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Multifractal surfaces and terrestrial topography

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Abstract. – We analyze terrestrial topographic data from four Digital Elevation Models which collectively span the range 20000 km to 50 cm. We use power spectra and trace moment analysis techniques to show that topography is multifractal from planetary scales down to around 40 m, where the multiscaling is broken by trees. We show that over this range the topography is reasonably described by a global nonlinear moment scaling function, itself determined by three parameters. We argue that these isotropic analyses are insensitive to the anisotropies and are hence compatible with different geomorphologies.

Natural and artificial surfaces display complex structures over wide ranges of scale. Since ref. [1], and especially ref. [2], there has been an explosion of fractal models and analyses of surfaces based on a self-affine geometric model in which the scaling of surface gradients is determined by a unique exponent: the fractal codimension of the surface. This “monofractal” framework was given impetus by the success of the Kardar, Parisi and Zhang surface growth model [3], a soluble nonlinear stochastic PDE with self-affine solutions. Since then, work in the physics of interfaces, deposited metal films, fracture surfaces, contact mechanics, biomedical surfaces (*e.g.*, [4–10]) have tended to adopt this model. In geophysics, where the topography had long been known to have a power law spectrum [11], a similar trend occurred following Mandelbrot’s Gaussian self-affine model [12]. By the 1990s, there had been enough empirical work on the subject to generate a debate on the value of the supposed unique fractal dimension (see [13] for an early review).

This self-affine model assumes a scaling relation of the form $\Delta h_\lambda \stackrel{d}{=} \lambda^{-H} \Delta h_1$, where $\lambda = L/l$ is the ratio of the largest scale of the system L over the scale of observation l , $\Delta h_\lambda = h(\mathbf{x} + \lambda^{-1}\Delta\mathbf{x}) - h(\mathbf{x})$ are height increments at resolution λ , \mathbf{x} is a vector, H is a scaling exponent (it is a codimension) and $\stackrel{d}{=}$ means equality in the sense of probability distributions. This is called “simple scaling” because it involves only H : the fractal dimension of the surface is $3 - H$ and the fractal dimension of *any* isoline at altitude threshold T (*i.e.* the set of points \mathbf{x} such that $h(\mathbf{x}) = T$) is $D(T) = 2 - H$, independent of T . Alternatively, the

TABLE I – Characteristics of the DEMs and regions analyzed.

Data set	Horizontal resolution	Vertical discretization (m)	Number of transects analyzed	Length of transects (km)	α	C_1	H
ETOPO5	5' (≈ 10 km)	1	500	40000	1.72	0.14	0.71
GTOPO30	30' (≈ 1 km)	1	1225	4096	1.77	0.08	0.60
U.S.	90 m	1	2500	5898	1.51	0.09	0.61
Lower Saxony	50 cm	0.1	3000	3	2.00	0.17	0.60

structure functions and the energy spectrum scale as $\langle |\Delta h_\lambda|^q \rangle \sim \lambda^{-\xi(q)}$ and $E(\mathbf{k}) \sim |\mathbf{k}|^{-\beta}$, respectively, where $\xi(q) = qH$ and $\beta = 1 + \xi(2)$ (\mathbf{k} is a wave vector).

Although this framework is seductive—it reduces the relation of large and small surface patches to a simple geometrical transformation—it is in fact highly restrictive. Indeed, the full characterization of scale-invariant fields involves an infinite hierarchy of fractal dimensions: scale-invariant sets are fractals, scale-invariant fields are generally “multifractals” [14–16]. It would be a miracle if the fractal sets defined by different altitude thresholds all had exactly the same fractal dimension. Using functional box counting, ref. [17] directly confirmed this multifractality: $D(T)$ decreases to zero as the altitude increases; refs. [18, 19] showed that various statistical exponents (including those of the structure functions) were nonlinear with q . Similarly, in a paper otherwise devoted to demonstrating the validity of the monofractal/self-affine model, ref. [4] included a section showing that fracture surfaces were in fact multifractal. Unfortunately, they seem to have abandoned this insight in later publications. Similarly, in an analysis of fracture surfaces, ref. [7] observed a nonlinear $\xi(q)$, but attributed the effect not to multifractality but to monofractality combined with algebraic tails on the probability distributions (leading to a bilinear $\xi(q)$), concluding that, in spite of appearances, the self-affine picture was valid [20].

