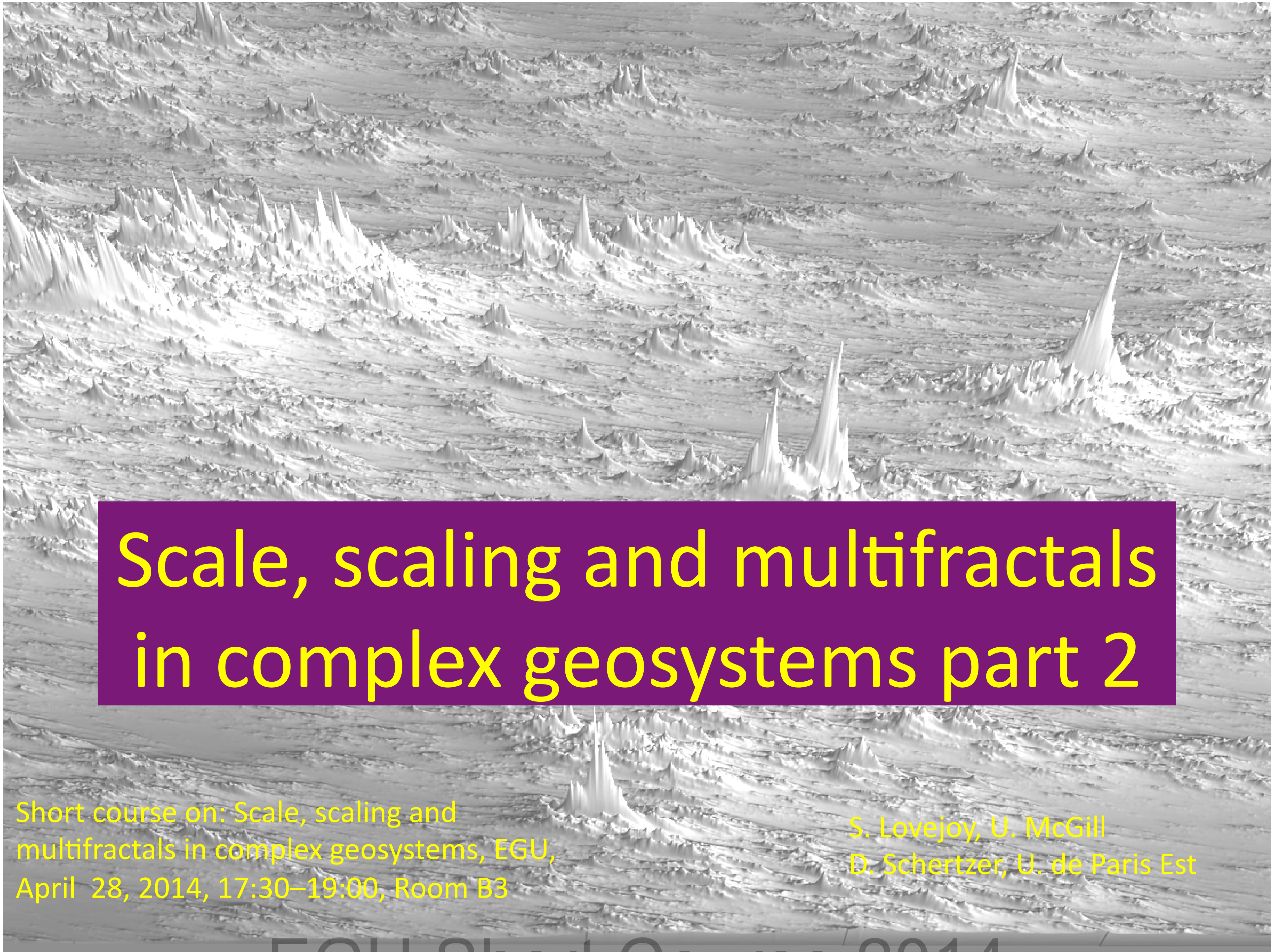


Scale, scaling and multifractals in complex geosystems part 2

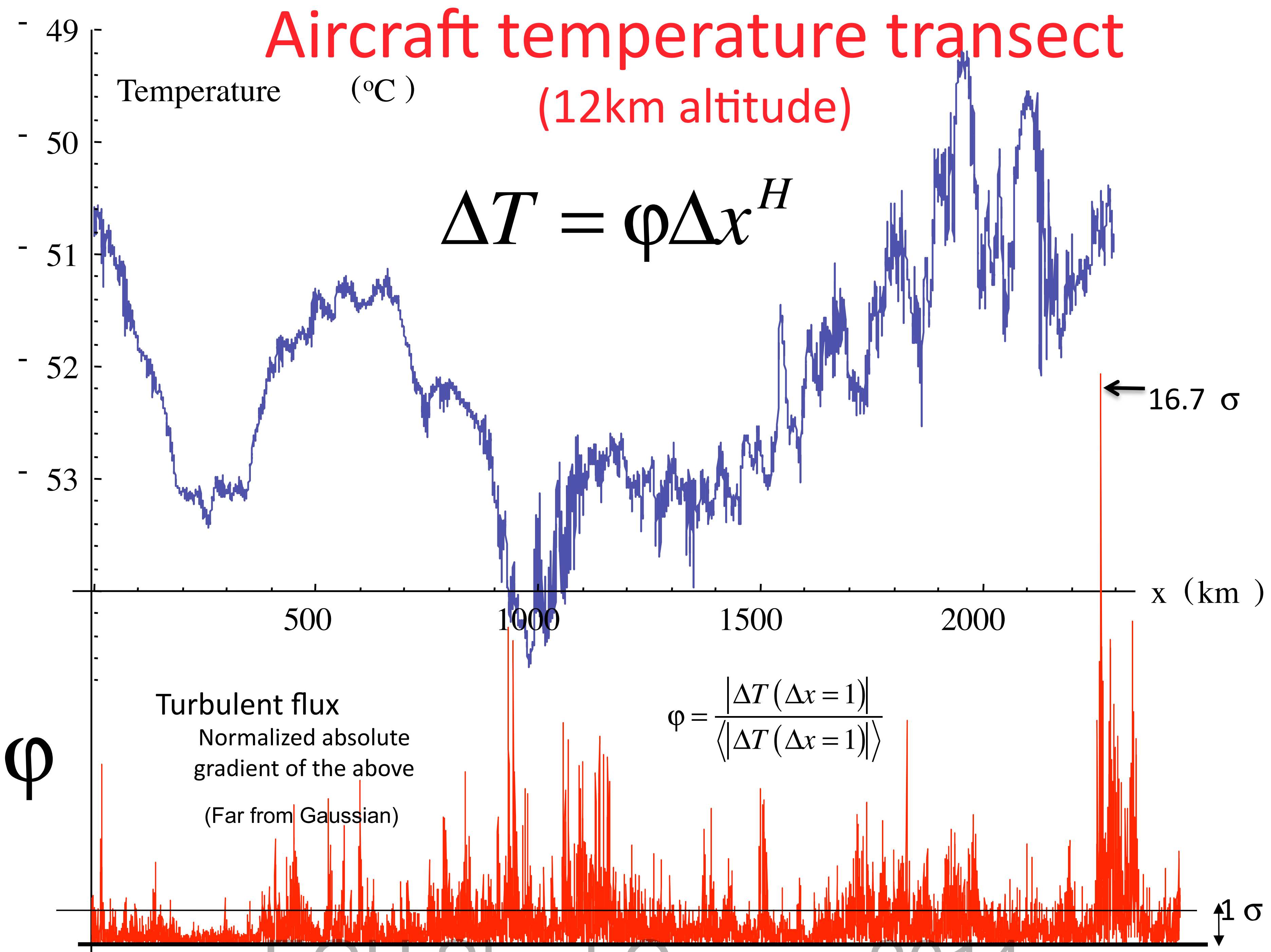
Short course on: Scale, scaling and multifractals in complex geosystems, EGU, April 28, 2014, 17:30–19:00, Room B3

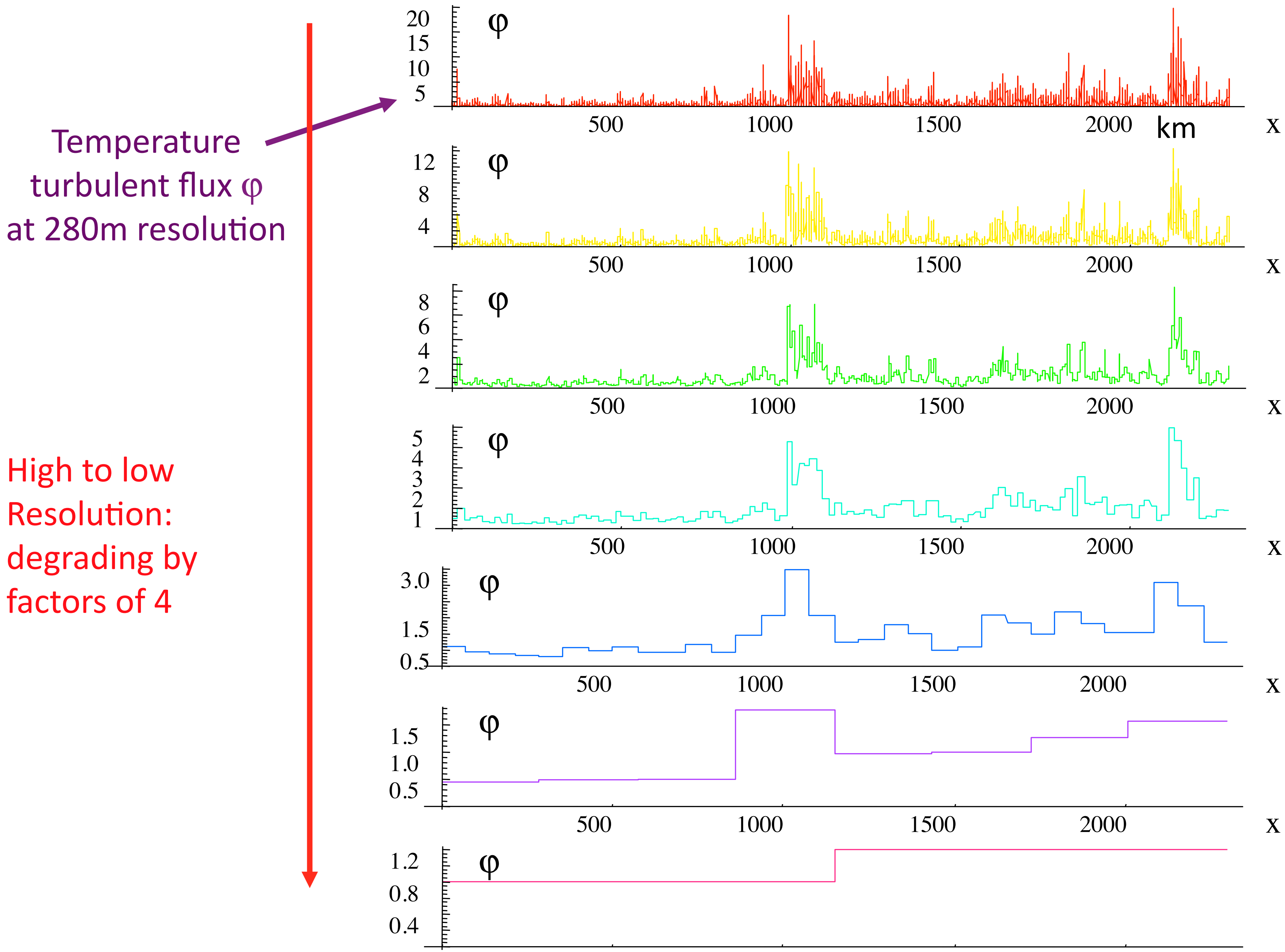
S. Lovejoy, U. McGill
D. Schertzer, U. de Paris Est

