JOURNAL OF GEOPHYSICAL RESEARCH, VOL. 101, NO. D21, PAGES 26,427–26,440, NOVEMBER 27, 1996

Multifractal analysis and modeling of rainfall and river flows and scaling, causal transfer functions

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Abstract. River flows have been known to be scaling for over 40 years and scaling notions have developed rapidly since the 1980s. Using the framework of universal multifractals and time series of rainfall and river runoff for 30 French catchments (basin sizes of 40 km^2 to 200 km^2) from 1 day to 30 years, we quantify types and extent of the scaling regimes. For both flow and rain series, we observed a scale break at roughly 16 days, which we associate with the "synoptic maximum"; the time scale of structures of planetary spatial extent. For the two scaling regimes in both series, we estimate the universal multifractal parameters as well as the critical exponents associated with multifractal phase transitions. Using these exponents, we perform (causal) multifractal time series simulations and show how a simple (linear) scaling transfer function can be used to relate the low-frequency rainfall series to the corresponding river flow series. The high-frequency regime requires nonlinear transforms.

Introduction

A Multifractal Scaling Framework for Rain and River Flows

In a celebrated paper, *Hurst* [1951] showed that the stream flows of various rivers exhibit long-range statistical dependencies indicating that water storage and runoff processes occur over a wide range of scales; i.e., they are scaling. In spite of this observation, it took over 30 years for its significance to be fully appreciated. On the contrary, the 1970s witnessed the widespread development of nonscaling autoregressive moving average (ARMA) type processes for modeling runoff, and the closely related compound Poisson point process models [e.g., *Bras and Rodriguez-Iturbe*, 1976; *Rodriguez-Iturbe et al.*, 1984] were used for rainfall itself. Not only were these models incompatible with the observed scaling, they usually involved only weak variability (e.g., exponential probability tails). Because of the existence of "outliers", this lead to various problems including recourse to subjective best fit criteria. In this framework, two or more different distributions are necessary to fit different regimes such as the "regular" and the "extreme" events. An early exception was the (mono)-scaling approach followed by *Mandelbrot and Wallis* [1968, 1969] who introduced the Biblical terms "Noah" and "Joseph" effects to denote nonclassical extreme (algebraic) probability tails and persistent (algebraically correlated) fluctuations, respectively, observed in river flow time series. The corresponding theoretical framework was that of scaling additive stochastic processes, essentially generalizations of Brownian motion. These involved two basic parameters: a scaling exponent H characterizing the persistence, and a probability exponent q_D (<2 for these additive models) characterizing the extreme events. In modern parlance, graphs of the corresponding time series were fractal sets having a single fractal dimension $D = 2-H$.

The fundamental source term driving runoff processes and hence river flow fluctuations is the rain field. The scaling properties of this field have indeed been the subject of considerable investigation in the last 15 years. At first, several attempts were made to estimate the (supposedly unique) fractal dimension of rainfall [e.g. *Lovejoy*, 1981; *Lovejoy and Mandelbrot*, 1985; *Hubert and Carbonnel*, 1988, 1989, 1991; *Olsson et al.*, 1990; *Bouquillon and Moussa*, 1991]. However, during the early 1980's it became clear that while the appropriate theoretical framework for scaling geometric sets was fractals, the framework for the more relevant scaling fields was multifractals. Since then, over 20 geophysical fields have been shown to be multifractal over various range of scales (see e.g. *Lovejoy and Schertzer* [1995a] for a review). Of specific relevance to rainfall and runoff were the studies using radar rain reflectivities [*Schertzer and Lovejoy*, 1985b; *Lovejoy et al.*, 1987; *Lovejoy and Schertzer*, 1990a; *Gupta and Waymire*, 1993], lidar reflectivities [*Lovejoy and Schertzer*, 1991], rain gauge series [*Duncan*, 1993; *Tessier*, 1993; *Tessier et al.*, 1993], in clouds [*Gabriel et al.*, 1988; *Lovejoy and Schertzer*, 1990a; *Tessier et al.*, 1993], and in topography [*Lovejoy and Schertzer*, 1990b; *Lavallée et al.*, 1993; *Lovejoy et al.*, 1995a].

One of the still underappreciated generic consequences of multifractality is that it leads to probability distributions with algebraic tails (with exponent $-q_D$). Contrary to the monofractal models where $q_D < 2$ (such as *Lovejoy and Mandelbrot* [1985]) or of runoff where $q_D < 1$ [Rodriguez-Iturbe et al, 1992], q_D can take any positive value. While this feature characteristic of scaling fields exhibiting sudden, violent extreme events was originally termed "hyperbolic

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Paper number 96JD01799.
0148-0227/96/96JD-01799 $09.00

intermittency" [*Schertzer and Lovejoy*, 1987b], it has more recently [*Schertzer and Lovejoy*, 1994, 1996b] become theorized as a multifractal phase transition route to self-organized criticality (SOC) [*Bak et al.*, 1987, 1988], the extreme events being self-organized critical events. Empirical confirmation of such strong intermittency in rain has come from many sources, see section on multifractal phase transitions below and the review by *Lovejoy and Schertzer* [1995b]. In river flows, similar behavior was reported by *Turcotte and Greene* [1993] (although they misleadingly call the probability exponent q_D a fractal dimension). Some authors [e.g. Rodriguez-Iturbe et al, 1992] have termed weakly varying gaussian processes SOC because the latter have power law distributions of zero-crossing lengths. We reserve the term for power law intensity fluctuations which corresponds to strongly varying avalanche like events.

The Statistical Relation Between Rainfall and River Flow and Implications for Runoff Processes

The physical basis of multifractality in rain is the dynamical turbulent cascade process which causes various conserved fluxes to be concentrated in smaller and smaller fractions of space, hence building extreme intermittency. *Schertzer and Lovejoy* [1987a], *Gupta and Waymire* [1993] and *Tessier et al.* [1993] discuss the corresponding multifractal rain models. However, other scaling fields are important for determining runoff and streamflows; the topography and porosity are also important. This has lead to the study of transport processes on multifractals (see *Lovejoy et al.* [1995a] for relevant diffusive and kinetic transport respectively). Also relevant are the findings by *Gupta et al.*, 1994 that streamflows are multiscaling functions of basin size. It is therefore now possible, at least in principle, to model river flows starting with the topography, (space-time) multifractal rain, and soil permeabilities fields as inputs. Such modeling will probably be indispensable in solving basic runoff problems.

However, before developing such sophisticated models, it is important to (1) empirically investigate the statistical properties of river flows in a multifractal framework (including the basic types and limits of (multi) scaling regimes, (2) empirically determine the statistical relations between the river flows and the corresponding rainfall (including possible transfer functions), and (3) to produce (multifractal) simulations of both rain and river flow which respect the directionality of time (i.e., which are causal). These three related points are the subject of the present paper; in particular, we determine under what conditions a linear transfer function connecting river flows and rainfall is possible, and we determine its form. The existence of a linear transfer function has been hypothesized for some time [e.g., *Sherman*, 1932; *Clark*, 1945; *Minshall*, 1960; *Nash*, 1960; *Dooge*, 1973]; this work gives it some empirical justification (although for longer time periods than usually considered). We will see that the transfer function must respect not only scale invariance (as in the space time rain models discussed by *Tessier et al.* [1993]), but, as developed in *Marsan et al.* [this issue], causality.

Data Description

We first concentrate on a multifractal analysis of river flows. River flows have several peculiarities that must be taken into account. For example, since we are interested in the natural variability, a system with minimal human intervention is highly desirable. Hence we chose a database of 30 time series of daily river flows with small basins (40 to 200 km^2) without artificial dams or reservoirs. The mean flow rates varied between 0.5 and 8.0 m^3/s and the mean daily rainfall accumulations varied between 1.8 and 5.5 mm/d. These data have been gathered by the hydrological division of the Centre National du Machinisme Agricole, du Génie Rural, des Eaux et des Forêts (CEMAGREF) as a contribution to the Flow Regimes from International Experimental and Network Data - Alpine Mediterranean Hydrology (FRIEND-AMHY) UNESCO regional hydrology project. In each of the basins we also disposed of daily rainfall accumulations as estimated by a single rain gauge. Each time series covered a period of time varying between 11 and 30 years. The rivers are located in different regions distributed all over France. An example of the daily stream flow records along with a corresponding daily rainfall accumulation record from a rain gauge situated in the basin is shown in Figure 1. Note that the individual spikes in river flow (and to a lesser extent in the rain series) are not symmetric under inversions of the time axis, and that the river flow seems to be roughly a "smoothed" version of the rain series. The former is a consequense of causality in rainfall and the latter suggests that the storage can be modelled by a transfer function between the rainfall field and the runoff field.

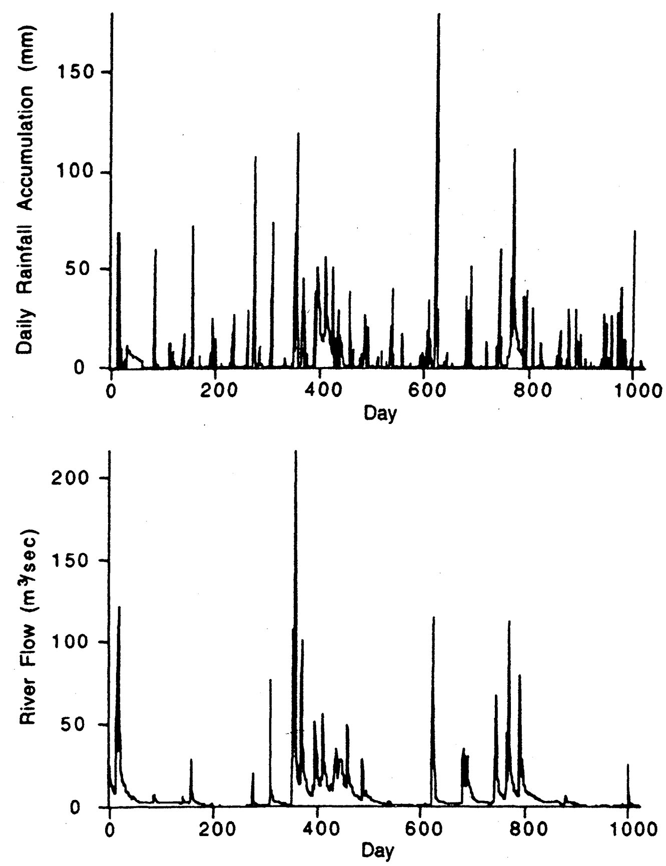

Figure 1. Simultaneous daily rainfall accumulation and river flow for the Le Gardon St-Jean river and the nearby station of Corbes Roc Courbes (France).

