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ESTIMATION OF UNIVERSAL MULTIFRACTAL INDICES FOR ATMOSPHERIC TURBULENT VELOCITY FIELDS

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Abstract

We study wind turbulence with the help of universal multifractals, using atmospheric high resolution time series. We empirically determine the three universal indices (H, C_1, and α) which are sufficient to characterize the statistics of turbulence. The first, H, which characterizes the conservation of the field, is theoretically and empirically known to be $\approx 1/3$, while C_1 corresponds to the inhomogeneity of the mean field ($C_1 = 0$ for homogeneous fields, and $C_1 > 0$ for inhomogeneous and intermittent fields). The most important index is the Lévy index α corresponding to the degree of multifractality ($0 \leq \alpha \leq 2$, $\alpha = 0$ for a monofractal). The two latter indices are directly obtained by applying the double trace moment technique (DTM) on the turbulent field. Analyzing various atmospheric velocity measurements we obtain: $\alpha \approx 1.45 \pm 0.1$ and $C_1 \approx 0.25 \pm 0.1$. These results show that atmospheric turbulence has nearly the same multifractal behavior everywhere in the boundary layer, corresponding to unconditionally hard multifractal ($\alpha \geq 1$) processes. This describes the entire hierarchy of singularities of the Navier-Stokes equations.

1. INTRODUCTION

In the limit of high Reynolds number, turbulence is known to be scale invariant.[1,2] One of the points which has been important in characterizing fully developed turbulence is to know whether it belongs to universal classes which could describe any high Reynolds number turbulent flow.[3,4] We test here such a scale invariant model of turbulence, called the *universal multifractal model*,[5,6] using different atmospheric turbulent datasets. This model is determined by only three indices, which completely describe the scaling behavior of the turbulent fields. We compare the values of these indices for various atmospheric datasets, in order to confirm the hypothesis of their constancy.

2. THE DATA AND THEIR SPECTRA

The datasets are turbulent velocity measurements made in the atmospheric boundary layer: the most important one was taken near Montreal, at 3 meters from the ground, with a high-resolution hot wire anemometer, and the others with sonic anemometers in Paris, and in Bordeaux (see Table 1 for a description of the experimental conditions).

Table 1 Description of the Characteristics of Three Atmospheric Datasets

Dataset #	Anemometer	Sampling Frequency (Hz)	Place of Acquisition	Altitude (m)	Total Number of Data Points used	β
1	Hot wire	2000	Montreal	3	720 000	1.70
2	Sonic	200	Paris	20	25 000	1.72
3	Sonic	10	Bordeaux	25	360 000	1.70

Their wind energy spectrum follows a power law over a wide range of scales:

$$E_\nu(k) \approx k^{-\beta} \tag{1}$$

with the rate of acquisition, the range of scaling, and β given in Table 1. Figure 1 shows the power spectrum of the dataset #1. The values of the spectral slopes are very close to the Kolmogorov-Obukhov phenomenological derivations predicting $\beta = 5/3$.[1,2] We then assume the validity of the refined[a] similarity hypothesis[3]:

$$\Delta\nu(\ell) \approx \varepsilon_\ell^{1/3} \ell^H \tag{2}$$

where $\Delta\nu(\ell)$ is the wind shear $|\nu(\mathrm{x}+\ell) - \nu(\mathrm{x})|$ at scale ℓ, ε_ℓ is the local rate of energy transfer (from scales larger to scales smaller than ℓ). The index H is the mean deviation from conservation of the velocity shears, and is related to β by $\beta \cong 2H + 1$ (e.g. see Ref. 7).

[a]Kolmogorov 1941 assumed ε to be homogeneous in space, i.e., $\varepsilon = \bar{\varepsilon}$, whereas Eq. (2) takes into account the small scale intermittency of turbulence with the introduction of a very intermittent energy transfer field.