In view of the success of multifractals in turbulence and other scale-invariant systems, it is perhaps surprising that the monofractal model persists in the field of rough surfaces. There are several reasons for the reluctance to consider multifractal models. First, multifractality apparently sacrifices simplicity since it seems to imply that even scale-invariant surfaces require an unmanageable infinity of parameters (*i.e.* the functions $D(T)$ or $\xi(q)$). Second, multifractality is apparently simplistic since it is not obviously compatible with the diversity of surface morphologies. Thirdly, since multifractal fields are the result of variability building up scale after scale over wide ranges, we anticipate that a convincing study will require huge quantities of data. We have already devoted several papers to the first two objections. Briefly, multifractal processes have stable attractive generators requiring only three “universal” parameters [22, 23]. In addition, scale invariance is a nonclassical symmetry principle and can readily be generalized to accommodate very general anisotropic notions of scale (“Generalized Scale Invariance” [22, 24]) which can—at least in principle via a group of scale changing operators—account for the observed ridges, textures and other surface morphologies.

The aim of this letter is to remove the third reason for hanging on to the monofractal picture: the inadequate empirical support for the multifractal model. Below, we use several orders of magnitude more data to show that over roughly six orders of magnitude in scale the Earth’s topography is not self-affine but rather multifractal/multiscaling. Table I summarizes the four different Digital Elevation Models (DEM) that span various ranges of scales: ETOPO5 (global topography), GTOPO30 (global continental topography), the United States continent

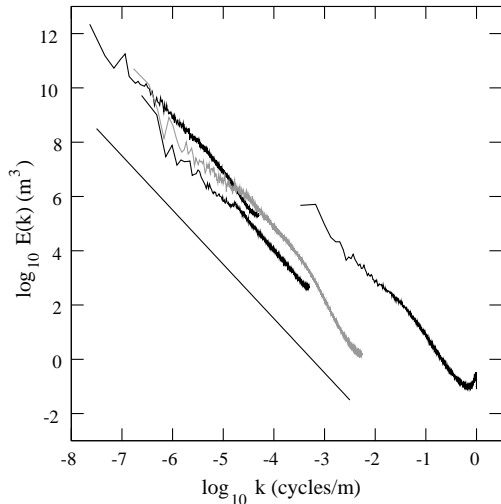


Fig. 1

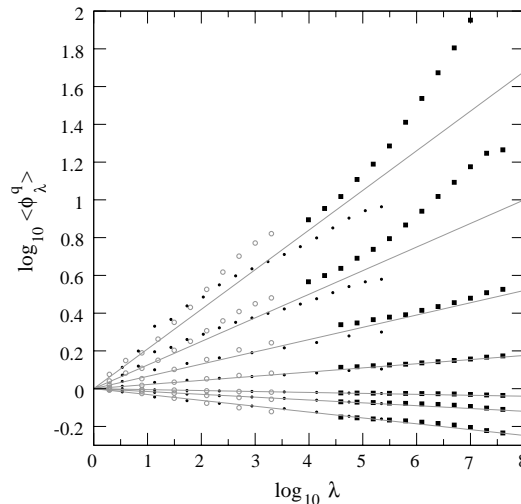


Fig. 2

Fig. 1 – Log/log plot of $E(k)$ vs. k for the four DEMs. From right to left: Lower Saxony, U.S. (in grey), GTOPO30 and ETOPO5. The reference line has slope -1.98 ($H = 0.60$ because the analysis is on continents, except ETOPO5).

Fig. 2 – Log/log plot of the normalized trace moments vs. $\lambda = L_{\text{global}}/l$ (with $L_{\text{global}} = 20000$ km, *i.e.* the largest great circle) for the three DEMs (empty grey circles correspond to ETOPO5, small black dots to U.S. and squares to Lower Saxony). The solid lines are for (top to bottom): $q = 2.18, 1.77, 1.44, 1.17, 0.04, 0.12$ and 0.51 . We did not show GTOPO30 because it is more or less a subset of the 90 m resolution data set and would make the graph unnecessarily crowded.