Paper Overview

In the next section we investigate the basic scaling properties of the time series using the standard and quite sensitive method of spectral analysis. This enables us to determine the basic scaling regimes. The section on universal multifractals presents a review of universal multifractals and the double trace moment technique that is used to estimate the universal multifractal exponents α and C_1 which characterize the type of scaling. We then proceed in the following section to the determination of a third parameter H which measures the degree of (scale by scale) conservation of the fields. In the section on multifractal phase transitions we investigate the divergence of statistical moments of the series and show the relations of these divergences (treated as multifractal phase transitions) with self organized criticality. In the final section we introduce causal scaling transfer functions and show their potential use in the derivation of river runoff from rain records and in the simulation of both series.

The Scaling Ranges and Spectral Exponents of Rain and River Flows

We now attempt to empirically estimate the scaling ranges of the observed rain and river flow time series. This is conveniently done by calculating the power spectrum $E(\omega)$ (in one dimension, ensemble average of the square of the Fourier amplitudes as a function of the frequency ω) and estimating the spectral exponent β in the scaling relation $E(\omega)\approx\omega^{-\beta}$. Although the spectrum is defined as an ensemble average quantity, if the series is long enough, it may be possible to get some idea of the scaling range and exponent on individual realizations. If the spectra are not too different from each other, this gives support to the hypothesis that the various series are indeed from the same statistical ensemble. Scaling is only an ensemble statistical property. Furthermore, due to the extreme intermittency characterizing these generally nonergodic processes, the scaling can be badly broken on individual realizations, even if the latter are very long. We therefore first estimated spectra on individual time series of daily river flows for all 30 rivers. The results for a few typical records are shown in Figure 2. Figure 3 presents the corresponding analysis for rain. Bearing in mind the above caveat that scaling is only a statistical symmetry, and that large numbers of realizations may be required to obtain adequate estimates of the ensemble statistics, a few features are readily observed on these graphs. All the spectra have broad scaling regions, although there appears to be a break, which is not always very pronounced, separating two regions with different β. In the rainfall records, this transition, when present, was always found at around 2 weeks. Such a break is presumably a manifestation of the "synoptic maximum" [*Kolesnikov and Monin*, 1965; *Lovejoy and Schertzer*, 1986; *Ladoy et al.*, 1991; *Fraedrich and Larnder*, 1993; *Tessier et al.*, 1993]. The synoptic maximum arises because in most geophysical systems, and especially in geophysical fluid

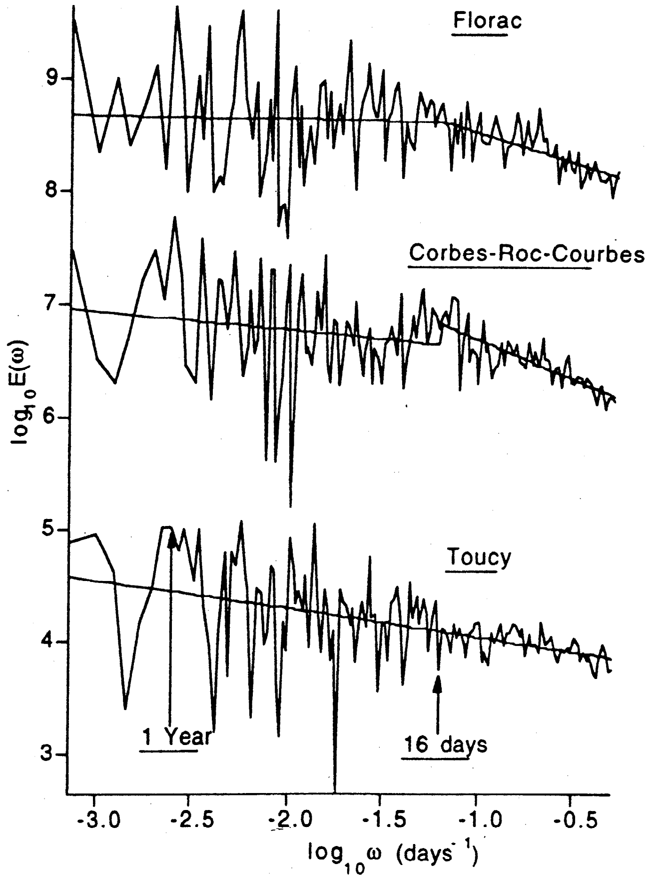

Figure 3. Power spectra for daily rainfall accumulations for three stations: Toucy, Florac, and Corbes Roc Courbes (France).

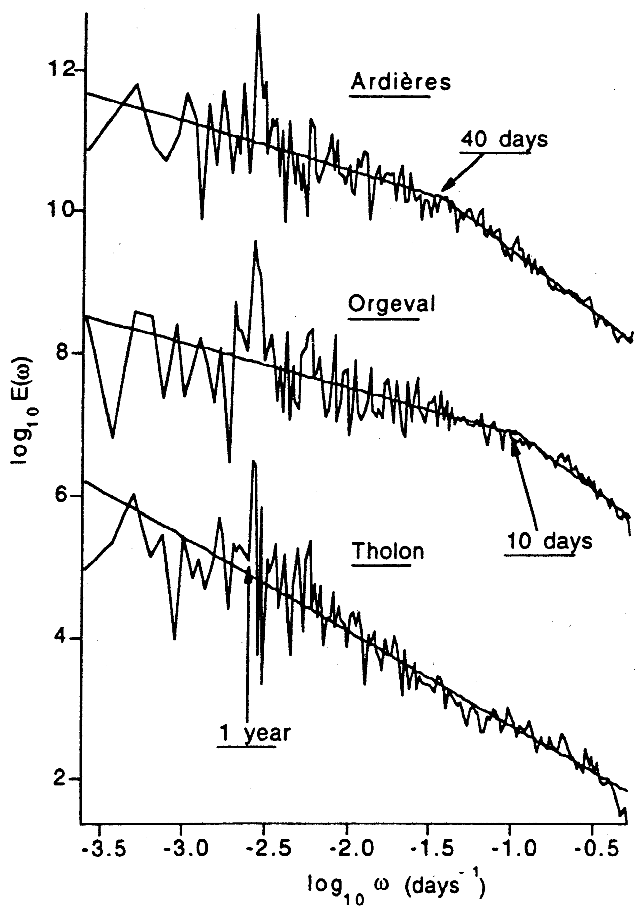

Figure 2. Power spectra for daily river flows for three different rivers in France.

dynamical systems, there is a fundamental dynamical velocity field which connects the spatial and temporal statistics. Larger and larger structures evolve, on average, more and more slowly and have longer and longer lifetimes. Indeed, *Lovejoy and Schertzer*, [1991] and *Tessier et al.* [1993] argued that for rain, the corresponding velocity is a turbulent (scale dependent) velocity whose scaling exponent was empirically estimated. The appropriate theoretical space-time framework called generalized scale invariance ([*Schertzer and Lovejoy*, 1985a, b] is essentially a statistical and anisotropic generalization of Taylor's [1938] hypothesis of "frozen turbulence", (i.e., "that the sequence of changes... at a fixed point are simply due to the passage of an unchanging pattern of turbulent motion over the point..."). The synoptic maximum is therefore the timescale associated with the evolution of structures of planetary spatial extents. This would imply that the typical velocity of the large-scale motions is roughly 10^7 m/10^6 s$\approx$10 m/s, which is indeed compatible with usual meteorological estimates [e.g., *Petterssen*, 1969]. Note that, owing to the wide range scaling of the velocity field, we do not expect this velocity to be independent of scale [see *Tessier et al.*, 1993].

We note that since the rain field itself is the source of the river flow, typical scales in the former will also be present in the latter. It is possible that a timescale analogous to the synoptic maximum exists for the field of rainfall runoff/surface water flow, associated with the spatial scales characteristic of the basin. In that case the main difference between the rain field and runoff water fields is the magnitudes of the corresponding velocities and the spatial scales. For surface/runoff water the largest spatial scale of relevance is the basin scale. Here, the basins ranged from 40 km^2 to 200 km^2; about 10^3 times smaller than planetary scales (10^4 m / 10^{-1} m/s). However, the corresponding velocity is the much smaller one associated with runoff processes; perhaps of the order of only 10^{-1} m/s. In this case, the corresponding timescale at which the river flow responds to the rain input will be of the order of a day or less. A timescale this short is indeed compatible with the observation that on the aggregation scales used here (a day or longer), the river flow is observed to increase essentially immediately with rain events. The corresponding high frequency break would thus not be observable with the daily series at our disposal. We therefore anticipate a transition time scale which will not depend noticeably on the river basin size, nor much on the geology or topology (which is scaling); - it would primarily modify the mean runoff water velocity used in the space-time

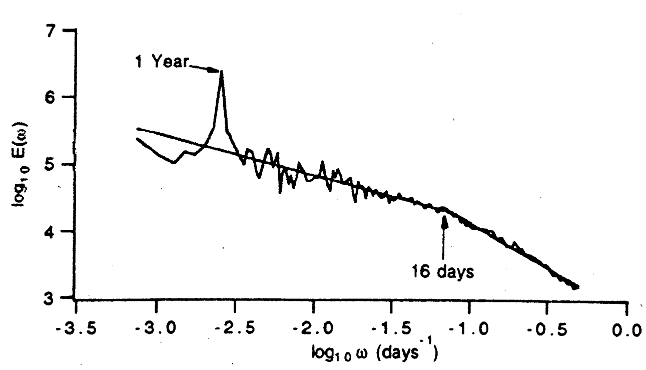

Figure 4. Power spectrum for daily river flow average for all stations.

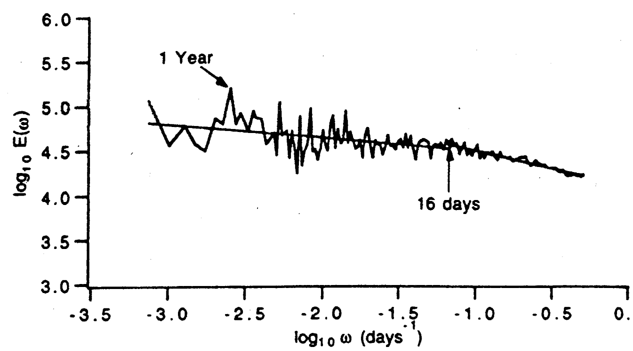

Figure 5. Power spectrum for daily rainfall accumulations average for all stations.

relation. Ultimately, these relationships should be clarified with the help of multifractal models of the rain field [*Tessier et al.*, 1993], coupled with topography [*Lavallée et al.*, 1993] and transport [*Lovejoy et al.*, 1995a]. Studying the spectra of streamflows for the 30 rivers described above, we found that the transition time (break in the scaling) varies from roughly 16 to 90 days (although only in three cases out of 30 were transition times greater than 30 days observed). We then attempted to confirm that the variation in the transition times was not caused by obvious variations in the size of the basins, or of the topology of the river network and the geology or topography of the region.