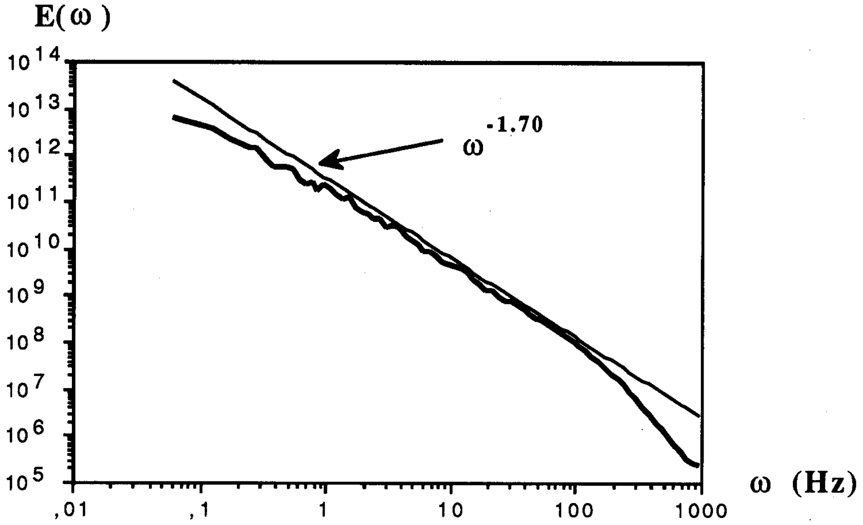

Fig. 1 The wind energy spectrum of atmospheric turbulence corresponding to dataset #1. For a wide range of scales, it follows a power law relation $E_\nu(k) \approx K^{-\beta}$ with $\beta \approx 1.68 \pm 0.05$ as predicted by Refs. 1 and 2.

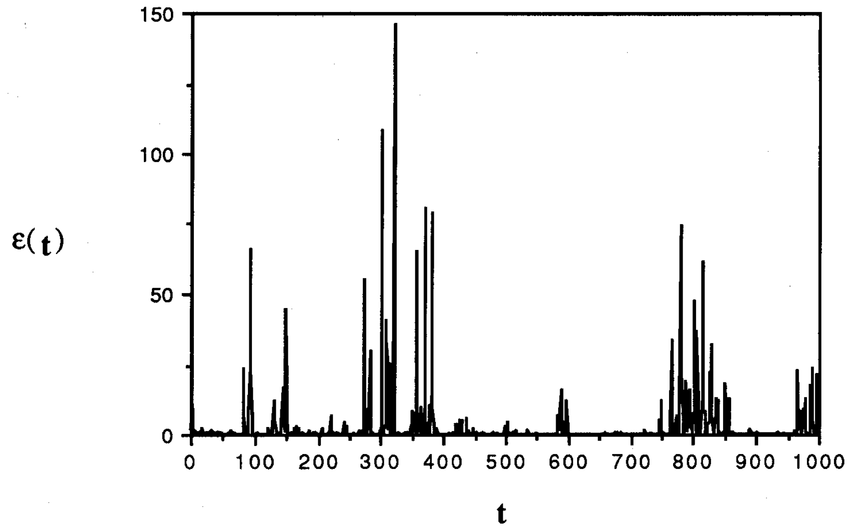

Fig. 2 A pattern of ε, the rate of energy transfer from large to small scales, for atmospheric turbulence: this is a very intermittent (and scaling) field.

This gives us $H = 0.33 \pm 0.03$ for the three datasets, as predicted by phenomenological models. In order to study the statistical properties of the very intermittent dissipation field ε_ℓ, we perform a fractional differentiation of order 1/3 of the wind field (i.e., a multiplication by $k^{1/3}$ in Fourier space,[6]) and then take the third power: the result is a very intermittent field, whose pattern can be seen in Fig. 2 for atmospheric measurements corresponding to dataset #1.

3. UNIVERSAL MULTIPLE SCALING OF THE FIELD ε

We have studied the energy flux ε using multifractal measures resulting from a multiplicative cascade process.[4,8–11] When the cascade has proceeded over a scale ratio $\lambda = \frac{L}{\ell}$ (the ratio of the largest scale of interest to the smallest scale) the density of the conserved energy flux has the singular behavior[5,12,13]:

$$\varepsilon_\lambda \approx \lambda^\gamma \tag{3}$$

When $\lambda \to \infty$ (or $\ell \to 0$), $\gamma > 0$ is an order of singularity. The statistical moments of this field will have the following scaling behavior[5]:

$$\langle (\varepsilon_\lambda)^q \rangle \approx \lambda^{K(q)} \tag{4}$$

where $\langle\ \rangle$ indicates ensemble averaging, and $K(q)$ is a nonlinear function[b] which characterizes all the statistics of the field.

