

**SCALING LAWS FOR ASYMPTOTICALLY THICK CLOUDS
DIMENSIONAL DEPENDENCE - PHASE FUNCTION INDEPENDENCE**

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1. INTRODUCTION

Very thick horizontally homogeneous plane-parallel layers with conservative scattering are known to have transmittances (T) and albedoes (R) that scale as

$$T = 1-R \propto \tau^{-1} \quad \tau \gg 1 \quad (1)$$

when optical thickness τ becomes very large, the phase function and illumination geometry affecting only the common prefactor. This regime corresponds to the diffusion of photons through the layer.

This analysis is extended to finite homogeneous clouds in two and three dimensions, as well as inhomogeneous fractal clouds with a definite inner cut-off scale using a Discrete Angle (DA) approach to radiative transfer and borrowing generously from non-linear systems theory and lattice statistical mechanics. DA methods are reviewed below, the results pertaining to scaling can be summarized by the more general algebraic scaling laws

$$T \propto \tau^{-\nu_T} \quad 1-R \propto \tau^{-\nu_R} \quad \tau \gg 1 \quad (2)$$

- where (i) $\nu_T \leq 1$ the inequality applies to the fractal case where the holes will enhance transmittance, τ must then be interpreted as an average.
 (ii) $\nu_R \leq \nu_T$ the inequality applies to the finite case where light will escape through the sides.
 (iii) ν_R, ν_T are independent of the postulated scattering phase function.

The last point shows that these scaling exponents define universality classes amongst model clouds; this important feature justifies taking the asymptotic limit for the sake of its conceptual simplicity. Large scale Monte Carlo simulations show that (i), (ii) and (iii) carry over to the more realistic continuous angle phase functions and transfer theory with somewhat different numerical values. Finally, we discuss the significance of these results to meteorological and climatological modeling as well as remote sensing.

2. PLANE-PARALLEL CLOUDS: $\nu_R = \nu_T = 1$

Schuster (1905) was the first to show that, within the context of two-flux theory, transmittance through and reflection from horizontally extended homogeneous atmospheres obey algebraic (rather than exponential) laws in absence of true absorption

$$1-R = T = \frac{1}{1+A\tau} \quad (3)$$

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where A is a simple function of the optical properties of the medium - say g , the asymmetry factor - and (possibly) the illumination conditions - namely μ_0 , the cosine of the incidence angle. Thus the only characteristic optical length in the problem is $1/A \approx 1/(1-g)$; as soon as $(1-g)\tau$, which may be considered as a rescaled optical thickness, is large, 1-R and T scale as in eq.(1). This behavior is universal in the sense that it does not depend on μ_0 or g (except for the common prefactor, A^{-1}). Accounting for the directly transmitted beam introduces transient exponential terms with characteristic optical depth μ_0 , see Meador and Weaver (1980).

Asymptotic theory for the complete radiance distribution reviewed by van de Hulst (1980) gives (3) as the leading term with an extra dependence of the prefactor on μ , cosine of the viewing angle. By definition, DA radiative transfer cannot make a statement about μ_0 or μ ; nevertheless Gabriel (1988) retrieves (3) as an exact solution to the functional equation obtained by "adding" two non-absorbing layers of arbitrary optical thickness.

3. FINITE HOMOGENEOUS CLOUDS: $\nu_R < \nu_T = 1$

DA radiative transfer may be thought of as a model for describing systems where photon propagation is confined to a lattice. Adopting space-filling cells, such as squares or triangles in 2-D, cubes in 3-D, one can apply "doubling" ideas to DA radiative transfer. Lovejoy et al. (1989b) interpret the resulting formulae as a bidimensional iterated map in (R,T) space. Using methods developed to investigate non-linear dynamical systems (e.g. fixed point stability analysis), they obtain a closed-form approximation for ν_R . This estimate is improved numerically by considering ever larger systems (i.e. quadrupling, etc.). The scaling exponents are numerically estimated to be $\nu_R = 3/4, 1/2$ in 2-D and 3-D respectively.