Using a 16-day break, the estimated value of β for a low-frequency river flow regime was estimated as $\beta \approx 0.7\pm0.3$ and $\beta \approx 1.6 \pm0.5$ for the high-frequency regime. The cited errors correspond to 1 standard deviation from the values estimated from the individual series. For comparison, in rainfall we obtained $\beta \approx 0.2\pm0.1$ for the low frequencies and $\beta \approx 0.3\pm0.1$ for the high frequencies. We see below, by examining higher order statistics that the break in the scaling in the rain data is actually quite strong at this scale even though it is not so obvious from the spectrum. These results are very close to those obtained by *Ladoy et al.* [1991, 1993] at Nîmes (France) and for the mean of the global rain gauge network (for 1-64 days, $\beta \approx 0.2\pm0.3$ [*Tessier et al.*, 1993]). Making the assumption that there are common physical processes operating at the various locations under study, we averaged the spectrum of normalized time series for all locations and obtained Figure 4 for river flows and Figure 5 for rainfall. It is interesting to note that for rainfall a small annual peak begins to emerge above the background fluctuations. The peak in river records is probably stronger because of the seasonal variations in temperatures resulting in the accumulation of "nonflowing" water such as snow or ice and a reduced capacity of soils to absorb water during winter months. The amount of evaporation will also vary seasonally. We also note that as anticipated above, after averaging, the scale breaks occur at roughly the same timescale for rain and river flows. The 16-day period seems to be associated with a fundamental change in the scaling properties, but the 1-year cycle seems to be primarily a "regular" oscillation superposed on an otherwise scaling background, since on both sides of the peak the spectral slope remains unchanged (this is true at each site as well as for the average spectrum). Because spectral analysis separates the statistics so well according to frequency, this annual cycle does not seriously affect our spectral slope estimate. However, it does cause a significant break in the scaling when other (real space) scaling analyses are performed.

From the power spectrum we obtained the values of β presented in Table 1. The ensemble values of β will later be used to estimate the multifractal parameter H, which quantifies the degree of nonconservation in the cascade process and which is necessary to obtain the exponent of the transfer function. For the reasons given above, we decided to divide the following analyses in two regions: one from 1 to 16 days and one from 1 month to 30 years using monthly averages of daily records.

Universal Multifractals

The scaling of time series can be investigated by two equivalent routes: the probability distribution and the statistical moments. First, the resolution is expressed by a dimensionless ratio $\lambda = T/\tau$, where T is the longest duration of interest and τ is duration of the observation. In the case of multifractals, the probability distribution is given by *Schertzer and Lovejoy* [1987a]:

$$Pr\left(\varphi_\lambda \geq \lambda^\gamma\right) \approx \lambda^{-c(\gamma)} \tag{1}$$

where γ is the order of singularity and $c(\gamma)$ the codimension function of the singularities and φ_λ is the "conserved" multifractal process associated with rain or flow (see below) at resolution λ (normalized by the ensemble/climatological mean). Similarly, the statistical moments are given by

$$\left\langle \varphi_\lambda^q \right\rangle = \lambda^{K(q)} \quad \lambda > 1 \tag{2}$$

where $K(q)$ is the multiple scaling exponent for moments; the two are related to each other via a Legendre transform [*Parisi and Frisch*, 1985]:

$$\begin{aligned} c(\gamma) &= \max_q \left(q\gamma - K(q)\right) \\ K(q) &= \max_\gamma \left(q\gamma - c(\gamma)\right) \end{aligned} \tag{3}$$

The only restriction on $c(\gamma)$ and $K(q)$ is that they are convex. In actual dynamical systems involving nonlinear interactions over a continuum of scales (and/or involving multiplicative "mixing" of different processes), we generally obtain a considerable simplification. *Schertzer and Lovejoy* [1987a, 1991, 1996a] show that cascade processes possess stable (attractive) universal generators irrespective of the details of the dynamics [see also *Brax and Pechanski* [1991]; *Kida*, [1991]. Recently, it has been suggested that a weaker form of universality involving log-Poisson distribution might also be relevant [e.g., *She and Waymire*, 1995; *Schertzer et al.*, 1995]. The universal $K(q)$ functions for conservative processes are of the following forms:

$$K(q) = \begin{cases} \dfrac{C_1(q^\alpha - q)}{\alpha - 1} & \alpha \neq 1 \\ C_1 q \log(q) & \alpha = 1 \end{cases} \tag{4}$$

where $0 \leq \alpha \leq 2$ is the multifractal index, which quantifies the distance of the process from monofractality; $\alpha = 0$ is the monofractal β model of turbulence [*Novikov and Stewart*, 1964; *Mandelbrot*, 1974; *Frisch et al.*, 1978], $\alpha = 2$ (the maximum) is the lognormal model. $0 \leq C_1 \leq$ dimension of space is the codimension of the mean of the process; it quantifies the sparseness of the mean.

In order to directly test the universality hypothesis, and estimate α, C_1, we used the double trace moment (DTM) technique [*Lavallée*, 1991; *Lavallée et al.*, 1993]. The q, η double-trace moment at resolutions λ and Λ is defined as:

$$Tr_\lambda\left(\varphi_\Lambda^\eta\right)^q = \left\langle \sum_i \left(\int_{B_{\lambda,i}} \varphi_\Lambda^\eta d^D x \right)^q \right\rangle \propto \lambda^{K(q,\eta)-(q-1)D} \tag{5}$$

where the sum is over all the disjoint balls $B_{\lambda,i}$ (dimension D, indexed by i) required to cover the multifractal, $K(q,\eta)$ is the double-scaling exponent, and $K(q,1) = K(q)$ is the usual scaling exponent. Although it looks complicated, applying (4) to the field of interest simply consists of taking various powers η of the field at its highest resolution (Λ), then degrading the result to a lower resolution (λ), finally averaging the qth power of the result.

The scaling exponent $K(q,\eta)$ is related to $K(\eta,1) \equiv K(q)$ by

$$K(q,\eta) = K(q\eta,1) - qK(\eta,1) \tag{6}$$

The advantage of the DTM over other techniques is that in the case of universal multifractals, applying (6) to the form in (5), we find that $K(q,\eta)$ has a particularly simple dependence on η:

$$K(q,\eta) = \eta^\alpha K(q) \tag{7}$$

Therefore, α can be estimated on a simple plot of log $K(q,\eta)$ versus log η for fixed q.

The above applies to processes which are the direct result of a multiplicative cascade process in which some quantity (analogous to the energy flux in turbulent cascades) is conserved from scale to scale. However, a priori, there is no reason to expect the observed processes (rain, river flow) to be conserved, they will more generally be related to a conservative process by integrations of various (generally fractional) orders denoted H (differentiations are obtained for H<0; see *Schertzer and Lovejoy* [1987a] for more on this general multifractal geophysical framework, for a related problem, see *Naud et al.* [1996]). If the process is nonconserved ($H \neq 0$), it suffices to perform the inverse fractional integration or differentiation so as to return to the underlying conservative process. Since fractional integration of order H corresponds to power law filtering, this is conveniently done by Fourier methods (note that different types of fractional integration exist; see below).

Table 1. A Comparison of Estimates of the Spectral Exponent β

	<16 Days		>1 Month	
	Averaged β	β of the Average	Averaged β	β of the Average
Rain	0.3±0.1	0.4±0.1	0.2±0.1	0.1±0.1
Rivers	1.6±0.5	1.3±0.1	0.7±0.3	0.5±0.1

Note that the mean of individual exponent estimates (given by the standard deviations of the "averaged β" values) are compatible with the exponent estimate of the ensemble.

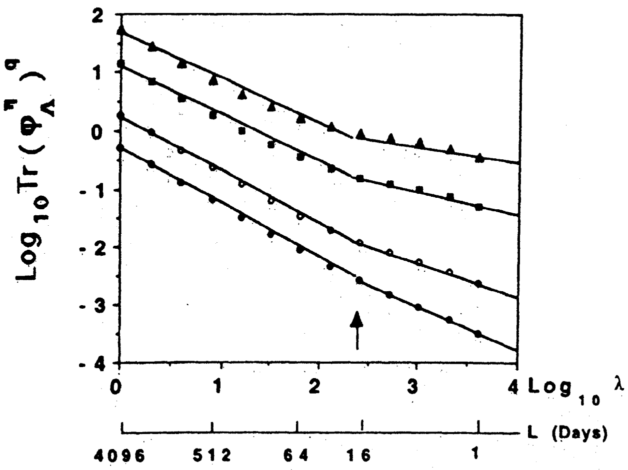

Figure 6. Log of the trace moments versus log λ, using $q = 2$ for daily rainfall accumulations for 30 stations in France. From top to bottom, η = 1.8, 1.5, 1.0, 0.5. The arrow points to the scaling break occurring at 16 days.

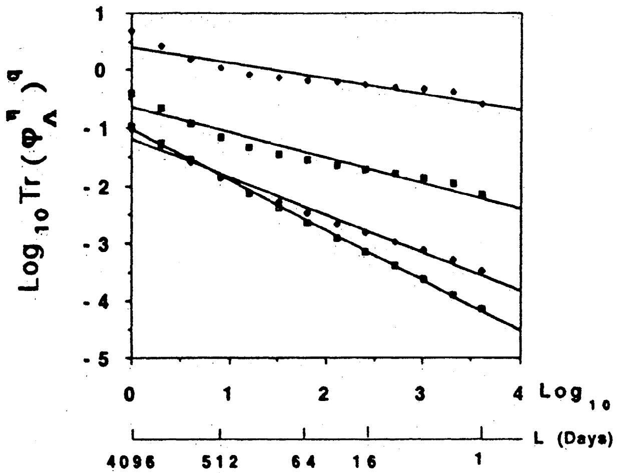

Figure 8. Log of the trace moments versus log λ, using $q = 2$ for normalized daily river flows of 30 rivers in France. From top to bottom, η = 1.8, 1.5, 1.0, 0.5.

Alternatively, since in practice it suffices to differentiate by an amount $-H$ [see *Lavallée et al.*, 1993], if $H<1$, one can use a first-order derivative approximated by using the (absolute) gradient of the series (roughly equivalent to filtering by a factor ω and taking the absolute value of the result).

We now consider the result of performing these analyses on the data. In conformity with the spectral results above, for rainfall, the double-trace moments (Figure 6) show a break at approximately 16 days. Estimating $K(q,\eta)$ (shown in Figure 7) over the period of 1 to 16 days yields $\alpha = 0.7\pm0.2$ and $C_1 = 0.4\pm0.1$ in reasonable agreement with other estimates ([*Hubert and Carbonnel*, 1991; *Hubert et al.*, 1993; *Ladoy et al.*, 1993; *Tessier et al.*, 1993; *Olsson*, 1995]. Performing the analysis on monthly averages (i.e., the low-frequency regime) leads to a different result. For rainfall, we obtained the values $\alpha = 1.6\pm0.2$ and $C_1 = 0.1\pm0.05$ (figure 7). In river flows we obtained (Figure 8) $\alpha = 1.45\pm0.25$ and $C_1 = 0.2\pm0.1$ for the 1 to 16 day period and virtually identical values for monthly averages (i.e. $\alpha = 1.45\pm0.2$ and $C_1 = 0.2\pm0.1$; see Table 2 for a comparison of the various values). In this case there is apparently no break in the double-trace moments (Figure 9; the series has been normalized by their respective means). This is compatible with the hypothesis that spectral break is purely caused by a change in the order of fractional integration (see the next section).