The unity of clouds and rocks:

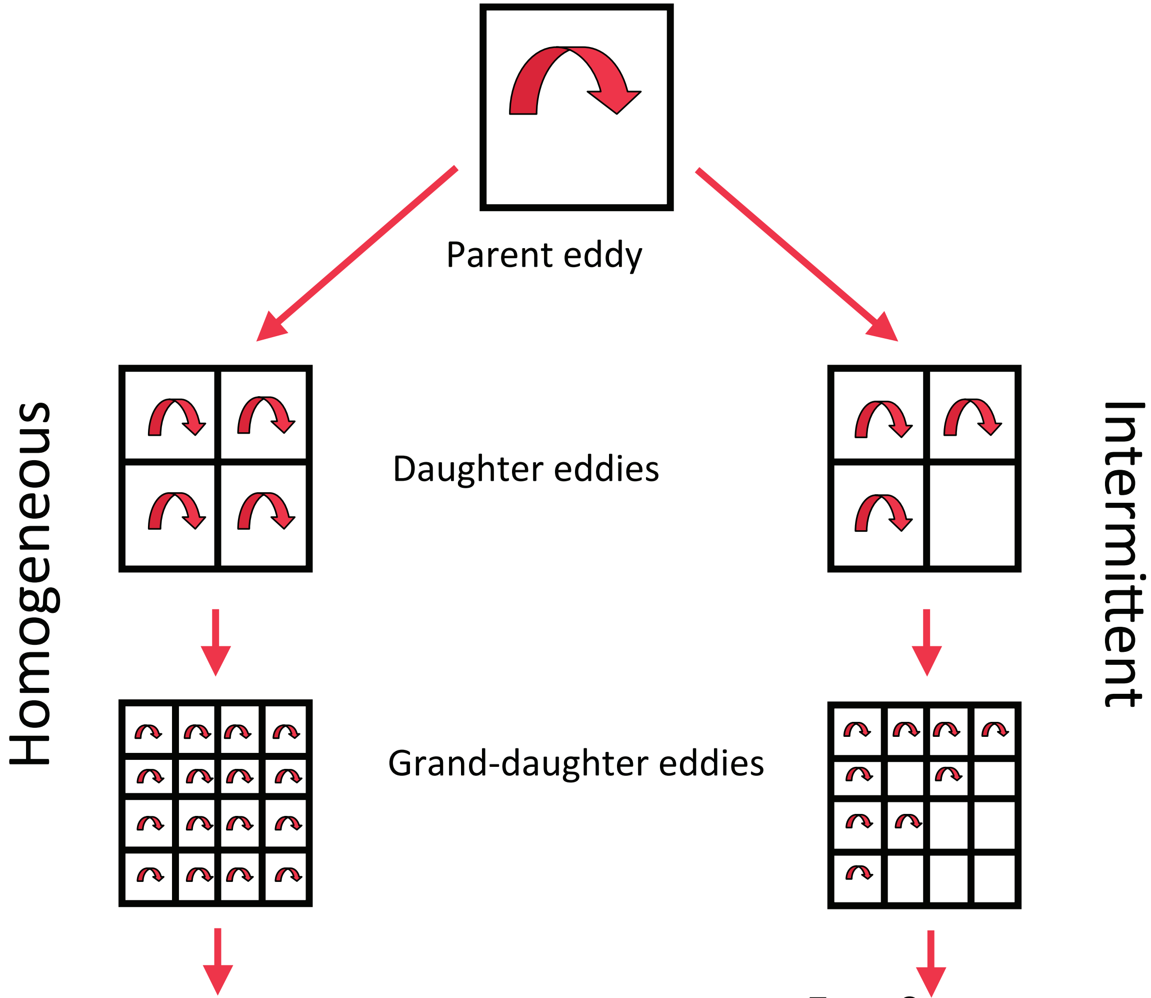
Multifractality

Multifractal simulation


Aircraft temperature transect
(12km altitude)
Temperature (°C)
49
50
51
52
53
$\Delta T = \varphi \Delta x^H$
16.7 σ
x (km)
500
1000
1500
2000
Turbulent flux
Normalized absolute
gradient of the above
(Far from Gaussian)
φ
$\varphi = \frac{|\Delta T(\Delta x = 1)|}{\langle |\Delta T(\Delta x = 1)| \rangle}$
1 σ


Temperature turbulent flux φ at 280m resolution
High to low Resolution: degrading by factors of 4
φ
km
x
500
1000
1500
2000
20
15
10
5
12
8
4
6
2
3
1
3.0
1.5
0.5
1.0
1.2
0.8
0.4

Cascades

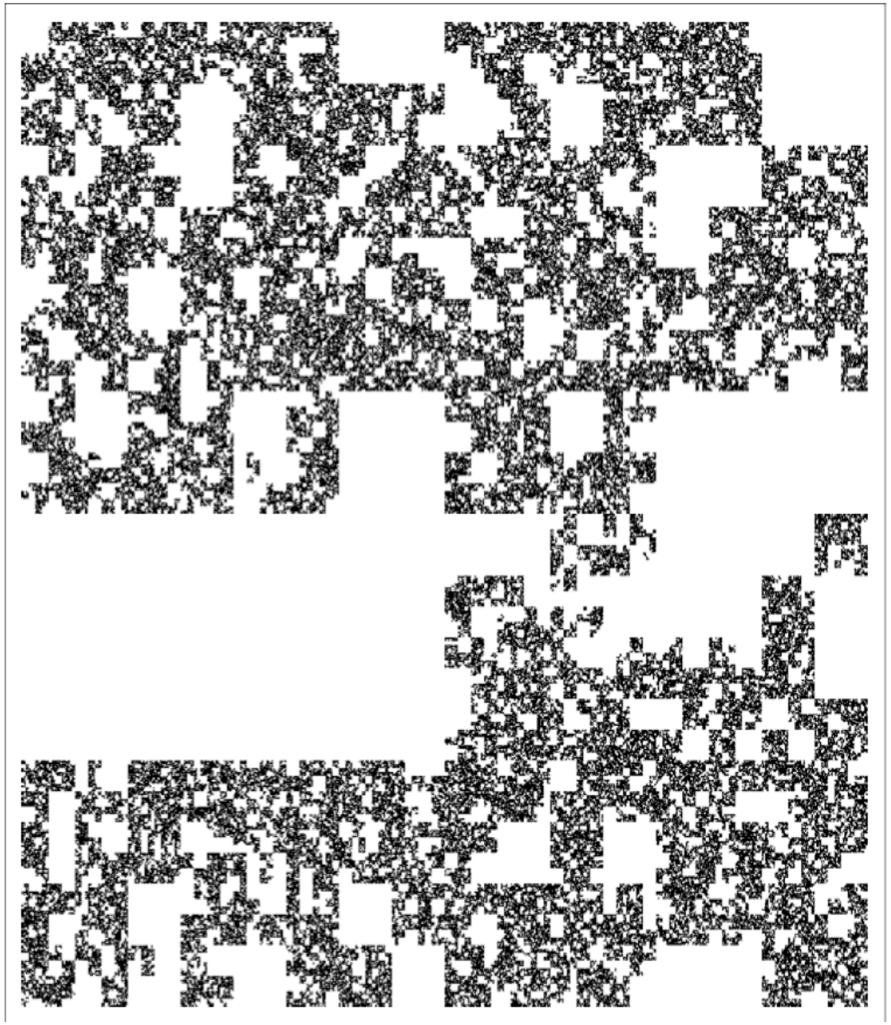


$$D = \frac{Log4}{Log2} = 2$$

$$D = \frac{Log3}{Log2} = 1.58...$$

Beta model

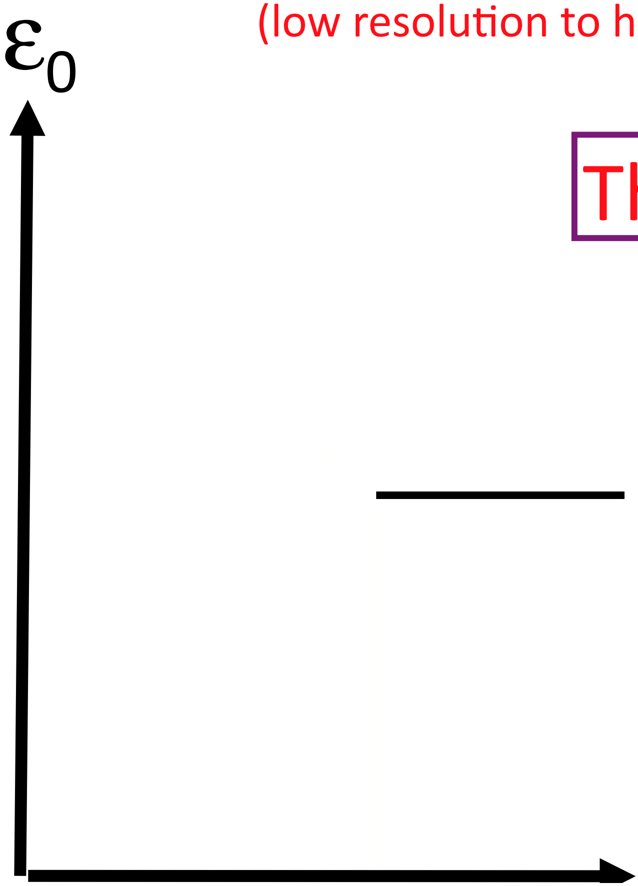
An initial attempt to handle intermittency reduces it to the simple notion of "on/off" intermittency, i.e. a cascade with the simple alternative alive/dead of the offspring.