and a $3 \text{ km} \times 3 \text{ km}$ section of Lower Saxony that was constructed with the help of the HRSC-A (High Resolution Stereo Camera Airborne) [25]. For the analyses, we chose 1D transects in order to maximize the range of scales analyzed on these generally rectangular shaped data sets (see table I). In 1D, the power spectrum is the ensemble average of the modulus squared of the usual Fourier transform (here we average over transects). The ETOPO5 data has a fixed angular (rather than spatial) resolution, so in order to compare with other data sets, we analyzed only the central 500 (out of 2160) latitude bands (which have a near constant spatial resolution). Figure 1 shows the results of the power spectrum analysis over the four DEMs. The log/log plot of the spectral energy $E(k)$ vs. the wave number k gives straight lines over 6 orders of magnitude, which means that scaling is well respected from planetary scales down to a few meters. Even if there are scale breaks at the highest wave numbers on each data set (these are artifacts of inadequate dynamical range), it is still possible to say that the scaling is respected because each data set spans two (or more) orders in magnitude, the scale ranges overlap and the curves are *not* adjusted in the vertical. In fact, it seems to be just one straight, unbroken line spanning six orders of magnitude: this is quite different from the results of [27], in which 23 data sets are used to cover 8 orders of magnitude and where the curves are shifted more or less arbitrarily in the vertical [28]. The theoretical spectral exponent predicted by universal multifractals (with the parameters estimated below) is $\beta = 1.98$ (see fig. 1), quite close to the value $\beta \approx 2$ first noted by [11]. The only clear break appears at around 40 m in the Lower Saxony DEM, but the cause of this scale break is beyond our present scope.

Consider a surface described by the multifractal “Fractionally Integrated Flux” (FIF)

model [22]. The basic element is the flux ϕ , which (in analogy with turbulence) is a multifractal noise directly resulting from a multiplicative cascade process which on average conserves ϕ from one scale to the next. The observed height field is then obtained by fractional integration of order H , *i.e.* $h(\mathbf{x}) = \phi^* |\mathbf{x}|^{-(D-H)}$. Such a surface has height increments given by

$$\Delta h_\lambda = \lambda^{-H} \phi_\lambda. \quad (1)$$

The multifractal flux ϕ_λ varies with resolution λ as

$$\langle (\phi_\lambda)^q \rangle = \lambda^{K(q)}; \quad (2)$$

$K(q)$ is generally convex (in the special case where it is linear or zero, we have “monoscaling”). A classical example of the latter results is when ϕ is taken to be a Gaussian white noise, we obtain the monofractal fractional Brownian motion model [12] with $K(q) = 0$. From eqs. (1) and (2), the moments of the height increments are simply given by

$$\langle (\Delta h_\lambda)^q \rangle \sim \lambda^{-qH+K(q)}, \quad (3)$$

i.e. $\xi(q) = qH - K(q)$ [29], and we still have $E(\mathbf{k}) \sim |\mathbf{k}|^{-\beta}$ with $\beta = 1 + \xi(2)$.

In order to demonstrate and quantify the multiscaling, we used the trace moments method based on eq. (2), *i.e.* on the analysis of the moments of the flux ϕ at resolution λ . The flux ϕ is not directly observable; to obtain it from the height h , a fractional differentiation of order H or greater must be performed [24]. In one dimension, the standard numerical approximation to this is to take the modulus of the finite difference gradient [18], which corresponds to a differentiation of order $H = 1$; this method (and minor variants) are quite standard in the multifractal turbulence literature [30]. Once ϕ has been obtained, we normalize (and nondimensionalize) it by dividing by the spatial mean so that $\langle \phi_\lambda \rangle = 1$; we then degrade the resolution by spatial averaging to scale $l = L/\lambda$, take the q -th moment and average over the data set to obtain $\langle \phi_\lambda^q \rangle$.