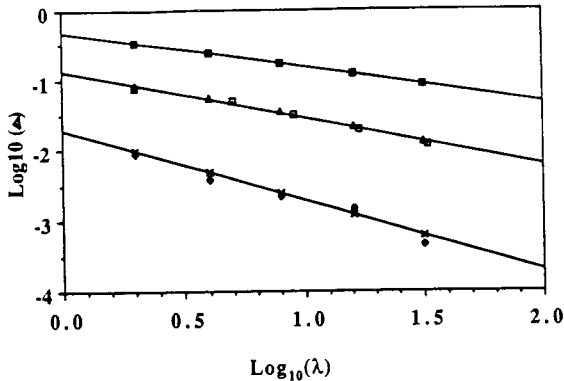
A further discretization, of space this time, allows DA radiative transfer problems to be solved exactly for (internally) homogeneous media, cf. Gabriel (1988). Figure 1 illustrates the results; the 1-R and T curves exhibit the theoretically predicted slopes. It is notable that ν_T retains its diffusional value of 1. Universality respective to the DA phase function is supported by the 2-D results as squares and triangles yield the same value for ν_R .

Extensive Monte Carlo simulations were performed in order to verify whether DA predictions carry over to the equivalent standard radiative transfer problems. In other words, do continuous angle phase functions belong to the same universality classes as their DA counterparts? Table 1 describes an experiment in 2-D that extends to $\tau = 10^{2.5} \approx 316$ and uses three different values of g which selects a 2-D analogue of the Henyey-Greenstein phase-function. Authors

such as Davies (1978) addressed equivalent 3-D problems using some model phase-function or even Mie coefficients usually based on Deirmendjian's C1 cloud droplet size distribution. While exploring various aspect ratios, it is not obvious that their results extend deep enough into the asymptotic regime which is notoriously time consuming for the Monte Carlo method.

The results, presented in fig. 2, are consistent with the DA prediction and positively exclude the plane-parallel exponent for 1-R, namely 1. At the same time, they suggest that 1-R and T are well approximated by a universal function of the optical thickness rescaled as $(1-g)\tau$, following van de Hulst and Grossmann (1968). Notice that our study of "scaling" focuses on the exponent of τ rather than this effect on the prefactor.

Fig 1: DA RESULTS FOR FINITE CLOUDS MADE OF SQUARE AND TRIANGULAR CELLS IN 2-D, CUBES IN 3-D.



The ordinate is \log_{10} of 1-R or T, the abscissa is \log_{10} of the size parameter λ which is proportional to optical thickness in this homogeneous case. The unit square coefficients were arbitrarily set at $R=0.9$, $T=0.05$ and $S=0.025$. Those of the unit cube were taken from Davies' (1976) Monte Carlo results for $\tau=100$: $R=0.5625$, $T=0.0089$ and $S=0.10715$. The slopes are, from top to bottom, -0.50 ($\Delta=1-R$, cubes), -0.75 ($\Delta=1-R$, squares and triangles) and -1.0 ($\Delta=T$, squares and cubes).

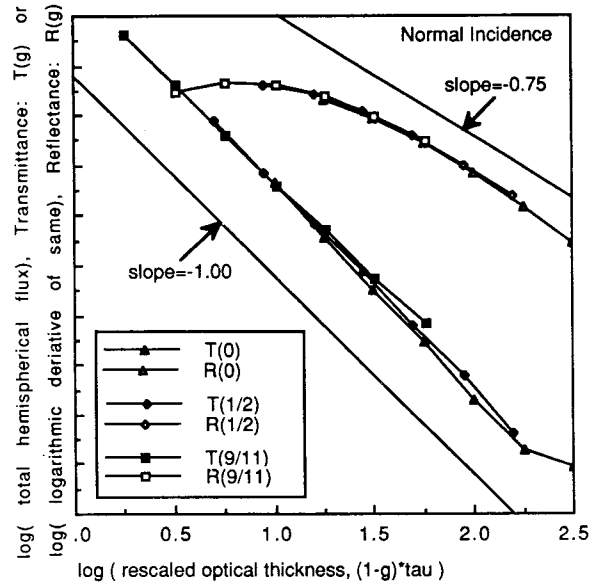
TABLE 1: PARAMETERS USED IN 2-D MONTE CARLO SIMULATION