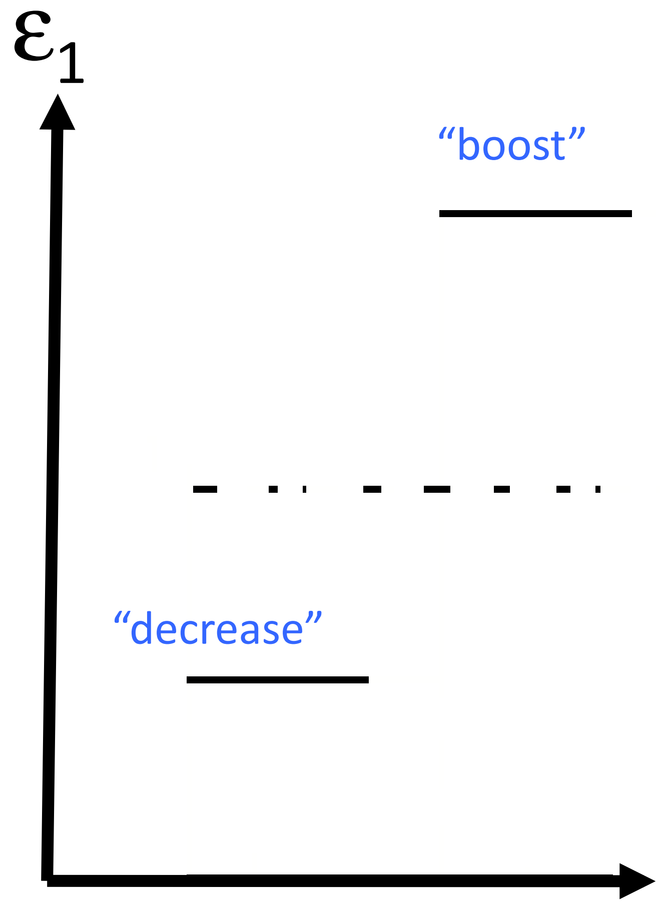
In this example, the probability that an eddy will remain alive is λ_0^{-C} =0.87 (using the scale ratio at each step λ_0 = 4 here and the codimension C = 0.2).

Cascades and Multifractals

Simulations: **multiplicative** introduction of small scale details (low resolution to high)



The α model



S+L 1983

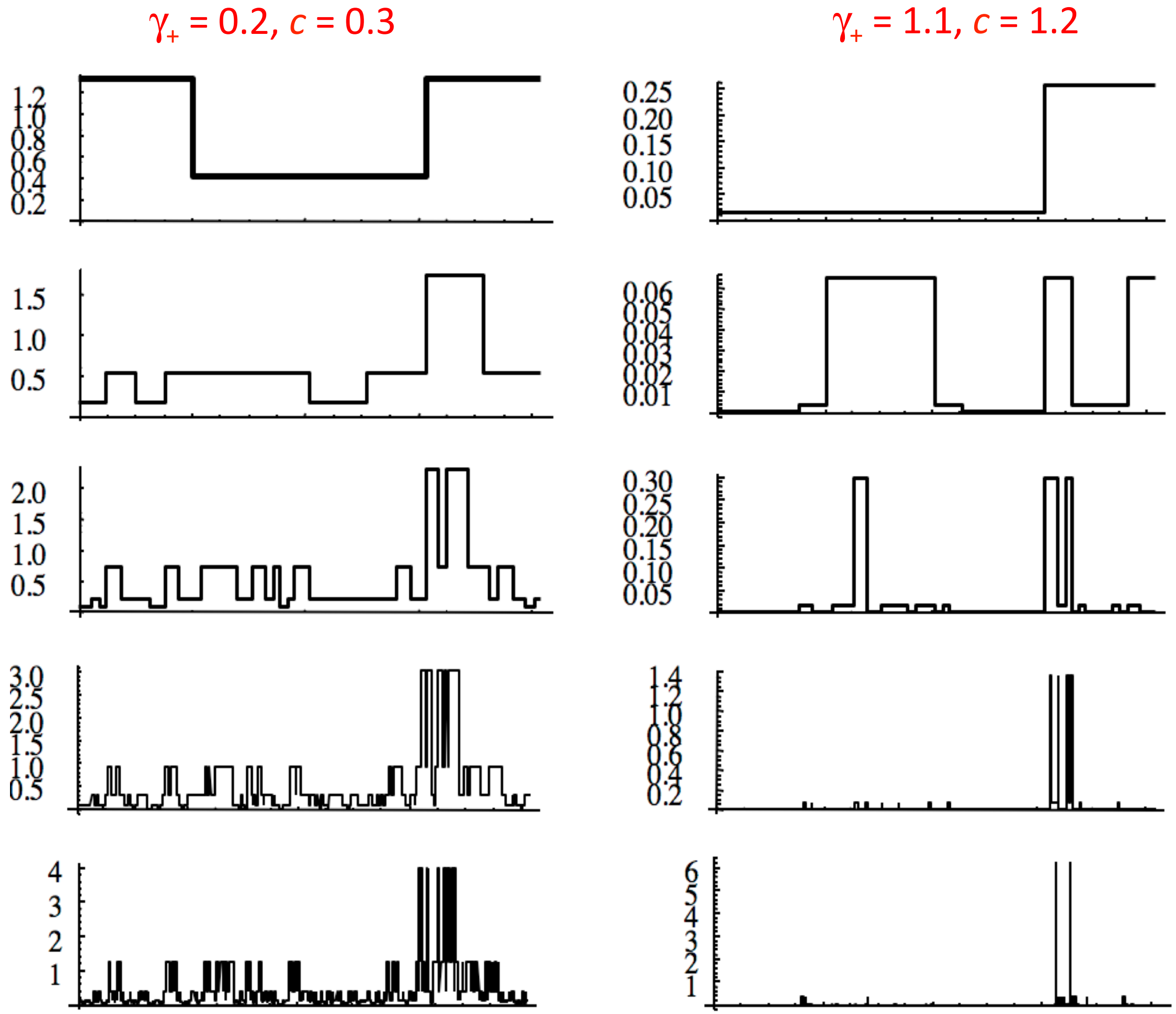
<table>
<tr><td colspan="8">+</td></tr>
<tr><td colspan="4">+</td><td colspan="4">−</td></tr>
<tr><td colspan="2">+</td><td colspan="2">−</td><td colspan="2">−</td><td colspan="2">−</td></tr>
<tr><td>−</td><td>+</td><td>+</td><td>+</td><td>−</td><td>+</td><td>+</td><td>−</td></tr>
</table>

The α model is a two state (binomial) process with $\mu\varepsilon$ = either $\lambda_0^{\gamma+}$ or $\lambda_0^{\gamma-}$ where $\gamma_+>0$ corresponds to a boost ($\mu\varepsilon>1$) and γ_- to a decrease ($\mu\varepsilon<1$) .

$$\Pr\left(\mu\varepsilon=\lambda_0^{\gamma_+}\right)=\lambda_0^{-c}$$

$$\Pr\left(\mu\varepsilon=\lambda_0^{\gamma_-}\right)=1-\lambda_0^{-c}$$

Alpha model



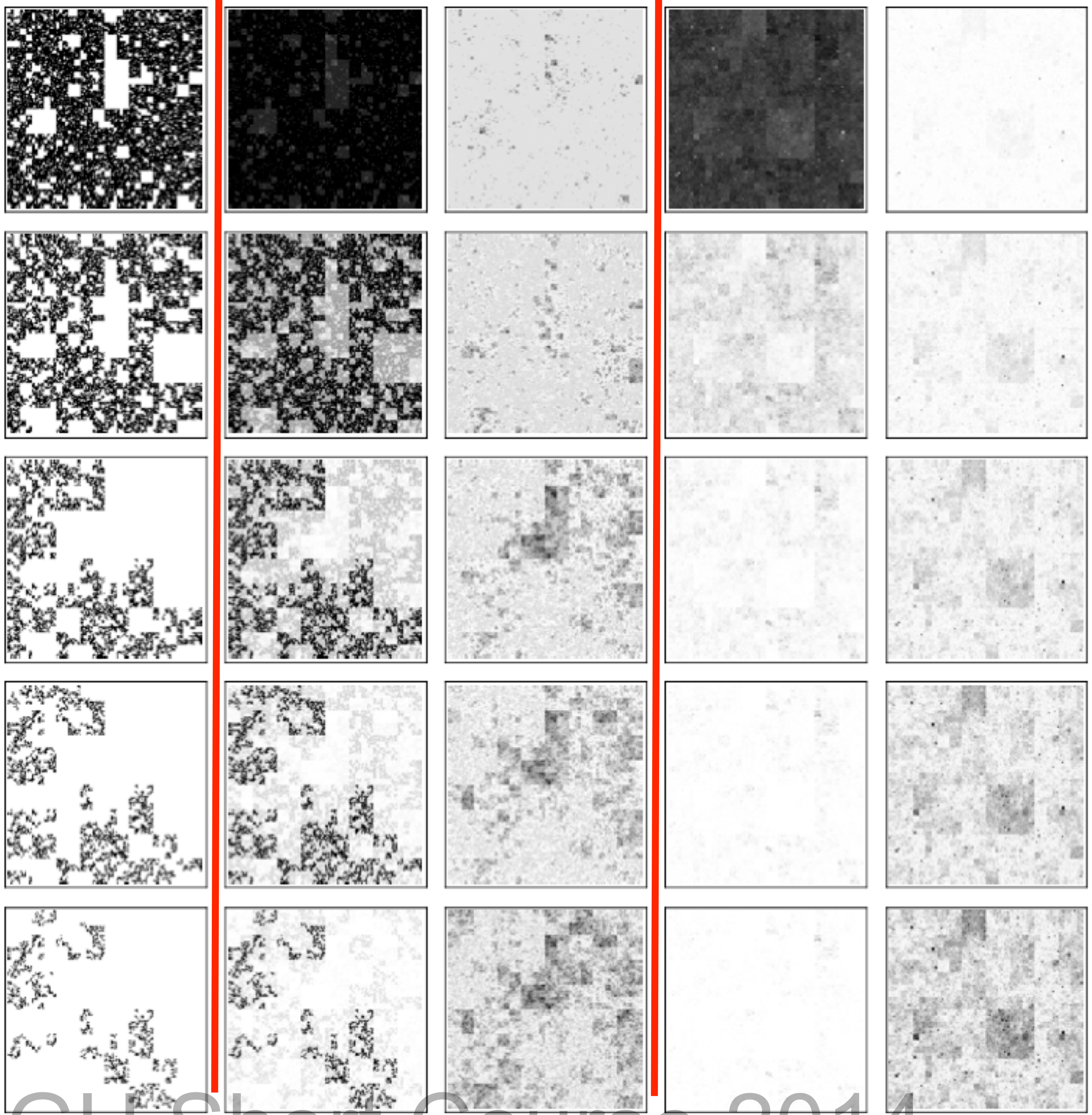
The α model is a two state (binomial) process with $\mu\varepsilon$ = either $\lambda_0^{\gamma_+}$ or $\lambda_0^{\gamma_-}$ where $\gamma_+>0$ corresponds to a boost ($\mu\varepsilon>1$) and γ_- to a decrease ($\mu\varepsilon<1$) .

$$\Pr\left(\mu\varepsilon=\lambda_0^{\gamma_+}\right)=\lambda_0^{-c}$$

$$\Pr\left(\mu\varepsilon=\lambda_0^{\gamma_-}\right)=1-\lambda_0^{-c}$$

From top to bottom every second cascade step is shown (a factor of λ_0^2) is shown, 10 steps in all, the total range of scales is 2^{10} = 1024). Notice the changing vertical scales