A log/log plot of these normalized trace moments *vs.* λ for three DEMs can be seen in fig. 2. Each line corresponds to a different q and is composed of the trace moments from the three different DEMs. First, to compare results, we nondimensionalize the lengths by dividing by an arbitrary reference length taken for convenience as the largest great circle, $L_{\text{global}} = 20000$ km. In comparing the different data sets, we first note that they are not independent of each other: each regional data set is simply a higher-resolution subsample of the low-resolution global data set. While clearly each individual regional data set has no variability at its largest scale (its nondimensional spatial mean at its largest scale is unity), this “missing variability” can be determined from the global data set using eq. (2). To demonstrate this, we can use the multifractal factorization property which follows from eq. (2). Let $\lambda_{\text{large}} = (\text{planetary scale})/(\text{intermediate data set scale})$ be the ratio of the missing range of scales. If $\lambda_{\text{small}} = (\text{intermediate data set scale})/(\text{pixel scale})$, then since $\lambda_{\text{tot}} = \lambda_{\text{large}} \lambda_{\text{small}} = (\text{planetary scale})/(\text{pixel scale})$, from eq. (2) we get

$$\langle \phi_{\lambda_{\text{total}}}^q \rangle = \langle \phi_{\lambda_{\text{large}}}^q \rangle \langle \phi_{\lambda_{\text{small}}}^q \rangle. \quad (4)$$

For each data set, we directly obtained $\langle \phi_{\lambda_{\text{small}}}^q \rangle$ by analysis, and estimated $\langle \phi_{\lambda_{\text{large}}}^q \rangle$ from $\lambda_{\text{large}}^{K(q)}$. Note that this adjustment of all the trace moments is a necessary consequence of the multiscaling: it is uniquely determined by the known range of scales and the exponent $K(q)$ (itself determined by two parameters). Figure 2 shows that —as for the power spectra— the trace moments are fairly linear down to roughly 40 m. The slight oscillations could in

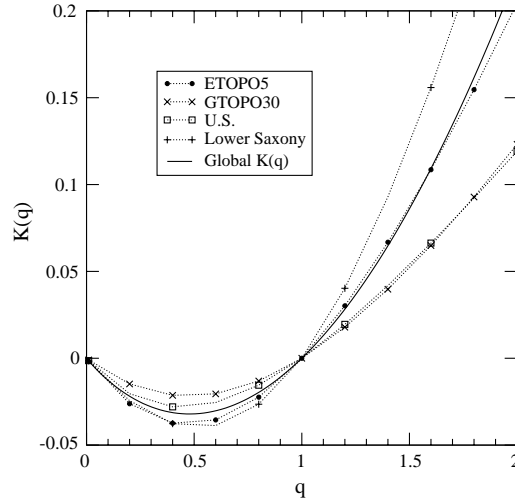


Fig. 3 – Plot of the moment scaling function $K(q)$ as a function of the moment q for the four DEMs (black circles correspond to ETOPO5, \times 's to GTOPO30, empty squares to U.S. and $+$'s to Lower Saxony). The solid line corresponds to the global $K(q)$ with parameters $\alpha = 1.75$ and $C_1 = 0.12$.

principle be explained by anisotropy using Generalized Scale Invariance [19, 22]. Below, after estimating a theoretical $K(q)$, we quantify the magnitude of the small deviations.

From the log-log slopes of the trace moments, we calculate the $K(q)$ functions of each data set (see eq. (2) and fig. 3). Without further knowledge or theory, the task of intercomparing the different data sets would be unmanageable, since a purely empirical determination of $K(q)$ would require an infinite number of slopes (one for each q). Fortunately, multifractal processes admit universality classes [22] (see the debate in [23]); this is essentially the multiplicative analogue of the (generalized) central-limit theorem. In the absence of other information, it is therefore logical to first postulate these universal forms. The corresponding universal $K(q)$ function is

$$K(q) = \frac{C_1}{\alpha - 1} (q^\alpha - q), \tag{5}$$

where $0 \leq \alpha \leq 2$ is the degree of multifractality (0 is the monofractal case and 2 is the lognormal case) and C_1 is the codimension of the mean of the process.

Using eq. (5) to fit the $K(q)$ over the interval $0 < q < 2$ (where the moments are most accurate [31]), we obtain the values α and C_1 ; using $K(2)$ and β , we then obtain H (see table I). The values of H obtained are consistent with those expected for continents (except for ETOPO5, which is a mix of continents and oceans and is a little higher); we discuss possible regional variations in [26]. Here we concentrate on the values of α and C_1 , which are sufficiently close that they may be the result of a unique (global) multifractal process. At first sight, the possibility of a single global multifractal process for such (apparently) disparate morphologies may seem surprising. However, we should recall that even if such a process were isotropic (self-similar), it would still imply huge intermittency with local regions being dominated by singularities of random order. This is already enough to account for the coexistence of smooth and rough regions at all scales; indeed, numerical simulations [19] show that one automatically obtains “smooth” regions and rough high regions, in qualitative agreement with the morphologies of oceans and continents. Furthermore, it may be that only the flux ϕ (rather than the topography h itself) is the result of a global process: this is independent