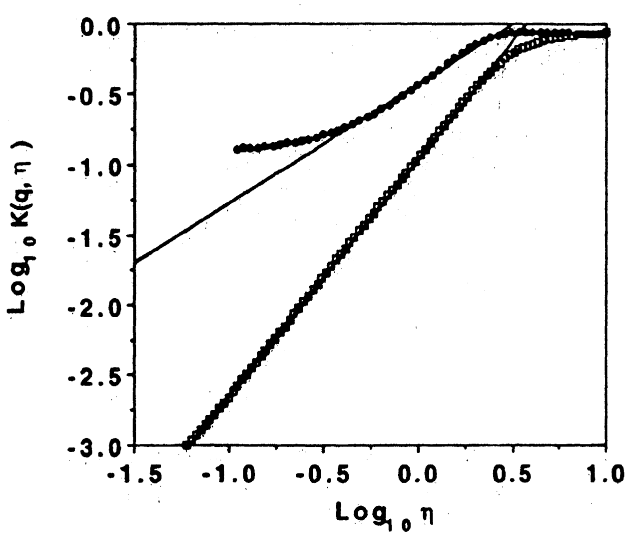

Figure 7. Log K(q,η) vs log η, using $q = 2$ for rainfall accumulation for 30 stations in France. Values are computed for scales of 1 to 16 days (solid circles) and for monthly averaged series (open squares).

Before continuing, one comment is necessary. The DTM is particularly sensitive to the weak events (which dominate the low η, low q statistics); α is the order of nonanalyticity of $K(q)$ at q =0. It will therefore be quite sensitive to biases or errors in estimating low values. Only the rain in the 1- to 16-day regime contains nominal zero values; this indeed corresponds to the estimate of α which is significantly lower than the others. Since many of the nominal zero rain rates were in actual fact simply below a minimum detection threshold, but were not true zero values, a model of rain and corresponding instrument response will be needed to fully resolve this problem. For example, *Larnder* [1995] showed that thresholding the values of a multifractal field (bringing to zero low values) can modify the scaling and thus influence the estimation of α and C_1. The problem is presently under study but since no clear cut answer has yet emerged we chose here to accept the zeroes as real.

Determination of H

We mentioned that a priori, the observed multifractal processes are not expected to be the direct results of cascade processes, but rather to be related to such a process by fractional integration/differentiation. Denoting the underlying (scale by scale) conservative cascade quantity by φ_λ such conservation implies $\langle\varphi_\lambda\rangle = \text{const}$ (independent of scale). This type of ensemble average conservation is called "canonical" conservation; more restrictive types of conservation (especially microcanonical conservation) are also possible, although they are artificial and are unlikely to be relevant here. In the scaling regime, the rain (or river) series (R_λ) is thus related to this series by

$$\Delta R_\lambda = \varphi_\lambda \lambda^{-H} \tag{8}$$

Table 2. The Different Multifractal Parameters Estimated From Ensembles of 30 Time Series

	α	C_1	H	q_S	q_D
River 1-16 days	1.45±0.25	0.2±0.1	0.4±0.3	5.3±0.5	2.7±1
River 30-4096 days	1.45±0.2	0.2±0.1	-0.05±0.2	4.2±0.5	3.2±1.5
Rain 1-16 days	0.7±0.2	0.4±0.1	-0.1±0.1	13.0±0.5	3.6±0.7
Rain 30-4096 days	1.6±0.2	0.1±0.05	-0.35±0.2	5.2±0.5	3.6±0.7

The single best fit parameters to the ensemble, the error estimates are the dispersions of the corresponding parameter estimates from individual series. The H values were estimated from α, C_1 and the ensemble averaged β values in Table 1. Note that with the exception of the rain ≤16 days, α, C_1 are compatible with the common values 1.5, 0.15, respectively. The columns q_s, q_D are explained in section 5.

In hydrodynamic turbulence, standard theory based on the nonlinear dynamical equations shows that the energy flux to smaller scales is conserved and that H=1/3 (the famous *Kolmogorov* [1941] value). However, at least for the moment, no corresponding theory of rainfall tells us which physical field φ_λ represents, nor the value of H. Various powers of φ can also be used, this is not fundamental [see *Schertzer and Lovejoy*, 1994]. However, the related cloud liquid water apparently has an exponent close to that predicted for passive scalars by *Corrsin* [1951] and *Obukhov* [1949] (see Lovejoy and Schertzer [1995a]). Since this parameter characterizes the long-term statistical dependency of the mean $\langle|\Delta R|_\lambda\rangle \sim \lambda^{-H}$, it has been denoted H in honor of Hurst. Hurst's original exponent is actually slightly different from this; indeed, it is not obvious how his original exponent is related to the multifractal exponents.

A direct method of estimating H is to use generalized structure functions [e.g., *Schmitt et al.*, 1995]. However, unfortunately, the latter are very sensitive to the presence of strong periodicities (such as the annual cycle found here). For this reason, a different technique called the spectral slope method was also used. This method uses the values of α and C_1 obtained from the DTM in order to estimate the spectral slope of the conserved process β_{con} from the formula [*Monin and Yaglom*, 1975] β_{con}=1-K(2). The order of fractional integration needed to obtain the non-conserved process from the conserved one is then given by

$$H = \frac{\beta-\beta_{con}}{2} = \frac{\beta-1+K(2)}{2} \qquad (9)$$

The value of K(2) can be estimated from α and C_1 using (4). The results of this analysis are shown in Table 2. Given the large statistical scatter (typical of multifractals, especially when relatively small sample sizes are analyzed), it is quite possible (with the exception of rain less than 16 days; see above comment about low values) that the results are compatible with $\alpha \approx 1.5$ and $C_1 \approx 0.15$ (median values of monthly rainfall and river flows). In this case, K(2)=0.25, and, using the ensemble average β values from Table 1, we find for rivers and rain for periods longer than 1 month, $H \approx$ -0.05, -0.35, respectively (and $H \approx 0.4$ for rivers for periods less than 16 days). With these parameters, for periods of 1 month to 30 years, river flow statistics are the same as the statistics of fractionally integrated rainfall of order $\Delta H = H_{river} - H_{rain}$ = (-0.05) - (-0.35) ≈ 0.3. However, for timescales of 1 to 16 days the rainfall and the river flows belong to two different multifractal classes (different α and C_1) but are both more or less compatible with quantities conserved by the cascade process (H≈0; although note that for the high frequencies, ΔH is roughly the same).

We can now compare our empirical results with those of Gupta et al [1994]. These authors assumed (without comment) that the river flow was the direct outcome of a multiplicative process (i.e. H = 0). They then fitted the log of the flow to various Levy distributions (including non-extremal Levy's which do not have any convergent statistical moments for the corresponding flows!) concluding that while a value of α in the range 1.5 to 2.0 provided the best fits (i.e. close to ours), that the fits were poor. Since we find H ≠ 0 (especially for rivers for periods less than 16 days) and theoretically H characterizes the long range correlations (hence storage), we do not expect the log of the flow to have a Levy distribution. We suspect that their results can be explained by the fact that H > 0.

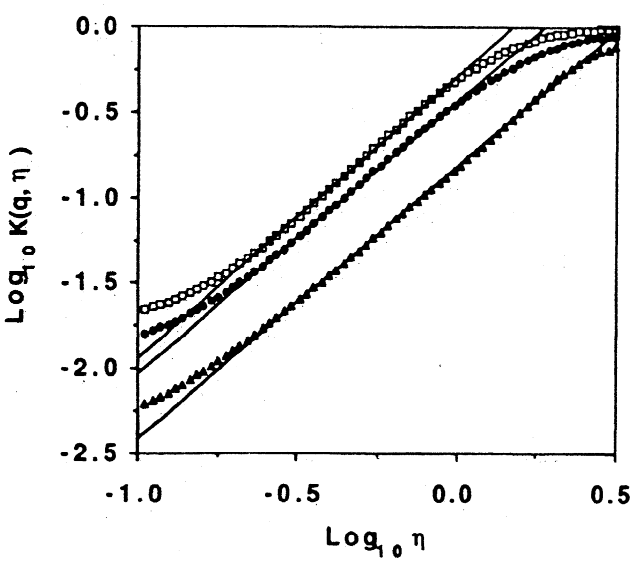

Figure 9. Log K(q,η) vs log η, using $q = 2$ for daily river flows of the Le Volp river (France). Values are shown for scales of 1 to 16 days (top), 1 month to 30 years (bottom) and 1 day to 30 years (middle).

Multifractal Phase Transitions

Before turning to the multifractal time series simulations and the transfer function, we will consider a final empirical aspect of the data; the statistics of the extremes. We have shown that high resolution (e.g., daily or less) rain or flow measurements and aggregated (monthly or yearly) measurements have different statistics. For multifractal processes the statistics of φ_λ resulting purely from dynamics at larger scales (smaller λ) are very different from those averaged at the same scale but whose dynamics continue to much smaller scales. The former are called the bare quantities, and the latter the dressed quantities. The main difference between the two is that the statistical moments of the bare field φ_λ will all converge, whereas the moments of the "dressed" $\varphi_{\lambda,d}$ will generally diverge if q is large enough:

$$\left\langle \varphi_{\lambda,d}^{q} \right\rangle \to \infty \quad q \geq q_D \tag{10}$$

This divergence corresponds to hyperbolic falloff of the probability distribution, i.e.,

$$Pr\left(\varphi_{\lambda,d} \geq s\right) \approx s^{-q_D} \quad (s >> 1) \tag{11}$$

The combination of divergence of statistical moments with spatial scaling can be taken as the defining property of self-organized criticality [*Bak et al.*, 1987, 1988]. Since finite samples will always yield finite moments, the divergence can be observed either via the divergence of empirical moments with increasing sample size, or via discontinuities in the derivatives of the observed scaling exponents K(q) and $c(\gamma)$. Since there is a formal analogy between multifractals and classical thermodynamics ([*Schuster*, 1988; *Schertzer et al.*,1993; *Schertzer and Lovejoy*, 1994, 1996b], these discontinuities can be considered as multifractal phase transitions.