CLOUD GEOMETRY	
optical thickness:	10.00, 17.78, 31.62, 56.23, 100.0, 177.8, 316.2
log(" "):	1.00, 1.25, 1.50, 1.75, 2.00, 2.25, 2.50
aspect ratio:	1.000...
BOUNDARY CONDITIONS	
collimated illumination:	0° (from normal above).
diffuse " :	nil (e.g. no reflecting ground below).
OPTICAL PARAMETERS	
single-scattering albedo:	1.000... (i.e. conservative scattering)
asymmetry-factor (g):	0, 1/2, 9/11=0.8181...
(1+g)/(1-g):	1, 3, 10.
ADOPTED 2-D PHASE FUNCTION	

$$p(\theta) = \frac{1-g^2}{1+g^2-2g\cos\theta}$$
 i.e. (co-focal) elliptical radiation diagrams
 with semi-major axis $= \frac{1+g^2}{1-g^2}$ & eccentricity $= \frac{2g}{1+g^2}$; this yields

$$\text{Probability}(\theta > \theta') = \frac{1}{2\pi} \int_{\theta'}^{\theta} p(\theta'') d\theta'' = \frac{1}{\pi} \tan^{-1} \left[\frac{1+g}{1-g} \tan \frac{\theta}{2} \right] + \frac{1}{2}$$

Fig 2: CONTINUOUS ANGLE 2-D MONTE CARLO SIMULATION FOR FINITE HOMOGENEOUS CLOUDS



The ordinate represents \log_{10} of photon-counts (for T) or \log_{10} of the differences of these counts (instead of 1-R) in arbitrary units. Note that the optical thickness' are equidistant on a logarithmic scale and that $1-R \propto \tau^{\nu_R}$ implies that $dR/d(\ln\tau) \propto \tau^{\nu_R}$ also. The reference slopes indicate DA predictions for the same geometrical set-up. The dispersion is explained by the statistics of the adopted Monte Carlo algorithm which is optimized for speed at the price of a more involved calibration (irrelevant to the slopes).

4. INTERNALLY INHOMOGENEOUS CLOUDS: $\nu_R \leq \nu_T < 1$

Spatial variability inside clouds has been modeled by geometrically self-similar fractal structures which, by construction, contain holes of all sizes down to some small inner length scale. Theoretically, these are of interest because they are the simplest examples of inhomogeneous but scaling geometries. Furthermore, geophysical fields are known to exhibit scaling over wide ranges, for instance King et al. (1981) report scaling power-spectra in cloud liquid water content (LWC) soundings. Interest in fractal-like clouds was spurred by Lovejoy's (1982) investigation of cloud perimeters using radar and satellite imagery. More recently, multifractal methods of analysis have been applied to similar data sets, see Gabriel et al. (1989a).

The inset in fig. 3 illustrates the generation of a deterministic fractal cloud used as a prototype in this study. As the individual cells are either filled or empty (a so-called "β-model"), it is described by a single "fractional dimension" $D = \log 3 / \log 2 = 1.58...$. This concept is defined in physical terms by

$$(\text{amount of matter}) \propto (\text{linear size})^D \quad (4)$$

Now, if the fractal is embedded in d-space ($d=1,2,3$), we have

$$(\text{space-averaged}) \text{ optical thickness} \propto \text{mass} / \text{size}^{d-1}$$

$$\therefore \tau \propto \lambda^{D-d+1} = \lambda^{1-C} \quad (5)$$

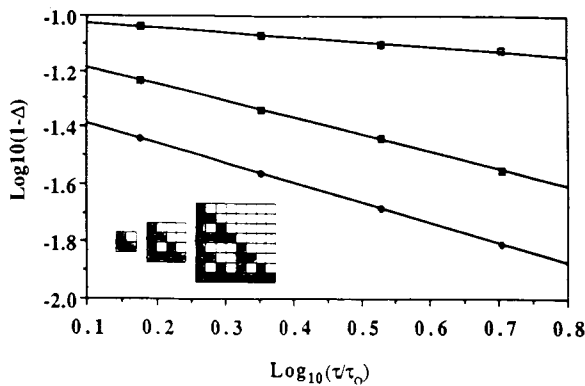
where $C=d-D$ is known as the codimension and λ is the size parameter, namely the ratio of the outer to the inner scales of the fractal. The latter corresponds to some unit cell of given optical thickness τ_0 which is the prefactor that determines τ completely. Homogeneous systems correspond to $C=0$ and

$C \geq 1$ is not of interest here since these structures are so sparse that horizontal (hyper-)planes usually don't intersect them.