For universality classes (the stable and attractive limit obtained when mixing cascade processes, see Refs. 5 and 6), the functions $K(q)$ depend only on two indices:

$$K(q) = \begin{cases} \dfrac{C_1}{\alpha - 1}(q^\alpha - q) & \alpha \neq 1 \\ C_1 q \ln(q) & \alpha = 1 \end{cases} \tag{5}$$

where the Lévy index α ($0 \leq \alpha \leq 2$) can be understood as an interpolation between the two extremes and well known cascade models of turbulence: the β-model ($\alpha = 0$)[4,8,9] and the lognormal model ($\alpha = 2$)[3,14,15]; and the second index corresponds to the mean inhomogeneity of the field: $C_1 = 0$ for homogeneous fields, and the larger C_1, the more the intermittent field. We have estimated these indices for wind tunnel and atmospheric energy fluxes, using a generalization of the (simple) scaling exponent $K(q)$.

4. ESTIMATION OF α AND C_1 WITH THE HELP OF THE DOUBLE TRACE MOMENT (DTM) ANALYSIS TECHNIQUE

The basic idea of the DTM technique[16,17] is to generalize the application of statistical methods to the quantity $(\varepsilon_\Lambda)^\eta$. This is done by taking the ηth power of ε_Λ at the scale ratio Λ (the outer or largest scale of interest to the smallest scale of homogeneity), and then studying its scaling behavior at decreasing values of the scale ratio $\lambda \leq \Lambda$:

$$\varepsilon_{\lambda,\Lambda}^{(\eta)} = \frac{(\varepsilon_\lambda)^\eta}{\langle (\varepsilon_\lambda)^\eta \rangle} \langle (\varepsilon_\Lambda)^\eta \rangle \tag{6}$$

The moments of this new field then have the following multiple scaling behavior:

$$\langle [\varepsilon_{\lambda,\Lambda}^{(\eta)}]^q \rangle \approx \lambda^{K(q,\eta)} \tag{7}$$

where $K(q, \eta)$ is the q, η double trace moment scaling exponent related to $K(q, 1) \equiv K(q)$ by [$\langle (\varepsilon_\Lambda)^\eta \rangle$ is a constant]:

$$K(q, \eta) = K(q\eta) - qK(\eta) \tag{8}$$

[b] $K(q)$ is related to the strange attractor notation[13] by: $K(q) = (q-1)D - \tau(q)$ where D is the dimension of the space.

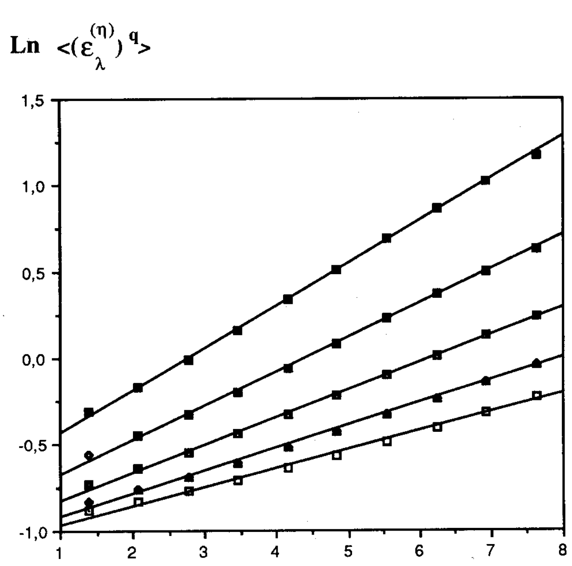

Fig. 3 A representation of $\langle[\varepsilon_{\lambda,\Lambda}^{(\eta)}]^q\rangle$, vs. λ in a log-log plot for atmospheric energy flux: the straight lines show that scaling of Eq. (4) is well respected. These straight lines correspond to $q = 2$ and $\eta = 0.43$, 0.49, 0.56, 0.65, 0.74 (from bottom to top).