The basic idea behind the multifractal phase transitions is that estimates of high enough order statistical moments will be dominated by the largest (most extreme) events in the sample, this leads to linear rather than convex $K(q)$. With more realizations (the number being denoted N_S) a larger and larger portion of the probability space will be explored and more extreme singularities will be encountered. $D_s = \log N_s / \log \lambda$ is the corresponding sampling dimension (*Schertzer and Lovejoy*, 1992; *Lavallée et al.*, 1991). For $q_s < q_D$ the maximum order of moment that will not be dominated by extremes (q_s) can then be estimated for universal multifractals as

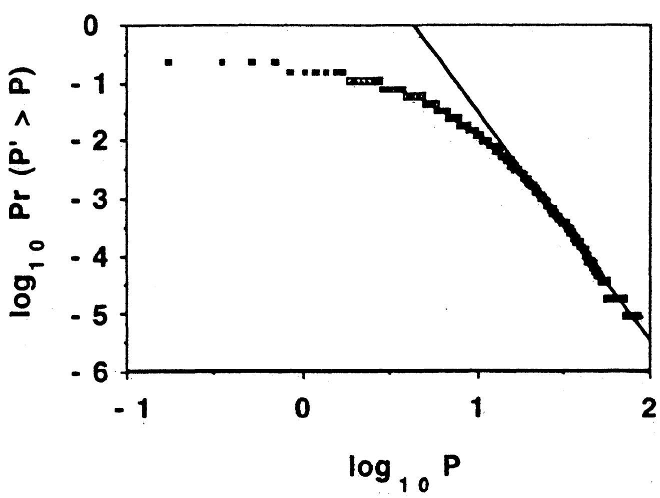

Figure 10. Log of the probability of getting a daily rainfall accumulation P' greater than a threshold p against the log of the threshold composed of 30 time series in different locations over France. The straight line has a slope $q_D = 3.6$.

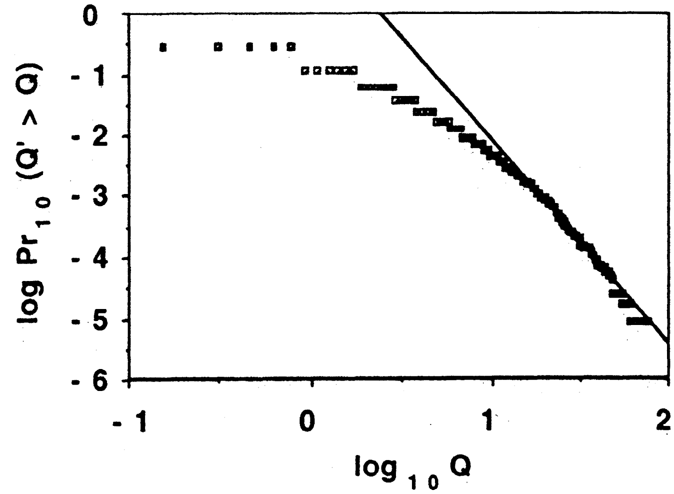

Figure 11. Log of the probability of getting a normalized daily river flow Q' greater than a threshold q against the log of the threshold composed of 30 time series in different rivers in France. The straight line has a slope $q_D = 2.7$.

$$q_s = \left[\frac{D + D_s}{C_1}\right]^{1/\alpha} \tag{12}$$

When $q_s < q_D$, the divergence is not observable and we obtain a second order phase transition; this becomes first order when N_S is large enough that $q_s > q_D$.

The probability distribution is shown for daily rainfall in Figure 10 and for daily river flows in Figure 11. In both cases each time series was normalized by its mean. In Table 2 we give the different estimates of q_s from (12) and the estimates of q_D from the algebraic falloff of the probability distributions. This tells us that the maximum statistical moments that can be reliably estimated with the available data vary between 4 and 13, depending on the scaling "regime" for both rain and river flows. Although more samples would be highly desirable, a quick glance at Table 2 shows that N_s (and hence q_s) was large enough so that these first-order transitions were indeed observable in both cases (since $q_D \approx 3$ which is < q_s in all cases).

Another way to study phase transitions is to calculate $K(q)$ for a different number of samples. Figure 12 shows $K(q)$ for 1 and 30 samples for the 1- to 16- day regime of rain, and Figure 13 shows the same graph for river flows. We see that in both cases for $q > q_D$ we obtain an asymptote that does not follow the tangent of $K(q)$ but which becomes steeper as the number of samples is increased starting at a specific point. This is direct evidence for a discontinuity in the first derivative at the corresponding value of q. This conclusion is in agreement with the results of *Ladoy et al.* [1993], who estimated that $q_D \approx 3$ for rainfall in the 1- to 16- day regime, and *Segal* [1979] who found $q_D \approx$ 2- 3 in 10-s rain gauge rates, and *Lovejoy* [1981] who found $q_D \approx 1.7$ for radar rain rates of isolated storms. It should also be mentioned that *Olsson,* [1995] obtained $q_D \approx 2.0$ for 8-min rainfall accumulations in Sweden and that *Fraedrich and Larnder*, 1993 obtained $q_D \approx 1.7$ with 1-min rainfall data in Germany. At present it is not known if the variations in these estimates are due to more extreme rain events in these regions, to the better sensitivity of their instruments, the process of obtaining rain rates from

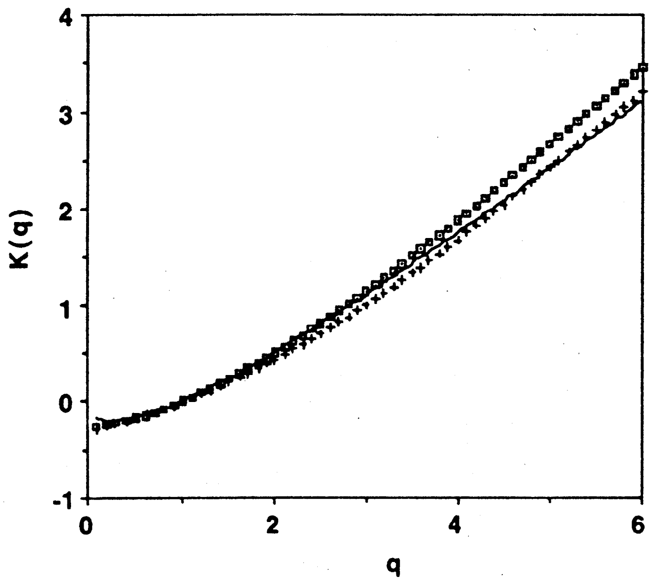

Figure 12. Log $K(q)$ vs q for 1-to 16- day rainfall accumulation over various stations in France. The empty squares were obtained considering 30 realizations (one per site) and the crosses were obtained from just one site. The continuous line is the theoretical curve calculated from the values of α and C_1 obtained by the DTM method (i.e. $\alpha = 0.7$ and $C_1 = 0.4$). For $q > q_D$, the assymptotes are straight, however , as predicted, they are steeper for the larger sample sizes.

tipping bucket records, or to different scaling regimes. In comparison, in river flows, the only other estimates of q_D of which we are aware are those of *Turcotte and Greene* [1993], who found values mostly in the range 3-6 for various U.S. river flows. The multifractal model can accommodate any value of q_D but monofractal models cannot. In particular since all the empirical results have $q_D > 1$ and all river flows have $q_D > 2$ the geometric monofractal model of Rodriguez-Iturbe et al [1992] can be excluded because it predicts $q_D < 1$.

Causal Simulations of Rain Series, River Flows, and Transfer Functions

Causality and Causal Multifractal Models

We have already mentioned that the natural framework for dynamical, space-time scaling processes is generalized scale invariance. By dimensional analysis, space and time are related by a velocity. Isotropic (self-similar) space-time corresponds to a generalization of Taylor's hypothesis involving a constant scale-independent velocity whereas a scale-dependent (e.g., turbulent) velocity involves anisotropic space-time (see *Tessier et al.* [1993] for a discussion), involving a group of scale changing operators ($T_\lambda=\lambda^{-G}$ changes scale by ratio λ, G is the corresponding semi-group generator). While knowledge of G places significant constraints on the process, it is not sufficient for a complete characterization. Indeed, the basic framework proposed by *Schertzer and Lovejoy* [1987a] for rain (and other turbulent atmospheric processes) was that of coupled cascade processes characterized by G as well as a second probability semigroup (discussed implicitly above) involving a generator characterized by H, α, C_1. However, in addition to these generators, dynamical models must also satisfy the basic physical principle of causality, which for stochastic models means that the statistical properties of the future state of any realization of the process must depend only on the past of the process. This causality principle is specifically violated by stochastic processes which (implicitly or explicitly) treat the time axis (i.e., past and future) symmetrically.

While the implementation of the causality constraint is somewhat involved in full space-time models [see *Marsan et al.*, this issue], here we will limit our discussion to simulations of time series, the discussion of which is much simpler. To illustrate our ideas, we recapitulate the basic steps involved in producing multifractal time series using continuous (in scale) cascade processes. The first step in the multifractal simulation procedure is to produce the subgenerator using a (white) Lévy-stable noise on the time axis, with the appropriate α and C_1. This noise is then fractionally integrated to produce a generator (log φ) with a logarithmic divergence with scale. The generator is then exponentiated to give the series corresponding to the conserved process. Finally, a further fractional integration of order H is performed to yield a simulation of the observed (nonconserved) process (see (8)).

We therefore generally require two fractional integrations. However, many different definitions of fractional integral exist. Up until now for simplicity, symmetric "Riemann-Liouville fractional integrals" ($I^H{}_{RL}$) were used (see *Schertzer and Lovejoy* [1991, appendix B] for more discussion and generalizations to higher dimensions; see also *Miller and Ross* [1993]). In real space for integrals of order H of the function f, this corresponds to the following:

$$I_{RL}^{H} f(t) = \frac{1}{\Gamma(H)} \int_{-\infty}^{\infty} |t-\tau|^{H-1} f(\tau) d\tau \qquad (13)$$

where Γ(H) is the usual gamma function. This corresponds to Fourier space filtering by

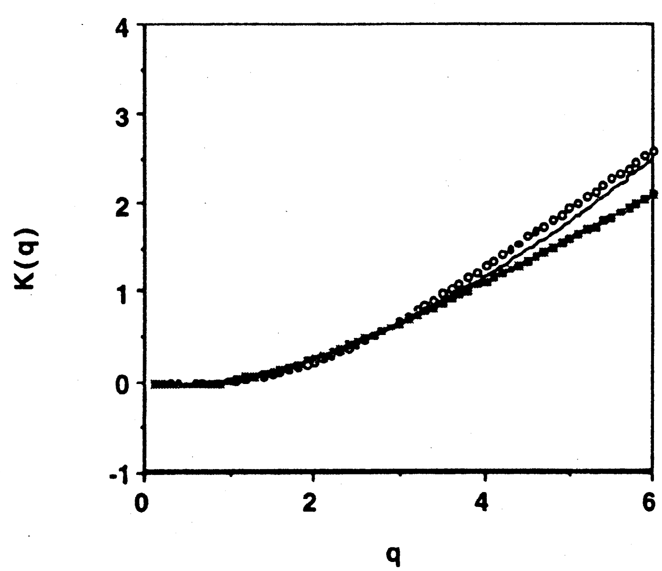

Figure 13. Log $K(q)$ vs q for 1-to 16- day flows for various rivers in France. The open circles were obtained considering 30 realizations (one per site) and the solid squares were obtained from just one site. The continuous line is the theoretical curve calculated from the values of α and C_1 obtained by the DTM method (i.e. $\alpha = 1.45$, $C_1 = 0.2$).