2-D Alpha model



Multiplicative Cascades

Generic statistical behaviour:

Statistical Moments:

scaling

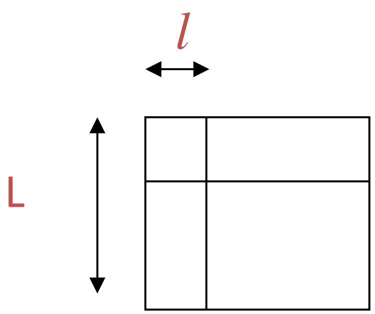
Turbulent flux

Scale invariant

$$\left\langle \varepsilon_{\lambda}^{q} \right\rangle \approx \lambda^{K(q)}$$

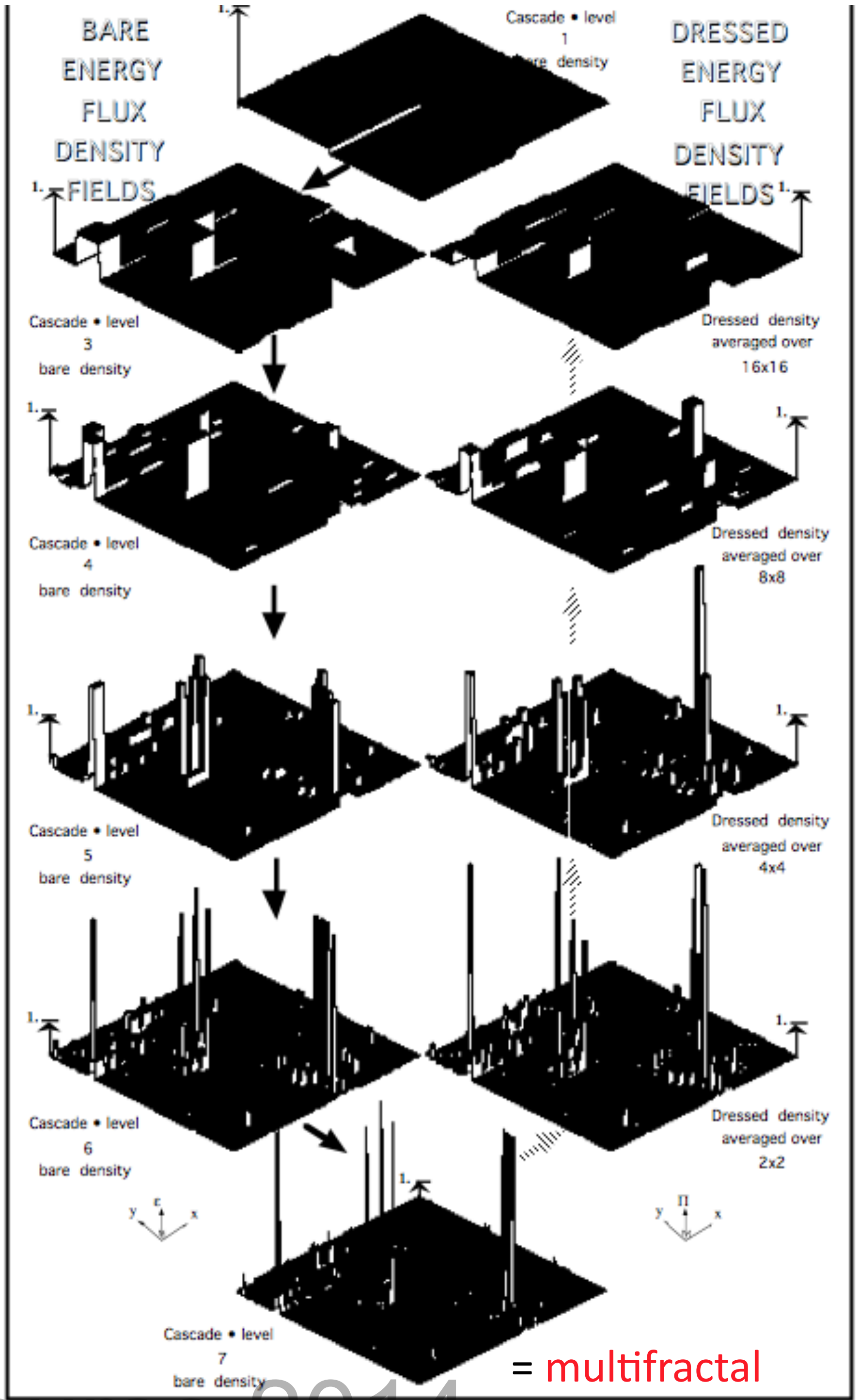
Statistical averaging

Resolution: ratio $\lambda = L/l$



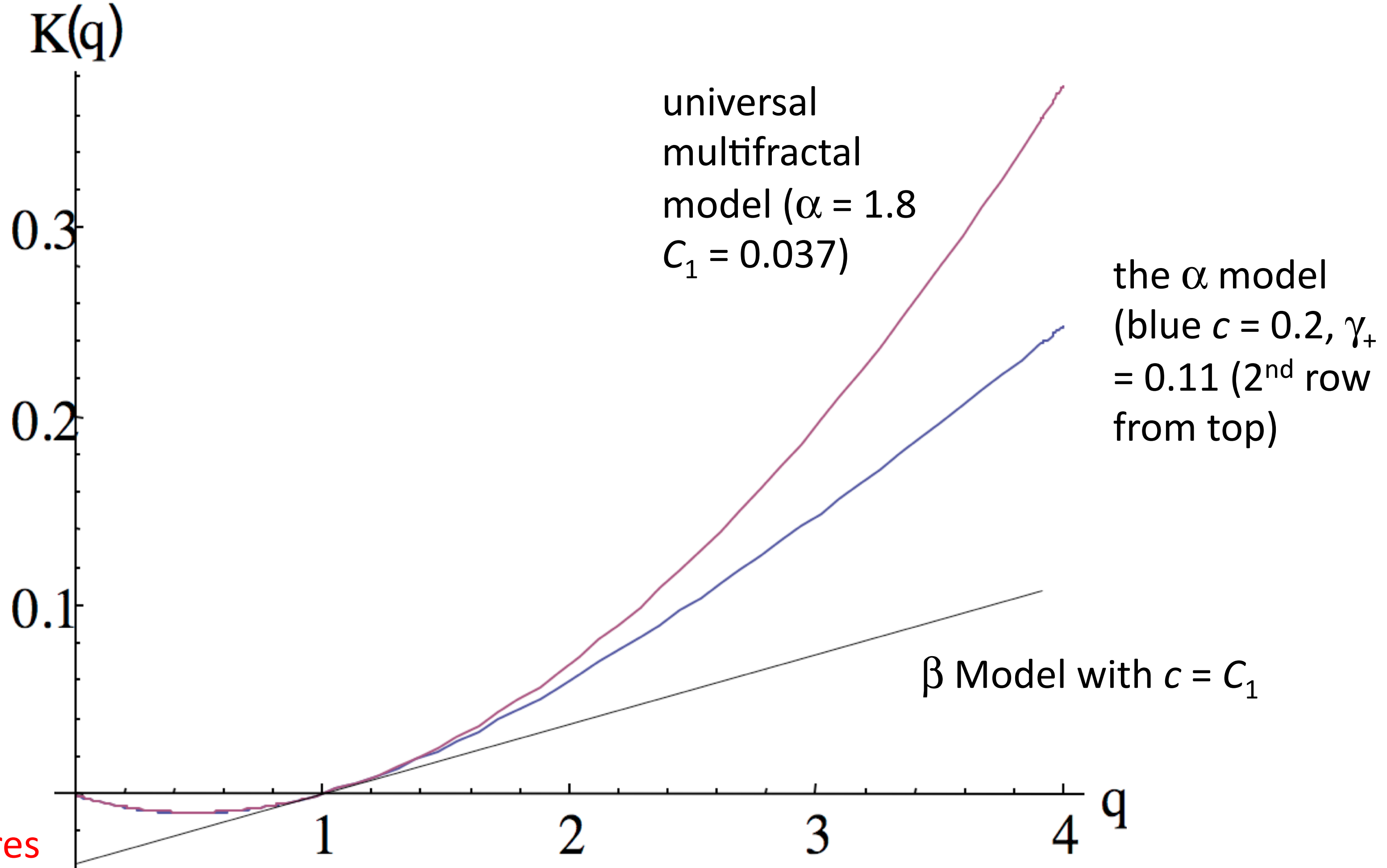
Probabilities:

$$\Pr\left(\varepsilon_{\lambda} > \lambda^{\gamma}\right) \approx \lambda^{-c(\gamma)}$$