DA radiative transfer allows an analytical approach to the multiple scattering problem on this medium, see Lovejoy et al. (1989a). Figure 3 shows the computational DA results for this fractal cloud with illumination from above, both open and cyclical boundary conditions were applied to the sides. For finite clouds, $v_R=0.162$ and $v_T=0.684$, whereas for extended clouds $v_R=v_T=0.601$. The fact that $v_T < 1$ reflects the impossibility, due to the presence of holes, of the photon flow to attain a diffusional regime anywhere in the cloud (except possibly in the sense of "diffusion on a fractal" used in solid state physics). D is not a one-to-one descriptor of universality classes (via v_R and v_T) since it depends on boundary conditions and d at the very least. These results are again obtained via space discretization with the (implicit) identification of the detailed radiation field with its average value at the scale of the elementary cell; this is called "real-space renormalization" in lattice statistical physics. In view of the results to be discussed below it seems that this approximation is not as good here as in the homogeneous case.

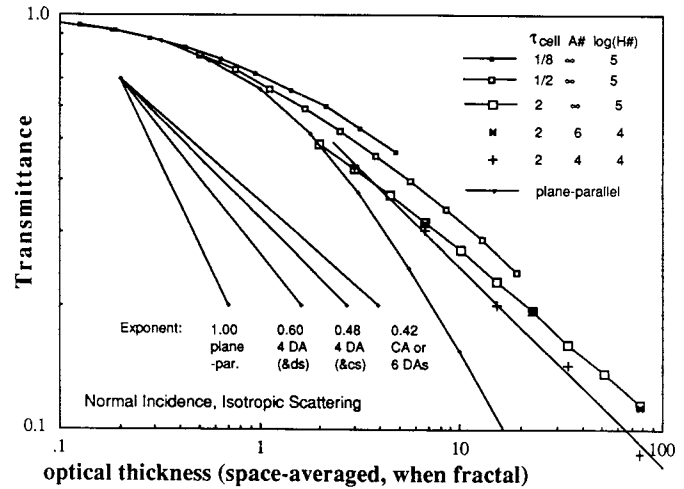
Figure 4 shows the Monte Carlo results for the same deterministic fractal cloud with cyclic boundary conditions (where $T=1-R$), various cell optical thicknesses and rather extreme types of phase-function were used: four or six discrete as well as continuously variable directions (all being isotropic within their respective angle spaces, in particular, this implies $g=0$). Fortunately, media with holes seem to reach their asymptotic regime sooner in terms of their (average) optical thickness. The clouds with semi-opaque cells ($\tau_0=2$) yield an accurate value of $v_T=0.42$ with six or more directions and somewhat less compelling evidence for $v_T=0.47$ with four directions only. The reason for this discrepancy within the continuous space approach is not obvious, some kind of resonance with the preferred directions of the grid was ruled out by a simulation with incident radiation inclined 30° to the normal (not illustrated). The calculations for clouds with semi-transparent cells ($\tau_0=1/2, 1/8$) are not incompatible with these results although it is doubtful they are in their asymptotic regimes. After the prescribed 9 cascade steps (512x512 cells), thus we cannot rule out a weak dependency of the exponents on the small scale limit of the cascade process.

Fig 3: DA RESULTS FOR THE DETERMINISTIC FRACTAL CLOUD WITH $D=1.58$ EMBEDDED IN 2-D SPACE.



The inset illustrates (the three first steps of) the construction of this cloud which is used as a prototype medium with holes at all scales. Using eq. (5), the (space-averaged) optical thickness is seen to be $(3/2)^n$ times that of the unit cell after n cascade steps. Light is incident from above. The ordinate is defined as in fig.1, the abscissa is \log_{10} of the average optical thickness expressed in units of cell optical thickness; same unit (square) cell coefficients as in fig.1 also. The slopes are, from top to bottom, -0.162 ($\Delta=1-R$, with sides), -0.690 ($\Delta=T$, with sides) and -0.601 ($\Delta=T=1-R$, horizontally extended).