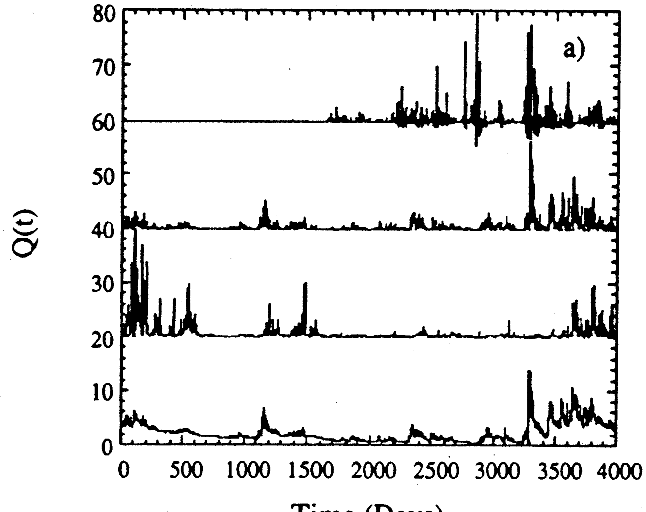

Figure 14a. Multifractal simulations of rain and river flow, using parameters α=1.5, C_1=0.15, and varying H. From top to bottom, H=-0.3, H=0.0, H=0.0 with a noncausal filter, and H=0.3.The same random seed was used throughout, and the fields have been offset for clarity. The three different parameter values correspond roughly to rainfall for periods >16 days, rivers >16 days, rivers <16 days, respectively.

$$\sqrt{\frac{2}{\pi}}\, sin\left(\frac{\pi}{2}(1-H)\right)|\omega|^{-H} \tag{14}$$

As can be seen by inspection of (13), the Riemann-Liouville fractional integral is determined symmetrically by the past and future values of $f(t)$. In order to obtain a causal "Liouville" fractional integral ($I^H{}_L$), we use the following definition:

$$I_L^H f(t) = \frac{1}{\Gamma(H)} \int_{-\infty}^{t} (t-\tau)^{H-1} f(\tau) d\tau \tag{15}$$

which by inspection only depends on the past of $f(t)$, and which is equivalent to the filter $(i\omega)^{-H}$.

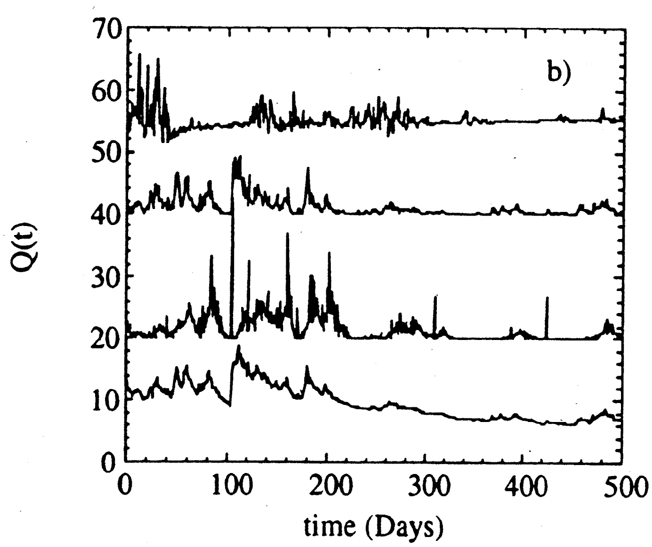

Figure 14b. As above, but showing a subsection of the simulated fields.

A priori, any H is permissible (including complex values!) as long as convergence is maintained. Figures 14a and 14b show the result of a continuous cascade multifractal simulation of rain and river flow series fully respecting the causality requirement discussed above and leading to the noticeable left-right asymmetry (compare the causal and acausal simulation for the same parameters and same random seed). The asymmetry arises from the asymmetry of the kernel in (15) contrary to the symmetry in (13). By using the one-sided fractional integral (15) that depends only on past times, the correlations induced on the white noise subgenerator are causal by construction.

Note that, at least in the usual interpretation, discrete multifractal cascade models of rain series are left-right symmetric; they cannot be used, except perhaps very artificially, for causal modeling of time series. Another causal approach for a space-time rain model has recently been proposed by *Over and Gupta* [this issue] involving discrete cascades for the spatial structure but causal (Markov) temporal structure. While this model is indeed causal, it has the unrealistic feature of being scaling and multifractal in space, but scale breaking in time. The velocity required to relate space and time is implicitly nonscaling which is contrary to mounting evidence from studies of the wind field [e.g., *Nastrom and Gage*, 1983], as well as contrary to direct space-time scale by scale analyses of rain [*Tessier et al.*, 1993]. A full scaling and causal space-time model using Liouville fractional integration in time, and Riemann-Liouville fractional integration in space has been developed by *Marsan et al.* [this issue]. Finally, we can also simulate the spectral break at periods of roughly 16 days by simply using a filter with two power law sections joined in a smoothly varying manner. The result is figures 15a-15c for river flows; the result looks very similar to the data series shown in Figure 1.

Scaling, Causal Transfer Functions

We have seen that empirically, at least for periods of 1 month or more, the conserved processes underlying both

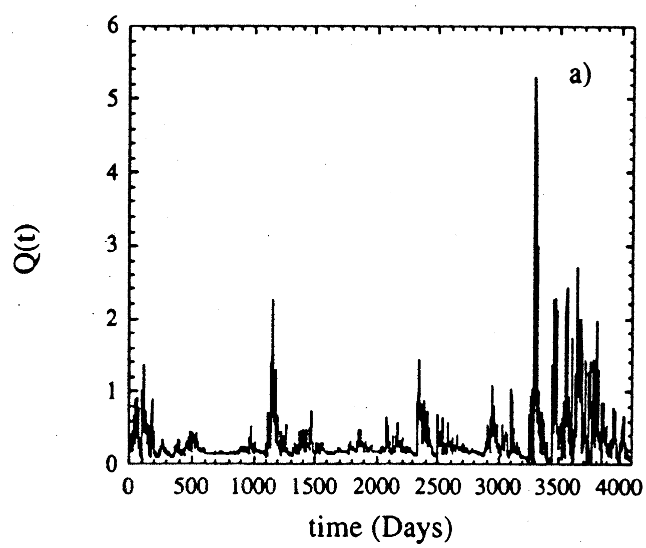

Figure 15a. A multifractal simulation of river flow, with parameters α=1.5, C_1=0.15, and fractionally integrated with a break in the scaling introduced only by the latter (H=0.3 at high frequencies, H=0 at low frequencies; see Figure 15c).

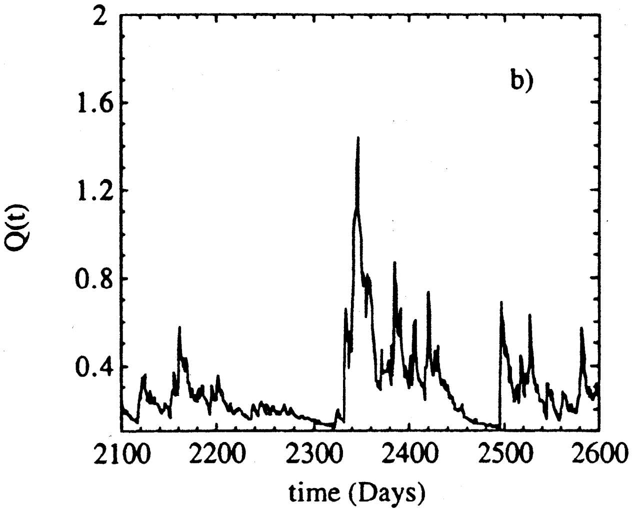

Figure 15b. Subsection of Figure 15a. Compare this with the data in Figure 1.

rainfall and river flow time series have nearly the same values of α and C_1, i.e., they have the same statistics over the entire corresponding range of scales (up to at least 11 years). Our analysis showed that over this range, the only significant difference is the corresponding degree of fractional integration. Therefore, if we fractionally integrate the rain series by an order $\Delta H = H_{\mathrm{river}} - H_{\mathrm{rain}} \approx 0.3$, we will obtain a series with the same statistics as the river flow series (i.e., filter by $(i\omega)^{\Delta H}$). Since fractional integration is simply a power law convolution, the two will be statistically related by a linear transfer function (although these will involve power laws rather than scale-dependent exponentials, as is usually assumed). Conversely, for periods less than 16 days (if the "zero value problem" has been adequately dealt with and hence the rain gauge estimates can be trusted) the α and C_1 values of the conserved multifractals are different. Hence in this case, both a linear transfer function and a nonlinear transformation of variables will be necessary. Note that in all cases, we establish only that series with the same statistics as the river series will be generated from the rain series; this does not in itself establish that actual river flows on specific realizations can be obtained from a single rain gauge in this way (the door is, however, open to the possibility of using more general linear functions; taking into account the spatial distribution of rain might be sufficient for the latter task).

Turning our attention to the determination of the linear transfer function for the low-frequency regime, and representing the rain rate as $R(t)$ and the stream flow as $Q(t)$, it is sufficient to consider the following convolution (represented by the asterisk):

$$Q(t) \overset{s}{=} I_{\mathrm{L}}^{\Delta H} R(t) \overset{s}{=} R(t) * t^{\Delta H - 1} \qquad (16)$$

where the notation $\overset{s}{=}$ indicates that the corresponding series have the same scaling properties. Using this representation gives a different understanding of the effect of the Liouville fractional integration or "causal transfer function". We see that the unit hydrograph will have a shape characterized by a steep rising limb, a peak and a milder sloping recession curve. This is because the response is integrating only past events. On the contrary, if we were to choose a Rieman-Liouville integration, the unit hydrograph would be a smooth rising slope, a peak and a smooth descent slope because the response would be incorporating past and future information. The basic Liouville asymmetry is indeed observed for river flows (see Figure 1). Using ΔH=0.3, Figures 16a and 16b shows an example indicating that using this single station transfer function approach does a surprisingly reasonable job of simulating the corresponding river series. Recall that the aim has only been to use a scaling, causal linear transformation on a rain gage series to yield a series with the same scale-by-scale statistics as for river flows. The fact that we get a surprising degree of agreement, even on a single realization, suggests that the method could profitably be extended to incorporate spatial rainfall information (e.g., via the use of more

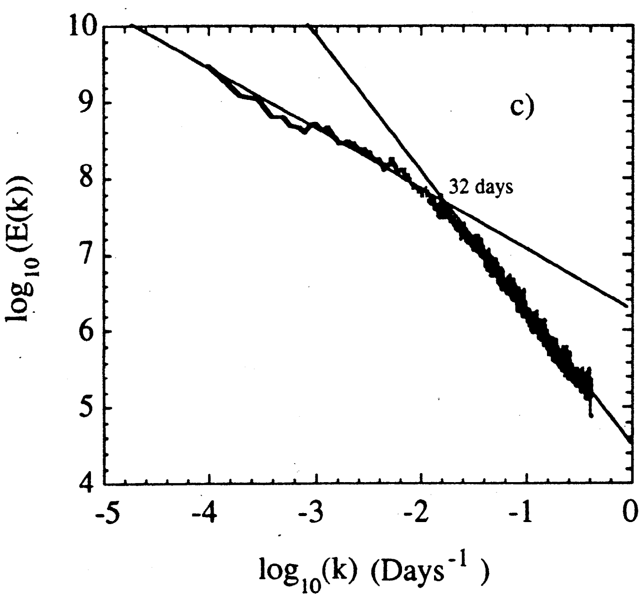

Figure 15c. Power spectrum of 400 realizations of a simulated causal multifractal field with a break in the scaling, using the parameters determined for river flow. The straight lines fit to the data have slopes of β=1.75 and β=0.68.