Characterizing K(q): parametrisation near the mean

$\langle \varepsilon_\lambda^q \rangle \approx \lambda^{K(q)}$



General features
K(q) convex
K(1)=0

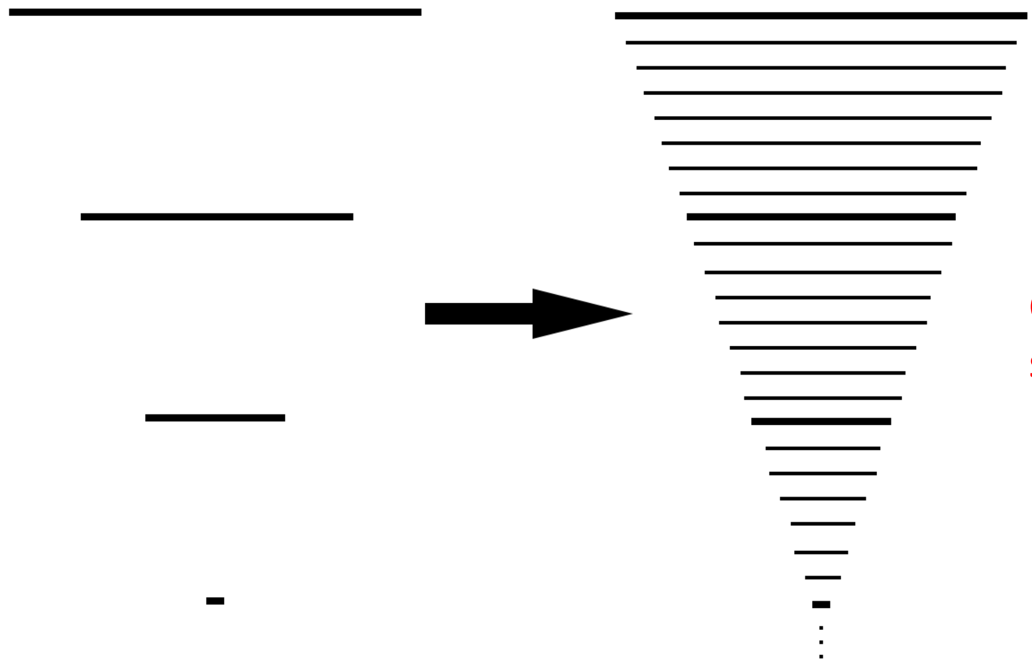
$C_1=K'(1)$
$\alpha = K''(1)/K'(1)$

Characterization near the mean: q=1

Characterizing K(q): universality

Route 1) Densification of scales

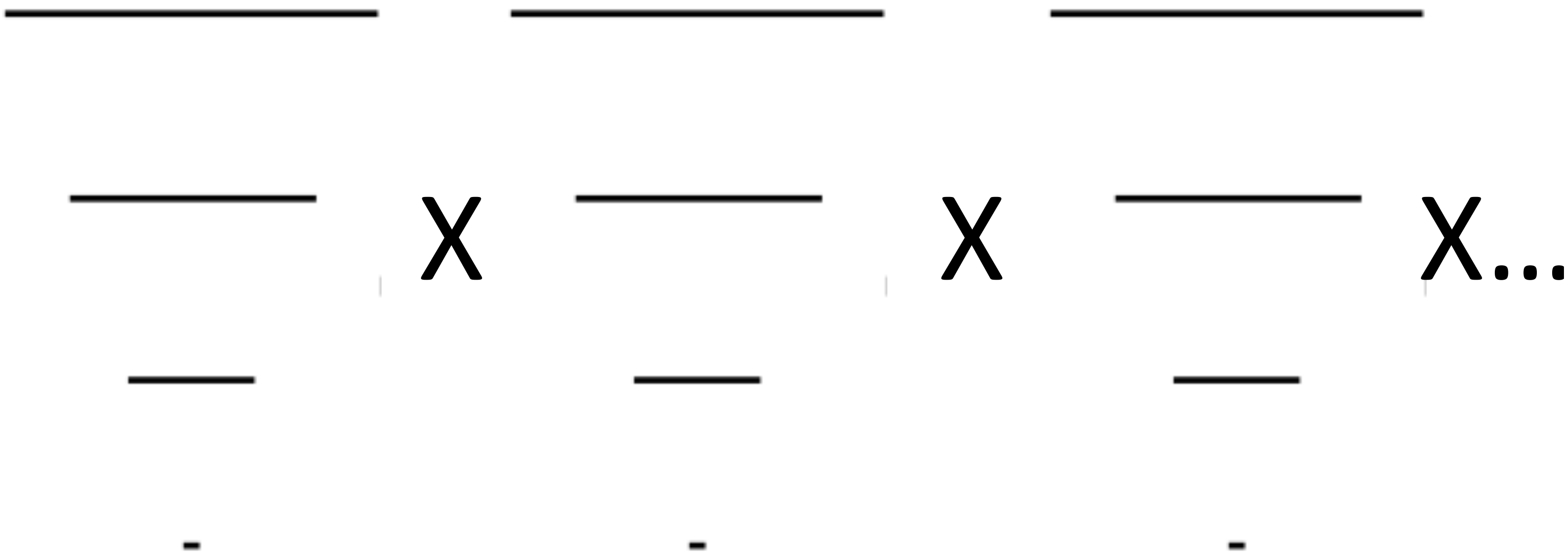
Discrete in scale
(ex. β, α models)



Continuous in scale

Characterizing K(q): universality

Route 2) “Mixing” of independent discrete cascades



$\varepsilon_1 \times \varepsilon_2 \times \varepsilon_3 \times$

$\Gamma=\log\varepsilon \quad \Gamma_1 + \Gamma_2 + \Gamma_3 +$

Universal Multifractals

"Multiplicative central limit theorem"

Hence:

$$K(q)=\frac{C_1}{\alpha-1}\left(q^\alpha-q\right);\quad 0\le\alpha\le 2$$

Codimension of mean

$K(q)=C_1 q\log q$ $(\alpha= 1)$

Levy index of generator

$$\left\langle \varepsilon_\lambda^q \right\rangle \to \infty$$ For $\alpha<2$, and $q<0$

Note:

$C_1=K'(1)$

$\alpha = K''(1)/K'(1)$ ← Characterisation of all statistics

Data Analysis

Fluctuation statistics and structure functions

The space-time variability of natural systems, can often be broken up into various "scaling ranges" over which the fluctuations vary in a power law manner with respect to scale. Over these ranges, the fluctuations follow

$$\Delta v = \varphi_{\Delta x} \Delta x^{H}$$

The flux at resolution Δx

Using Fluctuations:

$$S_q(\Delta x) = \left\langle \Delta v(\Delta x)^q \right\rangle = \left\langle \varphi_{\Delta x}^q \right\rangle \Delta x^{qH} \approx \Delta x^{\xi(q)}; \quad \left\langle \varphi_{\Delta x}^q \right\rangle = \left(\frac{L}{\Delta x} \right)^{K(q)}; \quad \xi(q) = qH - K(q)$$

(generalized, qth order) Structure function

Hence, we seek H, K(q)

With universality: $$K(q) = \frac{C_1}{\alpha - 1}\left(q^{\alpha} - q\right)$$ i.e. we seek H, C_1, α

Empirical analysis: Estimating fluxes from the fluctuations

Multifractal cascade equation: $\langle \varphi_\lambda^q \rangle = \lambda^{K(q)}$

Fluctuations: $\Delta I = \varphi_{\Delta x} \Delta x^H$

Estimating the fluxes from the fluctuations

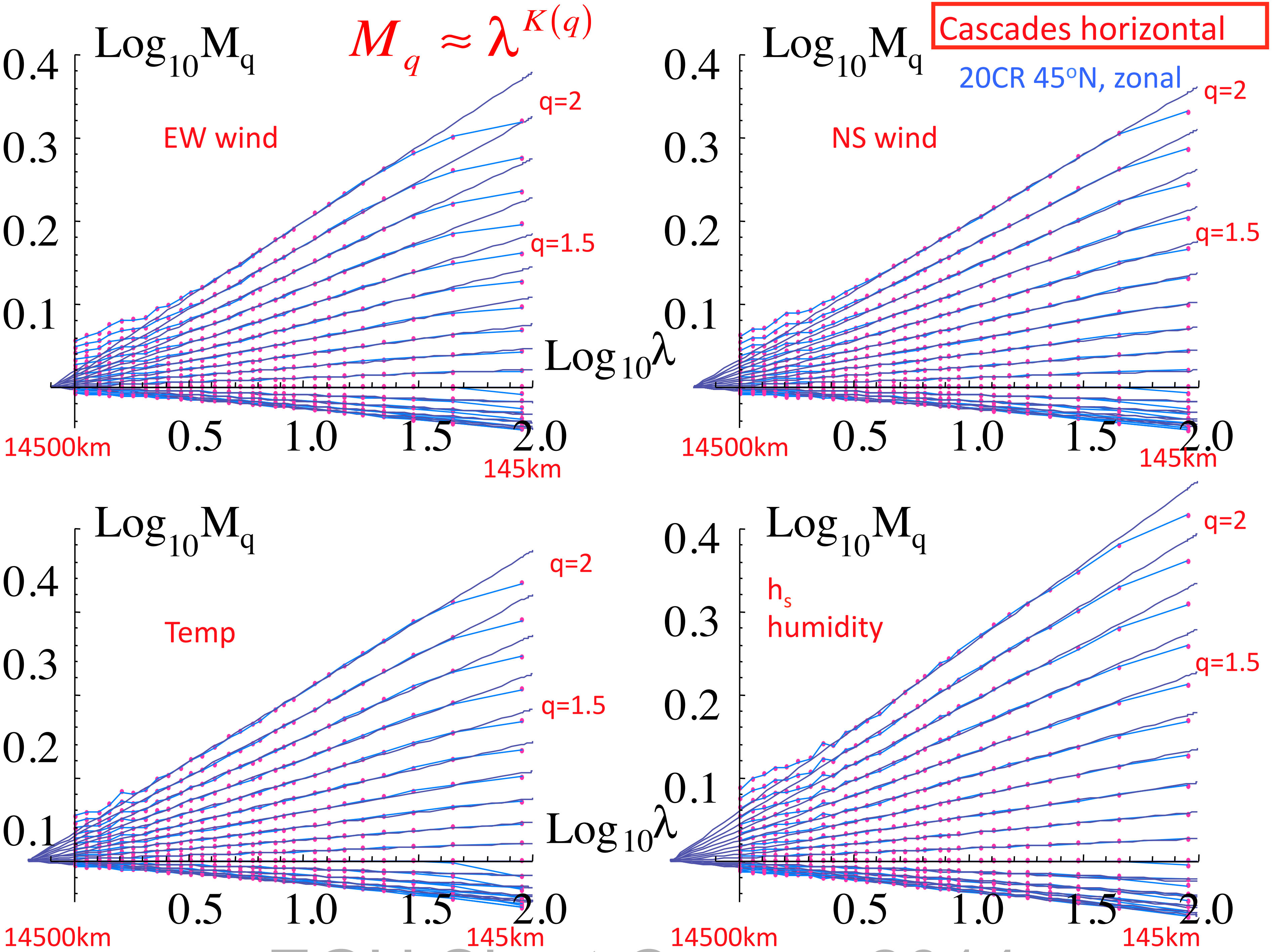
$$\varphi'_\lambda = \frac{\varphi_\lambda}{\langle \varphi_\lambda \rangle} \approx \frac{\Delta I(\Delta x)}{\langle \Delta I(\Delta x) \rangle}; \quad \lambda = \frac{L}{\Delta x}$$

outer cascade scale

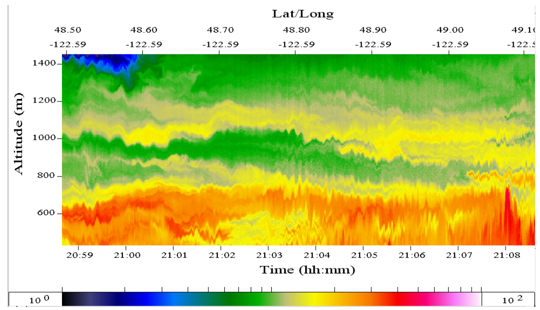
Normalized flux at resolution λ

$$M_q = \langle \varphi'^q_\lambda \rangle$$

Estimate at finest resolution, then degrade to intermediate resolutions by averaging

