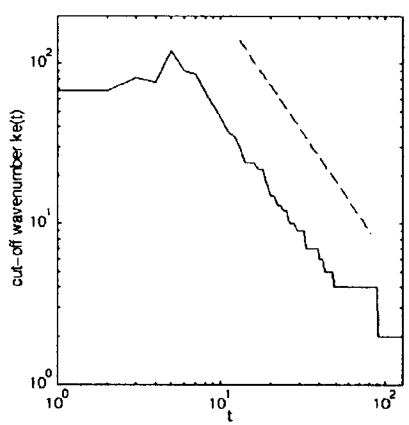

Figure 2: Cut-off wavenumber $k_e(t)$ deduced from the simulations of Figure 1. The dashed line corresponds to the $k_e(t) \sim t^{-1.5}$ law.

since they a priori allow one to explore the full range of singularities due to the fact they scale as the normalized η power of the field:

$$C_\Lambda(\epsilon, \eta, \Delta x, \Delta t) \sim \lambda^{K(2,\eta)}; \|(\Delta x, \Delta t)\| = \frac{L}{\lambda} \tag{35}$$

Indeed, the corresponding codimension function $c(\gamma, \eta)$ to $K(2, \eta)$, i.e., the scaling exponent function of the probability distribution of the singularities γ of the normalized η of the field:

$$\epsilon_\lambda^{(\eta)} \sim \lambda^{\gamma'} \; ; \; Pr(\gamma' \geq \gamma) \sim \lambda^{-c(\gamma,\eta)} \tag{36}$$

is:

$$c(\gamma, \eta) = c(\frac{\gamma + K(\eta}{\eta})) \tag{37}$$

$K(q, \eta), c(\gamma, \eta)$ are, as $K(q), c(\gamma)$, dual pairs for the Legendre transform. The nonlinearity of $K(q, \eta)$ (corresponding to non unicity of γ, i.e., multifractality), implies an intermittent behavior which is well beyond the scope of the theories or models of homogeneous turbulence. For instance, the existence of 'bursts' of impredictability is related to the fact that the estimates on a finite sample of the η^{th} order correlation structure function are more and more intermittent for increasing order η. This intermittency may be so extreme that it can induce two fundamental statistical problems ('multifractal phase transitions'[34] respectively of second and first order) for any D dimensional valued process ϵ: (i) a possible spurious maximum observable singularity $\gamma_s^{(\eta)}$ due to the finite size of sample ($N_s \sim \Lambda^{D_s}$ being the number of realizations of scale ratio Λ, D_s the 'sampling dimension') (ii) a possible statistical divergence for higher order: $\eta \geq \eta_D(q = 2)$, $\eta_D(q)$ being the critical order of divergence. In the case of universal multifractals ($\Delta_s = D + D_s$):

$$c(\frac{\gamma_s^{(\eta)}}{\eta^\alpha}) = \frac{\Delta_s}{\eta^\alpha} \tag{38}$$

$$K(q^{(\eta_D(q))}) = \frac{D}{\eta_D(q)^\alpha}(q^{(\eta_D(q))} - 1) \tag{39}$$

Conclusion

The decorrelation process of turbulent flows is characterized by a strong temporal and spatial variability, a consequence of scaling dynamics. Causal multifractal processes offer a relevant framework for the analysis of such variability.

They indeed reproduce the mean behavior obtained with other models, but also display properties going beyond it. We show that these latter properties can be explored with the help of extension of the classical correlation measures, and we demonstrate that indeed the latter have rather interesting general properties which should be investigated in detail.

Acknowledgments

We thank F. Schmitt , Y. Chigirinskaya, D. Kwak and C. Naud for stimulating discussions.