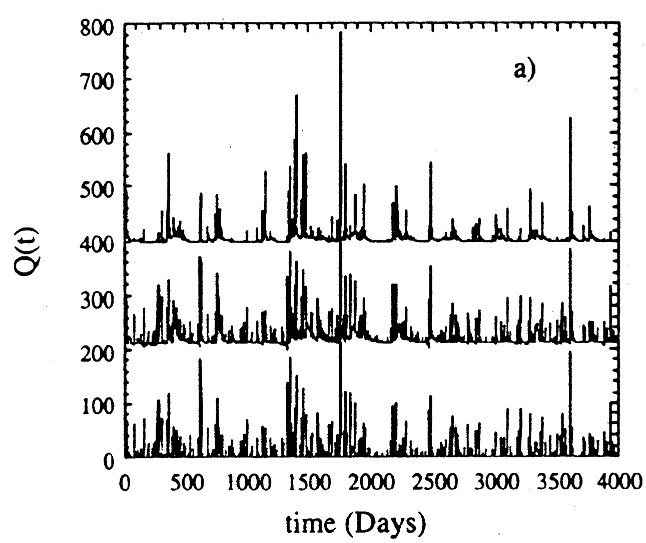

Figure 16a. Comparison of rainfall data (bottom), river flow data (top), and rainfall data fractionally integrated with a causal filter (transfer function) (middle). The fields have been offset for clarity.

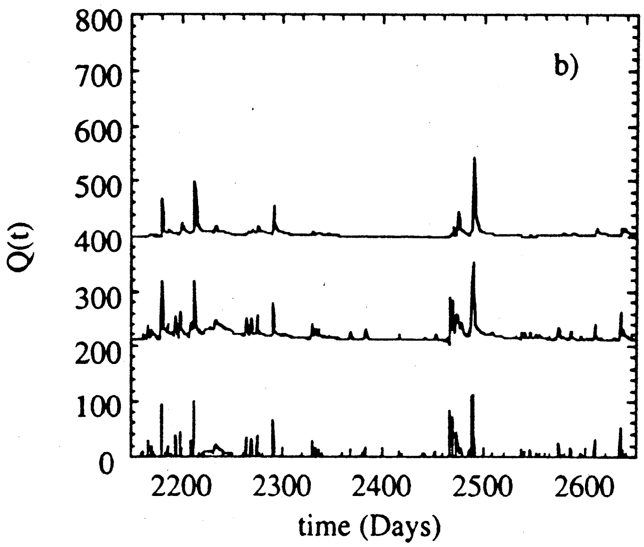

Figure 16b. Subsection of Figure 16a.

stations). Other limitations include the fact that the present transfer function does not account for a change in α and C_1 in the 1- to 16- day range nor has the annual cycle been considered.

Conclusion

A growing body of theoretical and empirical work indicates that space-time multifractals provide the natural framework for analyzing and modeling scale invariant geodynamical processes including rain, runoff and river flows. Using streamflow data from 30 rivers as well as the corresponding daily rainfall in France, we have provided the first multifractal analysis of river flows, estimating the range and types of scaling. Two basic regimes were identified (<16 days, >16 days), with the break corresponding to the atmosphere's synoptic maximum, the typical lifetime of planetary-scale atmospheric structures. Universal multifractal parameters characterizing the infinite hierarchy of scaling exponents were estimated, as well as critical exponents associated with extreme events, multifractal phase transitions, and self-organized criticality. These multifractal parameters (see Table 2) were also checked for possible systematic variations with regional climate types, the size of the basins, and the geology of the region. Since no systematic variations were found, we hypothesize that the measured properties are generic.

These parameters were then used to perform multifractal simulations of both rain and river flow time series. This involved an extension of previous multifractal modeling methods to take into account the requirement that the series are necessarily causal (the future depends on the past, but the past does not depend on the future). A causal series was obtained by using Liouville rather than Riemann-Liouville fractional integrals in the basic multifractal simulation algorithm. Since we found that, at least for the low-frequency regime, the basic conserved multifractal processes for both rain and river flows are nearly the same, a fractional integral of order ΔH (which corresponds to the Fourier filter by $(i\omega)^{\Delta H}$; $\Delta H \approx 0.3$) can be used to transform a single-gauge series into one having identical statistical properties to the corresponding river flow series; over the entire range 1 month to (at least) 11 years. To improve on this simple single-station linear response model, many stations could be added, and the cross-scaling properties of river flow and rainfall series could be analyzed with the help of Lie cascades [*Schertzer and Lovejoy*, 1995]. Alternatively, detailed understanding of the runoff processes could potentially be obtained by using multifractal models of topography, networks, and the corresponding transport. In the future, these methods promise to provide physically based hydrological models.

Acknowledgments. A significant amount of this work was made possible because of a fellowship given to Y. Tessier by CNRS in France. Jean-Pierre Carbonnel provided great support and made possible Tessier's stay at the Laboratoire de géologie appliquée, Université Pierre et Marie Curie, Paris (France). We thank G. Pandey for useful discussions.

References

Bak, P., C. Tang, and K. Wiessenfeld, Self-organized criticality: An explanation of 1/f noise, *Phys. Rev. Lett.*, *59*, 381-384, 1987.

Bak, P., C. Tang, and K. Wiessenfeld, Self-organized criticality, *Phys. Rev. Lett. A*,*38*, 364-374, 1988.

Bouquillon, C., and R. Moussa, Caractérisation fractale d' une série chronologique d' intensité de pluie, in *Rencontres Hydrologiques Franco-Roumaines*, pp. 35-36, Unesco, Paris, France, 1991.

Bras, R.L., and I. Rodriguez-Iturbe, Rainfall generation: A nonstationary time varying multidimensional model, *Water Resour. Res.*, *12*, 450-456, 1976.

Brax, P., and R. Pechanski, Levy stable law description on intermittent behaviour and quark-gluon phase transitions, *Phys. Lett. B*, 225-230, 1991.

Clark, C.O., Storage and the unit hydrograph, *Trans. Am. Soc. of Civ. Eng.*, *110*, 1945.

Corrsin, S., On the spectrum of isotropic temperature fluctuations in an isotropic turbulence, *J. Appl. Phys.*, *22*, 469-473, 1951.

Dooge, J.C.I., Linear theory of hydrologic systems, Agric. Res. Serv., U.S. Dept. of Agric., Washington, D.C., 1973.

Duncan, M., The Universal multifractal nature of radar echo fluctuations, Ph.D. thesis, McGill Univ., Montreal, Canada, 1993.

Fraedrich, K., and C. Larnder, Scaling regimes of composite rainfall time series, *Tellus ser. A*, *45*, 289-298, 1993.

Frisch, U., P.L. Sulem, and M. Nelkin, A simple dynamical model of intermittency in fully develop turbulence, *J. Fluid Mech.*, *87*, 719-724, 1978.

Gabriel, P., S. Lovejoy, D. Schertzer, and G.L. Austin, Multifractal analysis of resolution dependence in satellite imagery, *J. Geophys. Res.*, *15*, 1373-1376, 1988.

Gupta, V.K., and E. Waymire, A statistical analysis of mesoscale rainfall as a random cascade, *J. Appl. Meteorol.*, *32*, 251-267, 1993.

Gupta, V.K., O.J. Mesa, and D.R. Dawdy, Multiscaling theory of flood peaks: Regional quantile analysis, *Water Resour. Res.*, *30*(12), 3405-3421, 1994.

Hubert, P., and J.P. Carbonnel, Caracterisation fractale de la variabilité et de l' anisotropie des precipitations tropicales, *C. R. Acad. Sci. Paris*, *2*(307), 909-914, 1988.

Hubert, P., and J.P. Carbonnel, Dimensions fractales de l'occurence de pluie en climat Soudano-Sahélien, *Hydrol. Continentale*, *4*, 3-10, 1989.

Hubert, P., and J.P. Carbonnel, Fractal characterization of intertropical precipitations variability and anysotropy, in *Non-Linear Variability in Geophysics: Scaling and Fractals*, edited by D. Schertzer, and S. Lovejoy, pp. 209-213, Kluwer Acad., Norwell, Mass., 1991.

Hubert, P., Y. Tessier, P. Ladoy, S. Lovejoy, D. Schertzer, J.P. Carbonnel, S. Violette, I. Desurosne, and F. Schmitt, Multifractals and extreme rainfall events, *Geophys. Res. Lett.*, *20*(10), 931-934, 1993.

Hurst, H.E., Long-term storage capacity of reservoirs, *Trans. Am. Soc. Civ. Eng.*, *116*, 770-808, 1951.

Kida, S., Log stable distribution and intermittency of turbulence, *J. Phys. Soc. Jpn.*, *60*, 5-8, 1991.

Kolesnikov, V.N., and A.S. Monin, Spectra of meteorological field fluctuations, *Izv. Acad. Sci. USSR Atmos. Oceanic Phys.*, *1*, 653-669, 1965.

Kolmogorov, A.N., Local structure of turbulence in an incompressible liquid for very large Raynolds numbers, *Proc. Acad. Sci. URSS. Geochem. Sect.*, *30*, 299-303, 1941.

Ladoy, P., S. Lovejoy, and D. Schertzer, Extreme Variability of climatological data: Scaling and Intermittency, in *Non-Linear Variability in Geophysics: Scaling and Fractals,* edited by D. Schertzer, and S. Lovejoy, pp. 241-250, Kluwer Acad., Norwell, Mass., 1991.