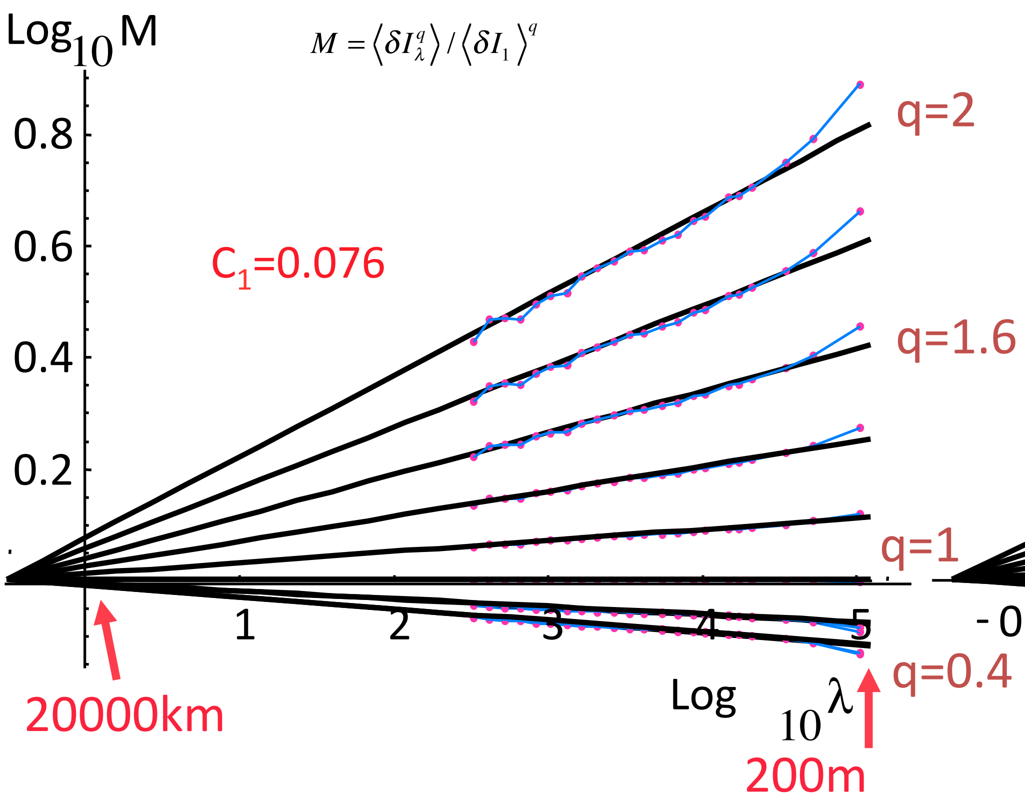

$M_q \approx \lambda^{K(q)}$
Cascades horizontal
20CR 45°N, zonal
Log10Mq
EW wind
NS wind
Temp
hs humidity
q=2
q=1.5
Log10λ
0.1
0.2
0.3
0.4
0.5
1.0
1.5
2.0
14500km
145km

Vertical cascades:
lidar backscatter

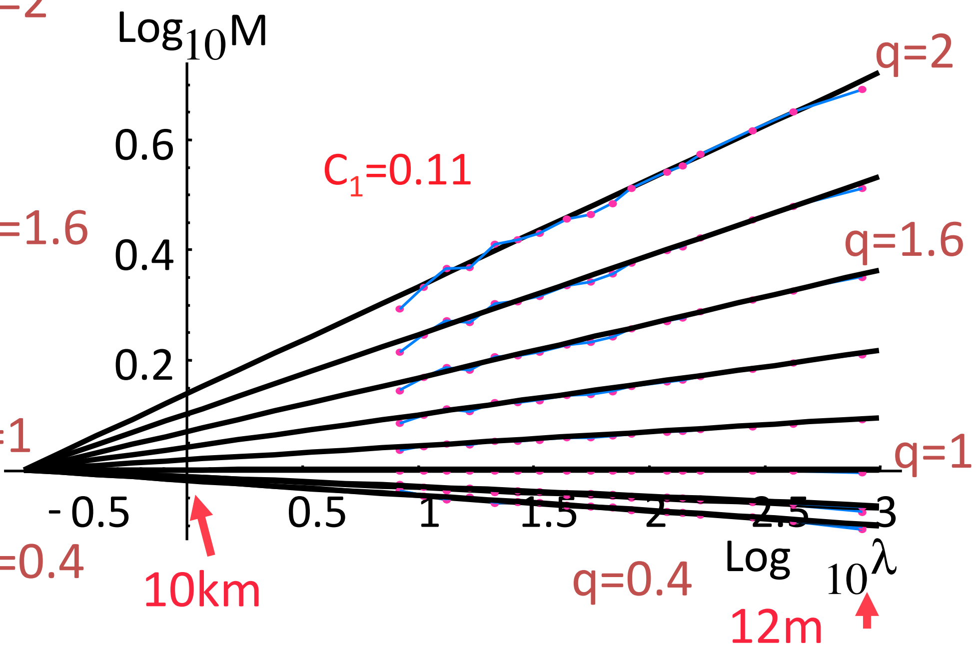


From 10 airborne lidar cross-sections near Vancouver B.C.