of H and hence less sensitive to gradients. However, perhaps the most important cause of morphological variation is that of anisotropy. By degrading ϕ over boxes which have the same shape at all scales (the boxes are related by isotropic scale change), we have washed out most of the effects of anisotropy. Numerical simulations of anisotropic multifractals [19] show that even the simplest (linear approximation) to anisotropic scale invariance can readily lead to highly diverse morphologies. We conclude that the possibility of a single global multifractal ϕ (or even h) is thus theoretically credible and, as we will see below, compatible with the data.

Since a global ϕ is the simplest hypothesis, we now examine its consequences. To do this, we estimate the global $K(q)$ by taking the average of the parameters for the different data sets [32] ($\alpha = 1.75 \pm 0.17$ and $C_1 = 0.12 \pm 0.04$, where the uncertainties are equal to one standard deviation) and plot the corresponding $K(q)$ (see fig. 3). This $K(q)$ was used to estimate the “missing” large-scale variability and to compute the reference lines in fig. 2. To quantify the statistical accuracy of the postulated global universal ϕ process determined by these α , C_1 values (assuming that the outer scale is the largest scale available, 20000 km), we calculated the deviation Δ between theory and data, given by $\Delta = |\log_{10}(\langle(\phi_\lambda)^q\rangle)^{1/q} - \log_{10}(\lambda^{K(q)})^{1/q}|$, where $K(q)$ is determined theoretically and we average over $0 < q < 2$ and the scale range 20000 km to 40 m. The overall result $\Delta = \pm 0.16$ corresponds to $\pm 45\%$ in the statistical moment; given the wide range of scales and the crude estimate of only two parameters, we feel that this makes the hypothesis of a unique global multifractal process for ϕ quite plausible.

By studying four large topography data sets, we have shown that the required scaling is multi-, not monofractal ($K(q)$ is nonlinear, $\alpha \neq 0$) over a range going from planetary scales down to around 40 m and that the (isotropic) statistics of the data can be reasonably well described by three universal multifractal parameters. The fact that the slope of $K'(1) = C_1$ is much smaller than H shows that a linear $\xi(q)$ (*e.g.*, simple scaling) may be a workable approximation around the mean but the large value of α indicates that it rapidly gets worse for extremes (*i.e.* large q). This shows the importance of using multifractals instead of monofractals to describe the true variability of a surface. The fracture surface estimates of [4] ($\alpha \approx 1.50$, $C_1 \approx 0.30$, $H \approx 0.8$) imply that the same qualitative conclusion applies to rough surfaces. A basic outstanding problem is the anisotropy which in general will vary from scale to scale and from place to place. The above analyses sidestep this issue by the use of isotropic statistics (*e.g.*, the resolution λ was defined isotropically by coarse graining over squares at all scales). In order to fully characterize surface morphology, Generalized Scale Invariance [19, 22] must be used: in principle, an entire vector field (the infinitesimal generator which defines the notion of scale) is needed to specify the anisotropy. The corresponding anisotropic analysis and modeling methods (see, for example, [33, 34]) open promising new avenues toward a complete statistical description of topography and other rough surfaces.

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- [30] In many data sets, including ours, the dynamical range was inadequate; many gradients are spuriously zero due to roundoff in the altitudes (*i.e.* many areas are artificially flat). In order to partially overcome this problem, a fractional integration of order 0.1 (a scale-invariant smoothing that changes H but does not affect $K(q)$) was used before taking the absolute differences.
- [31] This is inferred from the transition from nonlinear to linear regime in the $K(q)$ function, where there is a sample size determined multifractal phase transition [21] (here, this occurs near $q \approx 3$), so we restrict ourselves to $0 \leq \alpha \leq 2$.
- [32] Here we want the global $K(q)$, which is a characterization of the Earth's topography at all scales and intensities. So it is legitimate, as a first guess, to average the parameters of our four data sets, because they sample topography at all scales, which will give us a first approximation to the global $K(q)$.
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