Ladoy, P., F. Schmitt, D. Schertzer, and S. Lovejoy, Analyse multifractale de la variabilité temporelle des observations pluviometriques a Nîmes, *C. R. Acad. Sci. Paris, 317, II*, 775-782, 1993.

Larnder, C., Observer Problems in multifractals: the example of rain, M.Sc. thesis, McGill Univ., Montreal, Canada, 1995.

Lavallée, D., Multifractal techniques: Analysis and simulation of turbulent fields, Ph.D. thesis, McGill Univ., Montreal, Canada, 1991.

Lavallée, D., S. Lovejoy, and D. Schertzer, On the determination of the codimension function, in *Non-Linear Variability in Geophysics: Scaling and Fractals*, edited by D. Schertzer, and S. Lovejoy, pp. 99-110, Kluwer Acad., Norwell, Mass., 1991.

Lavallée, D., S. Lovejoy, and D. Schertzer, Nonlinear variability and landscape topography: Analysis and simulation, in *Fractals in geography*, edited by L. De Cola, and N. Lam, pp. 158-192, Prentice-Hall, Eaglewood Cliffs, N.J., 1993.

Lovejoy, S., Analysis of rain areas in terms of fractals, in *20th Conference on Radar Meteorology*, pp. 476-484, Am. Meteorol. Soc., Boston, Mass., 1981.

Lovejoy, S., and B.B. Mandelbrot, Fractal properties of rain and a fractal model, *Tellus Ser. A*, *37*, 209, 1985.

Lovejoy, S., D. Schertzer, Scale invariance in climatological temperatures and the spectral plateau *Ann. Geophys*, *4B* 401-410, 1986.

Lovejoy, S., and D. Schertzer, Multifractals, Universality classes and satellite and radar measurements of cloud and rain fields, *J. Geophys. Res.*, *95*, 2021, 1990a.

Lovejoy, S., and D. Schertzer, Our multifractal atmosphere: A unique laboratory for non-linear dynamics, *Can. J. Phys.*, *46*, 62, 1990b.

Lovejoy, S., and D. Schertzer, Multifractal analysis techniques and the rain and clouds fields from 10^{-1} to 10^{6}m, in *Non-Linear Variability in Geophysics: Scaling and Fractals*, edited by D. Schertzer, and S. Lovejoy, pp. 111-144, Kluwer Acad., Norwell, Mass., 1991.

Lovejoy, S., and D. Schertzer, How bright is the coast of Brittany?, in *Fractals in Geoscience and Remote Sensing*, edited by G. Wilkinson, I. Kanellopoulos and J. Mgier , pp. 102-151, Offi. for Official Publi. of the Eur. Communities, Luxembourg, 1995a.

Lovejoy, S., and D. Schertzer, Multifractals and Rain, in *New Uncertainty Concepts in Hydrology and Water Ressources*, edited by Z.W. Kunzewicz, pp. 62-103, Cambridge Univ. Press, New York, 1995b.

Lovejoy, S., D. Schertzer, and A.A. Tsonis, Functional Box-Counting and Multiple Dimensions in rain, *Science*, *235*, 1036-1038, 1987.

Lovejoy, S., D. Lavallée, D. Schertzer, and P. Ladoy, The $l^{1/2}$ law and multifractal topography: Theory and analysis, *Nonlinear Proc. Geophys.*, *2*(1), 16-22, 1995a.

Lovejoy, S., B. Watson, D. Schertzer, and G. Brosamlen, Scattering in multifractal media, in *Particle Transport in Stochastic Media*, edited by L. Briggs, pp. 750-760, Am. Nucl. Soc., Portland, Oreg., 1995b.

Mandelbrot, B.B., Intermittent turbulence in self-similar cascades: Divergence of high moments and dimension of the carrier, *J. Fluid Mech.*, *62*, 331-350, 1974.

Mandelbrot, B.B., and J.R. Wallis, Noah, Joseph and operational hydrology, *Water Resour. Res.*, *4*, 909-918, 1968.

Mandelbrot, B.B., and J.R. Wallis, Some long run properties of geophysical records, *Water Resour. Res.*, *5*, 228, 1969.

Marsan, D., D. Schertzer, and S. Lovejoy, Causal space-time multifractal processes: predictability and forecasting of rain fields, *J. Geophys. Res.*, this issue

Miller, S.M., and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, 366 pp., John Wiley, New York, 1993.

Minshall, N.E., Predicting storm runoff on small experimental watersheds, *J. Hydraul. Div. am. Soc. Civ. Eng.*, *86*, 1960.

Monin, A.S., and A.M. Yaglom, *Statistical Fluid Mechanics*, MIT Press, Cambridge Mass., 1975.

Nash, J.E., The form of the unit hydrograph, *Int. Assoc. Sci. Hydrol. Publ.*, 3, 1960.

Nastrom, G.D., and K.S. Gage, A first look at wave number spectra from GASP data, *Tellus*, *35*, 383, 1983.

Naud, C., D. Schertzer, and S. Lovejoy, Radiative transfer in multifractal atmospheres: fractional integration, multifractal phase transitions and inversion problems in *Stochastic Models in Geosystems, IMA volume in mathematics and its applications , vol 85,* edited by S.A. Molchanov and W.A. Woyczynski, pp. 239-267, Springer Verlag, New York, 1996.

Novikov, E.A., and R. Stewart, Intermittency of turbulence and spectrum of fluctuations in energy-disspation, *Izv. Akad. Nauk. SSSR. Ser. Geofiz.*, *3*, 408-412, 1964.

Obukhov, A., Structure of the temperature field in a turbulent flow, *Izv. Akad. Nauk. SSSR. Ser. Geogr. I Geofiz.*, *13*, 55-69, 1949.

Olsson, J., J. Niemczynowicz, R. Berndtsson, and M. Larson, Fractal properties of rainfall, *Ann. Geophys., spec. issue*, 142, 1990.

Olsson, J., Limits and Characteristics of the multifractal behavior of a high-resolution rainfall time series, *Nonlinear Proc. Geophys.*, *2*(1), 23-29, 1995.

Over, T.M., and VK. Gupta, A space-time theory of mesoscale rainfall using random cascades, *J. Geophys. Res.*, this issue.

Parisi, G., and U. Frisch, A multifractal model of intermittency, in *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*, edited by M. Ghil, R. Benzi, and G. Parisi, pp. 84-88, North-Holland, New-York, 1985.

Petterssen, S., *Introduction to Meteorology*, 333 pp., McGraw-Hill, New-York, 1969.

Rodriguez-Iturbe, I., V.K. Gupta, and E. Waymire, Scale considerations in the modelling of temporal rainfall, *Water Resour. Res.*, *20*, 1611-1619, 1984.

Rodriguez-Iturbe, I., E.J. Ijjasz-Vasquez, R.L. Bras and D.G. Tarbotton, Power law distribution of discharge mass and energy in river basins, *Water Resour. Res.*, *28* 1089-1093, 1992.

Schertzer, D., and S. Lovejoy, Generalised scale invariance in turbulent phenomena, *Phys.-Chem. Hydrodyn. J.*, *6*, 623-635, 1985a.

Schertzer, D., and S. Lovejoy, Generalized scale invariance, differential rotation and cloud texture, in *Turbulent Shear Flow 4*, edited by B. Launder, pp. 7-33, Springer-Verlag, New-York, 1985b.

Schertzer, D., and S. Lovejoy, Physical modeling and analysis of rain and clouds by anysotropic scaling of multiplicative processes, *J. Geophys. Res.*, *92*, 9693-9714, 1987a.

Schertzer, D., and S. Lovejoy, Singularités anysotrope, divergence des moments en turbulence, *Ann. Soc. Math. Québec*, *11*(1), 139-181, 1987b.

Schertzer, D., and S. Lovejoy, Nonlinear geodynamical variability: multiple singularities, universality and observables, in *Non-Linear Variability in Geophysics: Scaling and Fractals*, edited by D. Schertzer and S. Lovejoy, pp. 41-82, Kluwer Acad., Norwell, Mass., 1991.

Schertzer, D., and S. Lovejoy, Hard and soft multifractal processes, *Physica A*, *185*, 187-194, 1992.

Schertzer, D., and S. Lovejoy, Multifractal generation of self-organized criticality, in *Fractals In the natural and applied sciences*, edited by M.M. Novak, pp. 325-339, Elsevier, New York, 1994.

Schertzer, D., and S. Lovejoy, *Multifractals and Turbulence: Fundamentals and Applications*, 230 pp., World Scientific, River Edge, N.J., 1995.

Schertzer, D., and S. Lovejoy, Universal multifractals do exist: comments on "A statistical analysis of mesoscale rainfall as a random cascade" by V.G. Gupta and E. Waymire, *J. Appl. Meteor.*, in press, 1996a.

Schertzer, D., and S. Lovejoy, The multifractal phase transition route to self-organized criticality, *Phys. rep.*, in press, 1996b.

Schertzer, D., S. Lovejoy, and D. Lavallée, Generic multifractal phase transitions and self-organized criticality, in *Cellular Automata: Prospects in Astronomy and Astrophysics*, edited by J.M. Perdang and A. Lejeune, pp. 216-227, World Scientific, River Edge, N.J., 1993.

Schertzer, D., S. Lovejoy, and F. Schmitt, Structures in turbulence and multifractal universality, in *Small-Scale Structures in Three-Dimensional Hydro and Magnetohydrodynamic Turbulence*, edited by M. Meneguzzi, A. Pouquet, and P.L. Sulem, pp. 137-144, Springer-verlag, New-York, 1995.

Schmitt, F., S. Lovejoy, and D. Schertzer, Multifractal analysis of the Greenland Ice-core project climate data, *Geophys. Res. Lett.*, *22*, 1689-1692, 1995.

Schuster, H.G., *Deterministic Chaos*, VCH, New York, 1988.

Segal, B., High-intensity rainfall statistics for Canada, Comm. Res. Cent., Dept. of Commun., Ottawa, Ont., Canada, 1979.
She, Z.S., and E. Waymire, Quantized energy cascade and Log-Poisson statistics in fully developed turbulence, *Phys. Rev. Lett., 74*, 262-265, 1995.
Sherman, L.K., Streamflow from rainfall by the unit-graph method, *Eng. News Rec., 108*, 501-505, 1932.
Taylor, G.I., The spectrum of turbulence, *Proc. R. Soc. London A, 164*(919), 476-490, 1938.
Tessier, Y., Multifractal objective analysis, rain and clouds, Ph.D. thesis, McGill University, Montreal, Canada, 1993.
Tessier, Y., S. Lovejoy, and D. Schertzer, Universal Multifractals: theory and observations for rain and clouds, *J. Appl. Meteorol.*, *32*(2), 223-250, 1993.
Turcotte, D.L., and Greene, A. Scale-Invariant approach to flood-frequency analysis, *Stochastic Hydrol. Hydraul.*, *7*, 33-40, 1993.

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(Received March 14, 1996; revised May 29, 1996; accepted June 11, 1996.)