Horizontal cascade



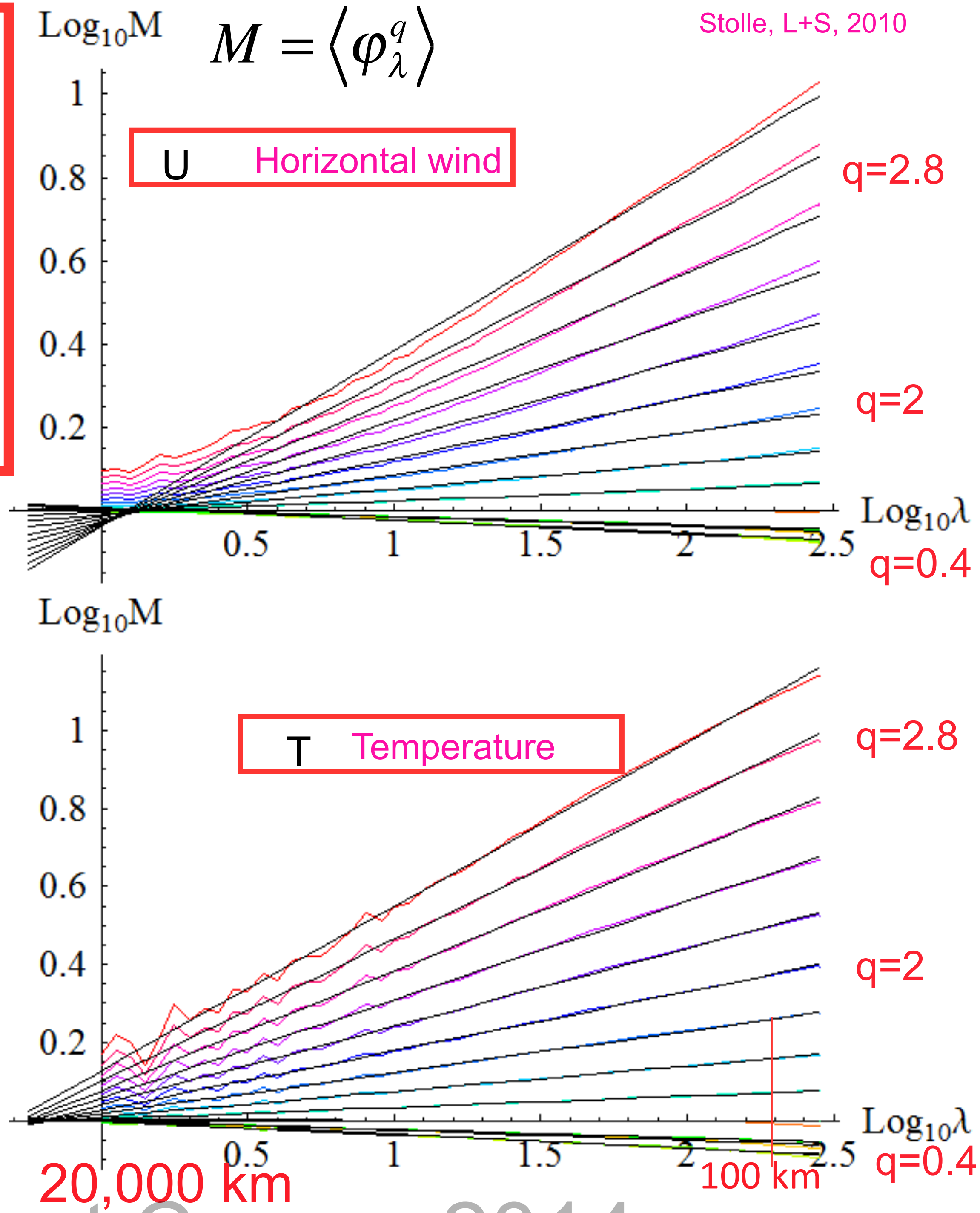
Vertical cascade



q=0, 0.2, 0.4..., 2

Global GEMS Model 00h

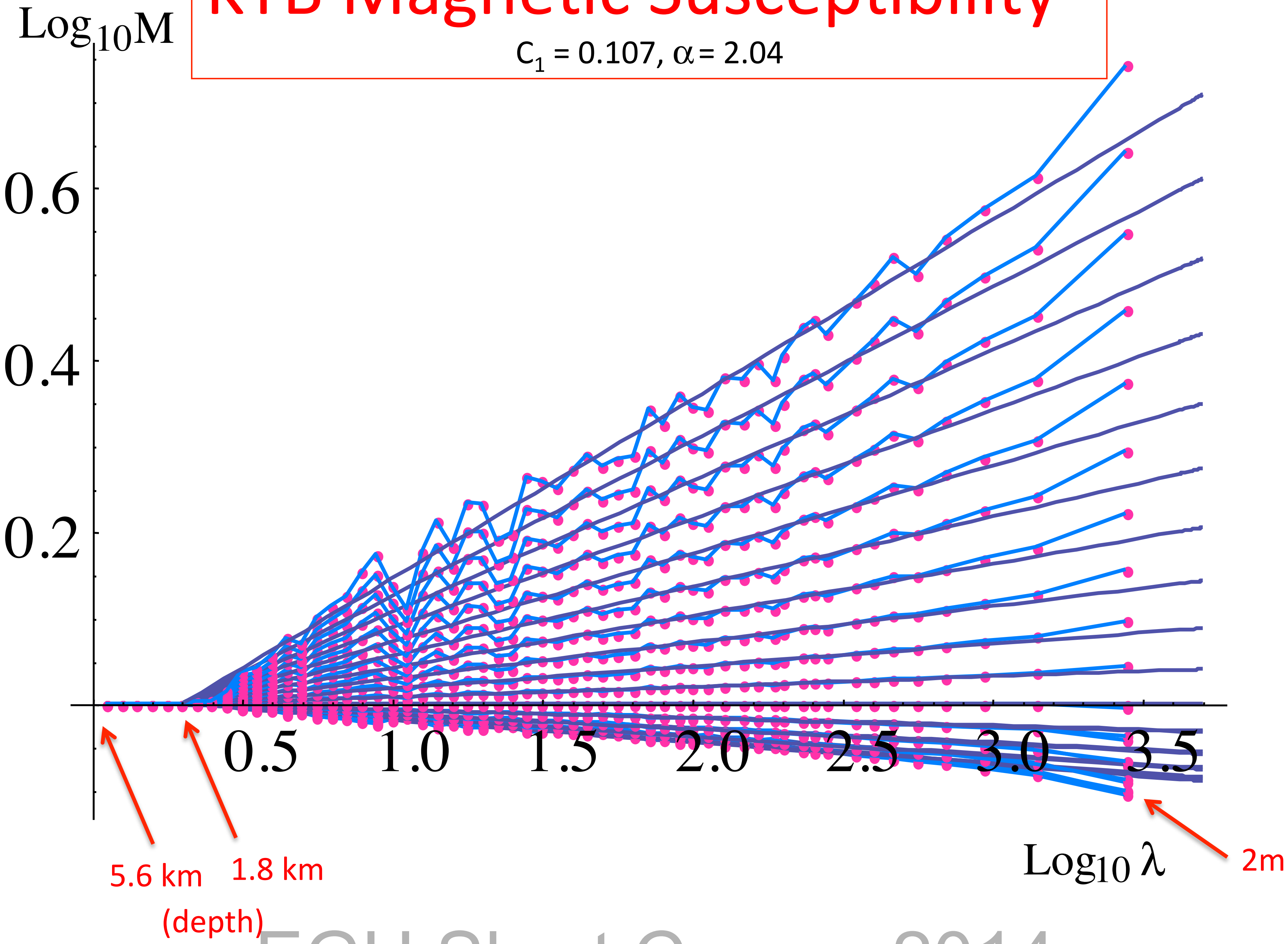
Analysis of four months
U,T at 1000 mb



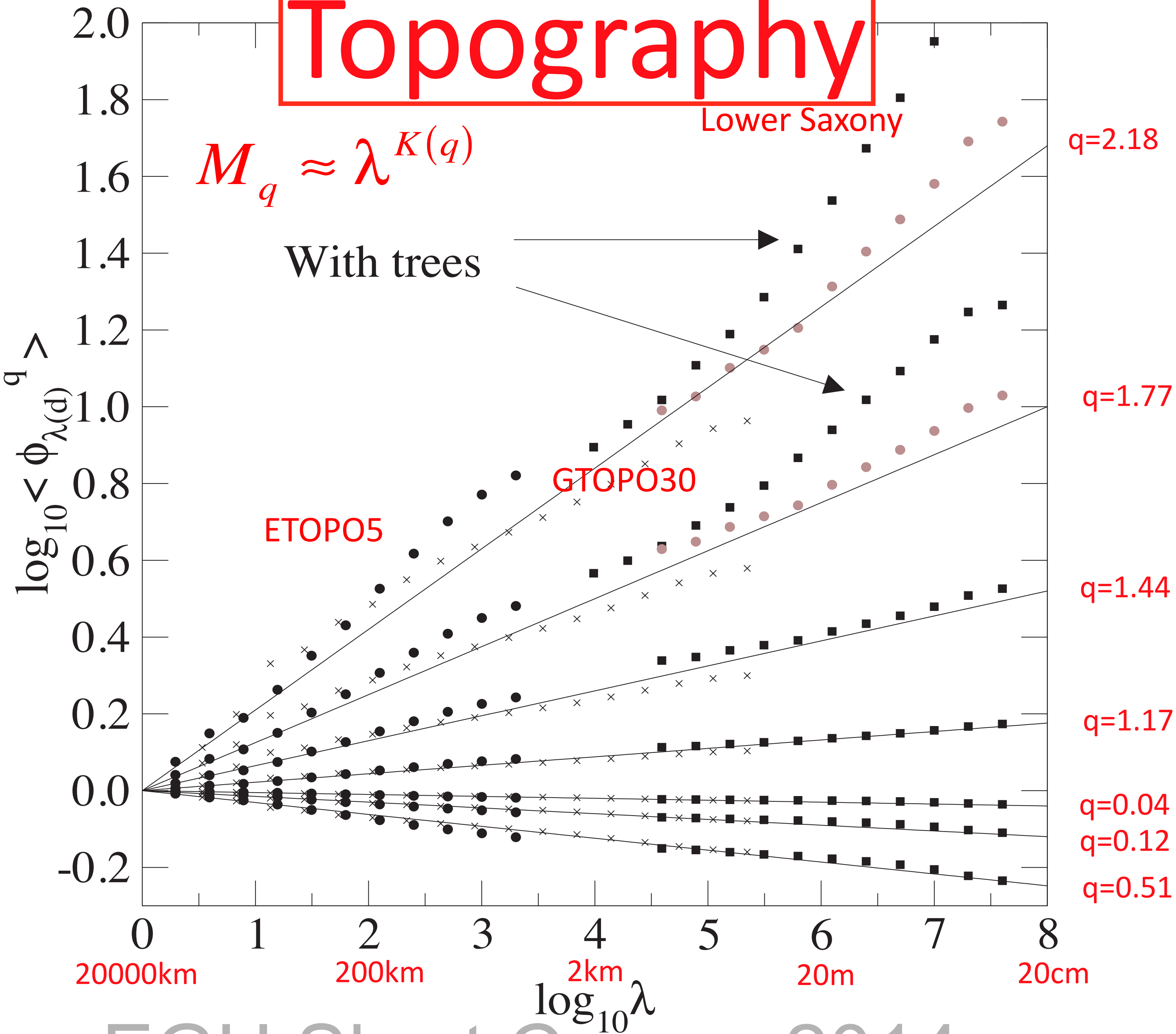
(48 h forecasts are almost the same)

KTB Magnetic Susceptibility

$C_1 = 0.107$, $\alpha = 2.04$



Topography



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Horizontal spatial Scaling exponents

		C_1	α	H	β	L_{eff}
State variables	*u, v*	0.09	1.9	1/3, (0.77)	1.6, (2.4)	(14 000)
	w	(0.12)	(1.9)	(−0.14)	(0.4)	(15 000)
	T	0.11, (0.08)	1.8	0.50, (0.77)	1.9, (2.4)	5000 (19 000)
	h	0.09	1.8	0.51	1.9	10 000
	z	(0.09)	(1.9)	(1.26)	(3.3)	(60 000)
Precipitation	*R*	0.4	1.5	0.00	0.2	32 000
Passive scalars	Aerosol concentration	0.08	1.8	0.33	1.6	25 000
Radiances	Infrared	0.08	1.5	0.3	1.5	15 000
	Visible	0.08	1.5	0.2	1.5	10 000
	Passive microwave	0.1–0.26	1.5	0.25–0.5	1.3–1.6	5000–15 000
Topography	Altitude	0.12	1.8	0.7	2.1	20 000
Sea surface temperature	SST (see Table 8.2)	0.12	1.9	0.50	1.8	16 000

$$\Delta I = \varphi \Delta x^{H} \qquad \left\langle \varphi_{\lambda}^{q} \right\rangle = \lambda^{K(q)} \qquad \lambda = L_{eff} / \Delta x \qquad K(q) = \frac{C_1}{\alpha - 1}\left(q^{\alpha} - q\right) \qquad E(k) \approx k^{-\beta}$$

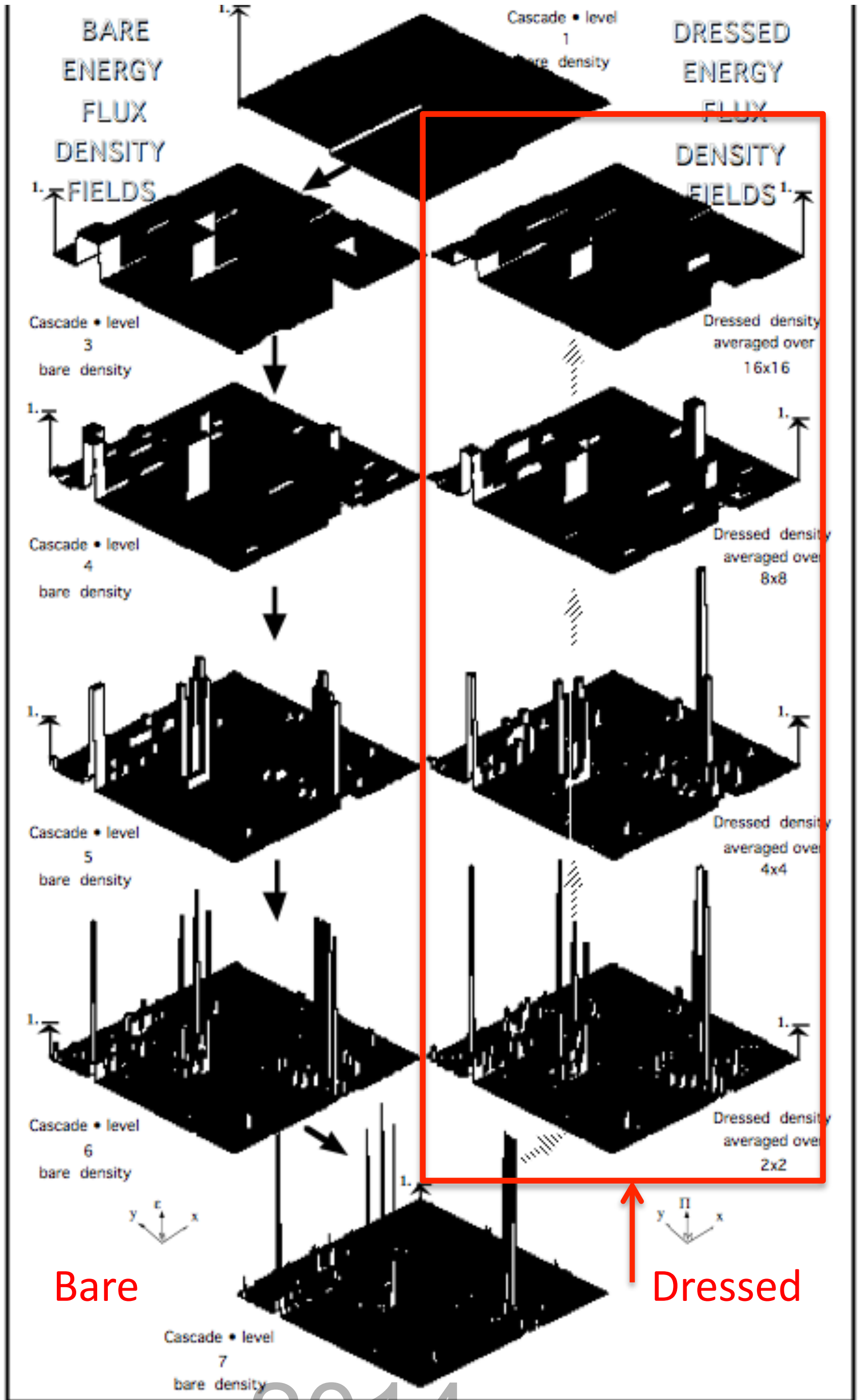
Surface, solid earth exponents

	C_1	α	H	β
Rock Density (vertical)	0.045	2.0	0.08	1.07
Magnetic susceptibility (vertical)	0.11	2.0	0.17	1.12
Topography	0.12	1.8	0.7	2.1
Vegetation index	0.064	2.0	0.16	1.19
Soil moisture index	0.053	2.0	0.14	1.17

$$\Delta I = \varphi \Delta x^{H} \quad \left\langle \varphi_{\lambda}^{q} \right\rangle = \lambda^{K(q)} \quad \lambda = L_{eff} / \Delta x \quad K(q) = \frac{C_1}{\alpha - 1}\left(q^{\alpha} - q\right) \quad E(k) \approx k^{-\beta}$$