It gives for universality classes:

$$K(q, \eta) = n^{\alpha} K(q) \qquad (9)$$

By keeping q fixed (but different from the special values 0 or 1), the slope of $|K(q, \eta)|$ as a function of η on a log-log graph gives the value of the index α, which with the help of the intersection with the line ($\eta = 1$) yields C_1. Varying q then allows for a systematic verification of Eq. (9), and hence the universality hypothesis.

Figure 3 shows $\langle[\varepsilon_{\lambda,\Lambda}^{(\eta)}]^q\rangle$ vs. λ in a log-log plot for atmospheric energy flux corresponding to dataset #1: the straight lines show that the scaling of Eq. (7) is well respected, and their slope are the estimates of $|K(q, \eta)|$. Figure 4 shows the curves $\log|K(q, \eta)|$ vs. $\log(\eta)$ for different values of the parameter q, for atmospheric energy flux corresponding to dataset #1: the straight lines show that Eq. (9) is well respected for a wide range of η-values, and their slopes give the estimates shown in Table 2.

Table 2 Estimates of the Indices α and C_1 obtained for the Different Datasets presented in Table 1, showing that their Values are likely to be Steady in the Atmosphere (at least in the boundary layer)

Dataset	Maximum Scale Ratio	α	C_1
1	2048	1.50 ± 0.05	0.25 ± 0.05
2	1024	1.45 ± 0.1	0.29 ± 0.1
3	512	1.4 ± 0.1	0.24 ± 0.1

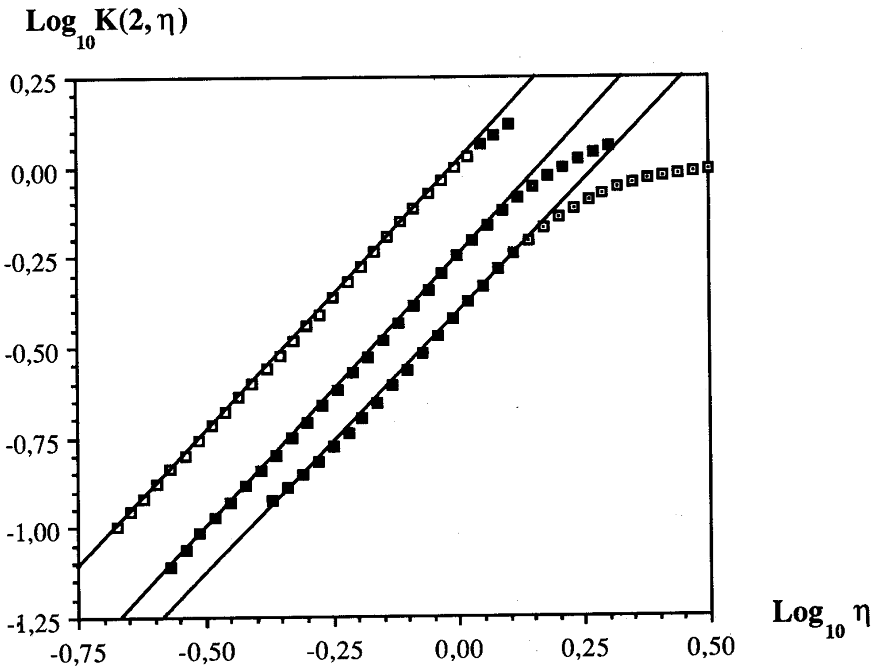

Fig. 4 The curves $\mathrm{Log}|K(q,\eta)|$ vs $\mathrm{Log}(\eta)$ for $q = 2$, 2.5 and 3 (from bottom to top) for atmospheric turbulence. The straight lines show that Eq. (9) is well respected for a wide range of η- values, and their slopes give the following estimates: $\alpha \approx 1.50 \pm 0.05$ for atmospheric turbulence corresponding to dataset #1. We can then estimate $C_1 \approx 0.25 \pm 0.05$.

According to error estimates presented in Table 2, we propose the following values for the indices α and C_1 in the atmospheric boundary layer: $\alpha \approx 1.45 \pm 0.1$ and $C_1 \approx 0.25 \pm 0.1$. We note that these values are in the same range that estimates obtained for wind tunnel turbulence[18–20]: $\alpha \approx 1.3 \pm 0.1$ and $C_1 \approx 0.25 \pm 0.05$. These values of α and C_1 show that turbulence is a hard multifractal process[18,21–23]: because $\alpha > 1$, divergence of moments[4,10,20,24] is expected to occur.