Fig 4: MONTE CARLO RESULTS FOR THE DETERMINISTIC FRACTAL CLOUD WITH $D=1.58$ IN 2-D SPACE



The ordinate is $\log_{10}(T) = \log_{10}(1-R)$ for cyclical boundary conditions. The abscissa is \log_{10} of the optical thickness $\tau = \tau_{cell}(3/2)^n$ with $\tau_{cell}=1/8, 1/2$ and 2 . $A\#$ is the number of angles considered and $H\#$ the number of photon histories followed. See text for the meaning of the reference slopes. Note that DA and continuous angle problems have numerically different plane-parallel limits (i.e. zero cascade steps)

Gabriel et al. (1989b) report numerical DA results on random fractal clouds embedded in 3-D space; their results are qualitatively and quantitatively different from those of plane-parallel theory even though the fractal behavior was restricted to an inner-to-outer scale ratio of merely 32 and relatively small (ensemble averaged) optical thickness' ($\langle \tau \rangle \approx 50$). Continuous angle radiative transfer studies are currently underway on random (multi-) fractals with (continuous) variability in all spatial dimensions.

5. APPLICATION TO ATMOSPHERIC RADIATION

DA radiative transfer is a self-consistent theory with discrete space and continuous space versions based on Priesendorfer's (1965) Interaction Principle or coupled (rather than integro-) differential equations respectively. Along with the numerical studies mentioned above, the formal connection between DA and continuous angle radiative transfer is presently under careful scrutiny. This is an important task in view of the consequences of the theory discussed below.

Supposing that our basic results hold - even just qualitatively - true in nature, the application of standard (plane-parallel) models to horizontally finite and/or inhomogeneous clouds can lead to arbitrarily large errors. Sides alone make $v_R < v_T$, hence $1 > T+R$ even for conservatively scattering (thick enough) clouds, potentially accounting for any anomalous absorption. Cloud transmittance is a critical quantity in the surface radiation budget; we see that inhomogenieties can enhance transmittance by a factor τ^{1-v_T} with respect to plane-parallel clouds of equal optical thickness. Thus, by themselves, holes modify the optical response of clouds drastically by making $v_T < 1$.

Let ξ be the ratio of actual LWC to the LWC predicted by plane-parallel theory, given the actual albedo. As τ (the space-averaged optical thickness) becomes very large, we have

$$\left(\frac{\tau}{\xi}\right)^{-1} \propto \tau^{-v_R}, \text{ hence } \xi \propto \tau^{1-v_R} \quad \tau \gg 1$$

This "packing factor" of the actual LWC respective to that retrieved from backscattered radiation diverges with τ as soon as $v_R < 1$. This in turn is seen to be an effect of finite size and/or the presence of holes. ξ also proves to be an increasing function of C (via v_R), see Gabriel et al. (1989b) for details.

6. SUMMARY AND DISCUSSION

This paper focuses on the asymptotic behavior of optically thick non-absorbing clouds in terms of reflectance and transmittance; various global geometries and local optical properties are considered. It is found that only the former affect the scaling exponents defined in eq. (2). The results presented here are largely derived from a relatively unexplored approach to radiative transfer using angular discretization and methods used in statistical physics. Taking the asymptotic limit allows analytic estimates to be obtained for the exponents. Monte Carlo simulations support the generalization of DA predictions to continuous angle radiative transfer. In either case, universality of these exponents respective to phase function is stressed. The results show that many important aspects of the interaction between clouds and shortwave radiation are not captured by plane-parallel radiative transfer and far more realistic models (than those presented here) must be developed in order to meet the needs of quantitative meteorology, climatology and remote-sensing.

At the same time, the idea is not to introduce a large number of new degrees of freedom in order to emulate any given dataset. Universal behavior, described by a reasonably small number of variable, is a prerequisite for any progress towards statistically robust data reduction. This applies to both density (of LWC) and radiation fields, in Preisendorfer's (1976) words, the "inherent" and "apparent" radiative properties of the optical medium respectively. In connecting the two kinds of properties, we have relied heavily on Monte Carlo techniques or idealisations such as DA radiative transfer, both of which have their limitations; fortunately, Stephens (1988) presents us with a penetrating analysis of the most general multiple scattering problem based on spatial Fourier transforms. In parallel, Schertzer and Lovejoy (1987) propose a two-parameter family of multiply scaling (probability) distributions for density modeled as a passive scalar in a turbulent cascade. They also suggest a three-parameter family for observed (i.e. non-linearly smoothed) quantities and indeed Gabriel et al. (1988) find representative samples of the Earth's radiation field to be well described by multifractals.

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