References

1. A. M. Yaglom, *Sov. Phys. Dokl.* **2**, 26 (1966).
2. E. A. Novikov and R. Stewart, *Izv. Akad. Nauk. SSSR Ser. Geofiz*, **3**, 408 (1964).
3. U. Frisch, P. L. Sulem and M. Nelkin, *J. Fluid Mech.*, **87**, 719 (1978).
4. D. Schertzer and S. Lovejoy in *Turbulence and chaotic phenomena in fluids*, ed. T. Tatsumi (North-Holland, 1984).
5. R. Benzi, G. Paladin, G. Parisi and A. Vulpiani A., *J. Phys. A*, **17**, 3521 (1984).
6. D. Schertzer and S. Lovejoy, *J. Geoph. Res.*, **92**, 9693 (1987).
7. Z. She and E. Leveque, *Phys. Rev. Letters*, **72**, 336 (1994).
8. B. Dubrulle, *Phys. Rev. Letters*, **73**, 959 (1994).
9. Z. She and E. Waymire, *Phys. Rev. Letters*, **74**, 262 (1995).
10. P. Brenier, D. Schertzer and S. Lovejoy, *Annales Geophysicae*, **Sup. 8**, 320 (1991).
11. Y. Tessier, S. Lovejoy and D. Schertzer, *J. Appl. Meteor.*, **32**, 223 (1993).
12. D. Marsan, D. Schertzer and S. Lovejoy, *J. Geoph. Res.*, **101**, D21 (1996).
13. D. K. Lilly in *Turbulence and predictability in geophysical fluid dynamics and climate dynamics*, eds. M. Ghil, R. Benzi and G. Parisi (North-Holland, 1985).
14. P. L. Houtekamer, *PhD thesis*, University of Wageningen, Holland (1992).
15. E. N. Lorenz, *Tellus*, **3** (1969).
16. C. E. Leith and R. H. Kraichnan, *J. Atm. Sci.*, **29** (1972).
17. O. Métais and M. Lesieur, *J. Atm. Sci.*, **43**, 9 (1986).
18. A. Crisanti, M. H. Jensen and A. Vulpiani, *Phys. Rev. Letters*, **70**, 166 (1993).
19. A. Crisanti, M. H. Jensen, G. Paladin and A. Vulpiani, *J. Phys. A*, **26**, 6943 (1993).

20. Y. Chigirinskaya and D. Schertzer in *Stochastic Models in Geosystems*, eds. A. Molchanov and W. A. Woyczynski (Springer-Verlag, 1997).
21. D. Schertzer and S. Lovejoy, *Phys. - Chem. Hyd. J.*, **6**, 623 (1985).
22. D. Schertzer and S. Lovejoy, *P. A. Geo.*, **130**, 209 (1989).
23. A. N. Kolmogorov, *Dokl. Acad. Sci. USSR*, **30**, 299 (1941).
24. A. Obukhov, *Dokl. Akad. Nauk. SSS*, **32**, 22 (1941).
25. A. N. Kolmogorov, *J. Fluid Mech.*, **83**, 349 (1962).
26. A. Obukhov, *J. Fluid. Mech.*, **13** (1962).
27. D. Schertzer and S. Lovejoy , *J. Appl. Meteor.*, (in press).
28. D. Schertzer, S. Lovejoy and F. Schmitt in *Small-scale structures in 3D hydro and MHD turbulence*, eds. M. Meneguzzi et al. (Springer-Verlag, 1995).
29. J. Wilson, D. Schertzer and S. Lovejoy *Nonlinear Variability in Geophysics: Scaling and Fractals*, eds. D. Schertzer and S. Lovejoy, Kluwer (1991).
30. F. Schmitt, D. Lavallée, D. Schertzer and S. Lovejoy, *Phys. Rev. Letters*, **68**, 305 (1992).
31. F. Schmitt, D. Lavallée, S. Lovejoy, D. Schertzer and C. Hooge, *Comptes-Rendus de l'Académie des Sciences*, **314**, 749 (1992).
32. F. Schmitt, D. Schertzer, S. Lovejoy and Y. Brunet, *Fractals*, **1**, 568 (1993).
33. G. Parisi and U. Frisch in *Turbulence and predictability in geophysical fluid dynamics and climate dynamics*, eds. M. Ghil, R. Benzi and G. Parisi (North-Holland, 1985).
34. D. Schertzer and S. Lovejoy, *Fractals in the Natural and Applied Sciences*, ed. M. M. Novak, (North Holland, 1994).