5. MOMENTS OF THE ENERGY FLUXES AND STRUCTURE FUNCTIONS OF THE WIND FIELD

With the help of these results, we compare empirical statistical moments with $K(q)$ given by Eq. (5), in Fig. 5: for moments order $q < 3.0 \pm 0.5$[c] the empirical and "theoretical" scaling exponents are in excellent agreement. This critical moment[21,24] corresponds either to a second order phase transition analogue (a maximum moment computable due to the finite size of the dataset), or to a first order phase transition[27] analogue

[c]Equations (2) and (4) give the scaling velocity structure functions: $\langle[\delta v(l)]^q\rangle \approx \lambda^{-\zeta(q)} \approx \langle(\varepsilon_\lambda)^{q/3}\rangle\lambda^{-Hq}$, which gives the relation $\zeta(q) = Hq - K(\frac{q}{3})$. Thus, in the case of database #1, it is useful to study the structure function $\zeta(q)$ only up to moment order $3q_{\max} \approx 9.0 \pm 1.5$, unlike Ref. 25, which computed structure functions up to order 18, and Ref. 26 which used the latter structure function to evaluate α.

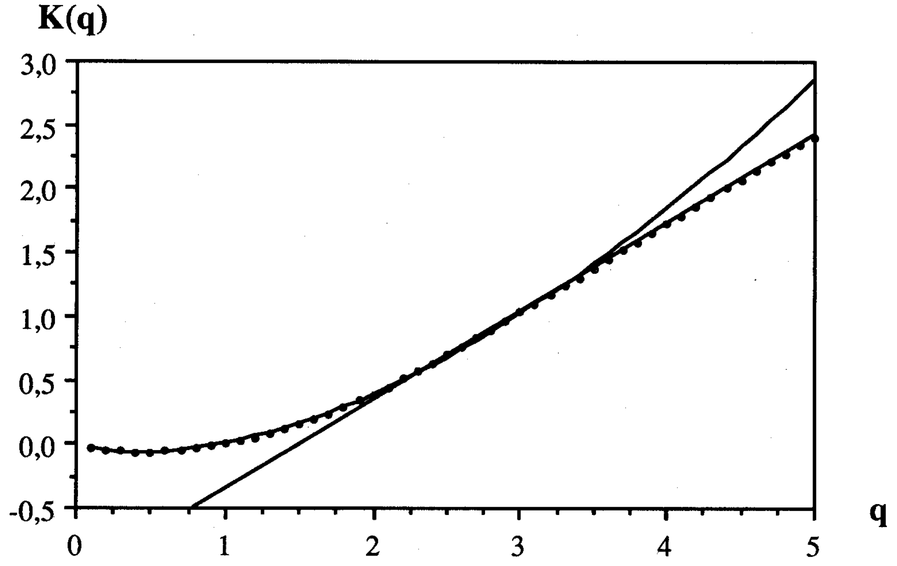

Fig. 5 A comparison between empirical statistical moments (squares) with $K(q)$ given by Eq. (4) (universality hypothesis, continuous line); for moments order $q < 3.0 \pm 0.5$ for atmospheric turbulence the empirical and "theoretical" scaling exponents are in excellent agreement. For larger moments, an empirical linear behavior is observed which can be explained by multifractal phase transitions.

(corresponding to a critical order of divergence of moments[d]). This question is studied elsewhere.[28]

6. CONCLUSION

Using various atmospheric datasets, we obtain similar values of the three indices (H, C_1, α) which completely characterize the statistics of turbulence: $H = 0.33 \pm 0.03$, $C_1 \approx 0.25 \pm 0.1$ and $\alpha \approx 1.45 \pm 0.1$ for the three atmospheric datasets studied. These results show that turbulence is a "hard" multifractal process ($\alpha \geq 1$), and that it belongs to theoretically predicted universality classes.

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[d]This divergence of moments is expected to occur because of some very violent "head" small scale singularities, which are not smoothed by integration: if the number of samples increases, more and more of such singularities are encountered, and the statistical moments diverges.

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