Extremes,
Divergence of moments,
Self-organized criticality

Bare and dressed Cascades



Multifractal Butterfly effect

Full cascade
averaged at scale λ^{-1}

$$\varepsilon_{\lambda(d)} = \varepsilon_{\lambda}\varepsilon_{\infty(h)}$$

large scales
(scale range λ)

small scales:
integrated fully
developped cascade

The hidden moments diverge:

$$\left\langle \varepsilon^{q}_{\infty,(h)} \right\rangle \approx \begin{matrix} O(1); & q < q_D \\ \infty; & q \geq q_D \end{matrix}$$

Divergence due to small scales: the multifractal butterfly effect

q_D is the solution to the implicit equation

$$K(q_D) = D(q-1)$$

Divergence of dressed moments:

$$\left\langle \varepsilon^{q}_{\lambda(d)} \right\rangle = \lambda^{K_d(q)}$$

where:

$$K_d(q) = \begin{cases} K(q); & q < q_D \\ \infty; & q \geq q_D \end{cases}$$

Discontinuity in first derivative = first order multifractal phase transition

Probability distributions

Long range dependencies place this outside the framework of Extreme Value Theory

$$\langle \varepsilon^{q}_{\lambda(d)} \rangle = \infty,\ q \geq q_D \iff \Pr(\varepsilon_{\lambda(d)} > s) \sim s^{-q_D},\ \ s >> 1$$

Mandelbrot 1974, S+L 1987

Divergence of moments in Laboratory turbulence

$$\Pr(\varepsilon > s) \approx s^{-q_{D,\varepsilon}}$$

Dissipation Range:

$$\varepsilon \approx \nu \underline{v} \cdot \nabla^2 \underline{v} \approx \nu \frac{\Delta v^2}{\Delta x^2} \qquad \Pr(\varepsilon > s) = \Pr\left(\frac{\nu \Delta v^2}{\Delta x^2} > s\right) \qquad q_{D,\varepsilon} = q_{D,v(diss)}/2$$

Inertial Range:

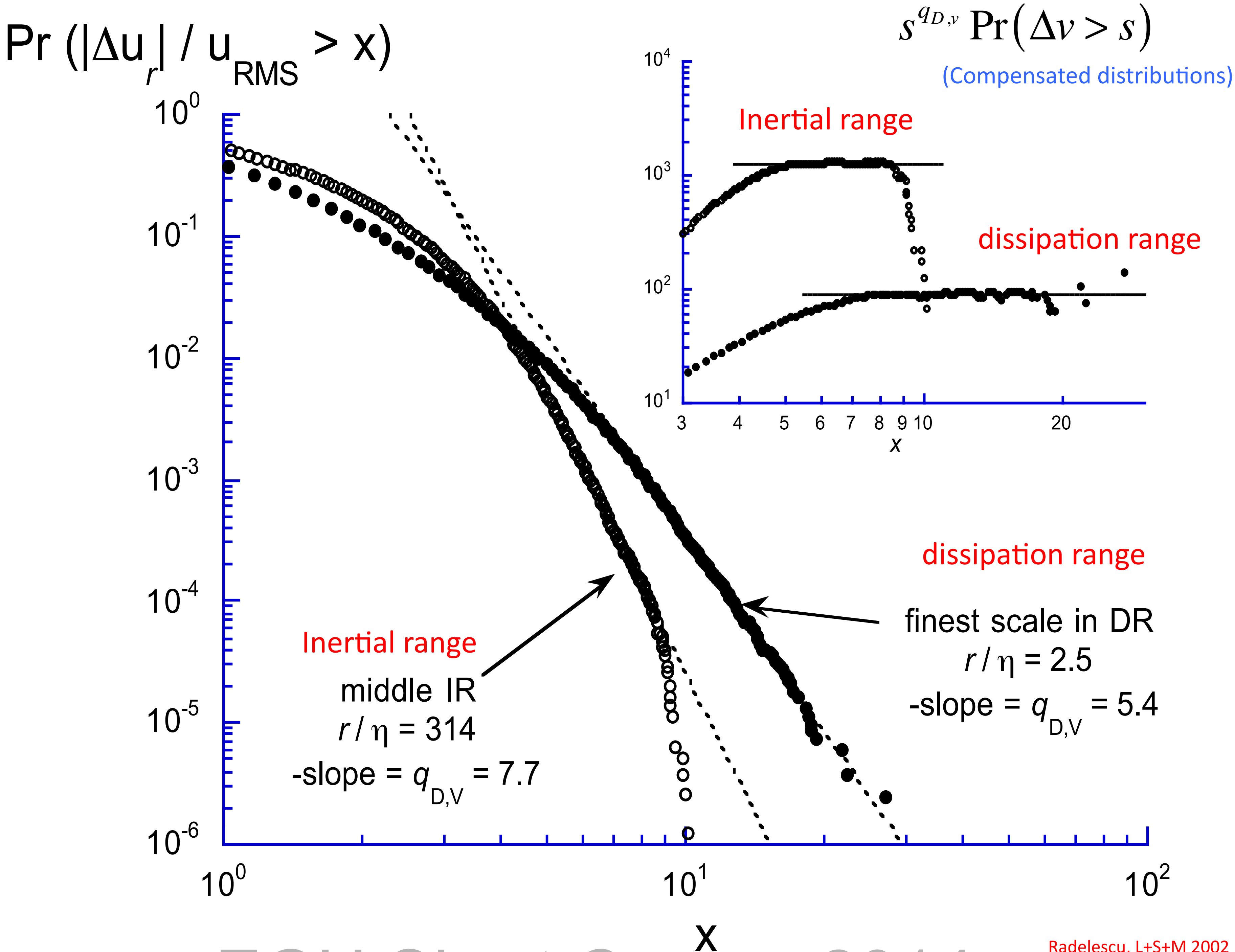
$$\varepsilon \approx \frac{\Delta v^3}{\Delta x} \qquad \Pr(\varepsilon > s) = \Pr\left(\frac{\Delta v^3}{\Delta x} > s\right) \qquad q_{D,\varepsilon} = q_{D,v(inertial)}/3$$

Laboratory Data:

Dissipation range estimate: $q_{D,v(diss)} \approx 5.4;$ $q_{D,\varepsilon} \approx 2.7$

Inertial range estimate: $q_{D,v(inertial)} \approx 7.7;$ $q_{D,\varepsilon} \approx 2.6$

Radelescu, L+S+M 2002



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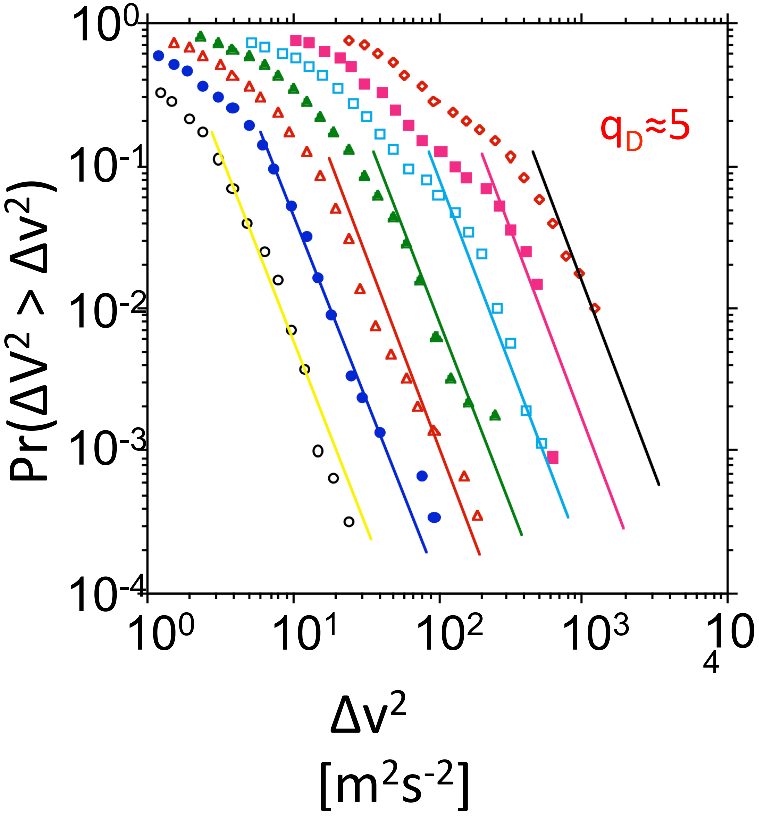
Divergence of moments in the horizontal wind field

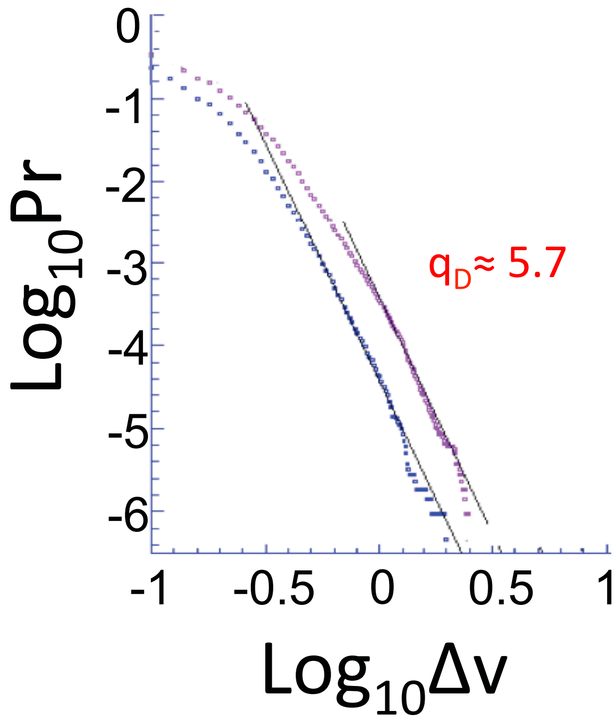


Across vertical layers

radiosondes

Layers 50,100, 200, 400,...3200m

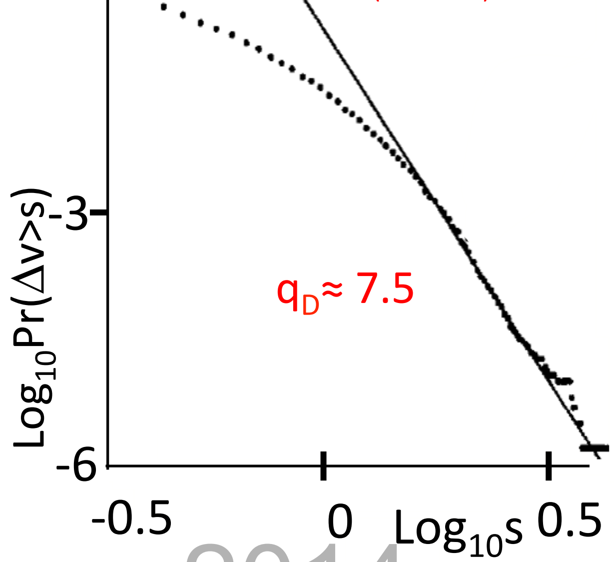
(S+L1985)



In the horizontal

aircraft: 40, 80 m

(L+S 2007)



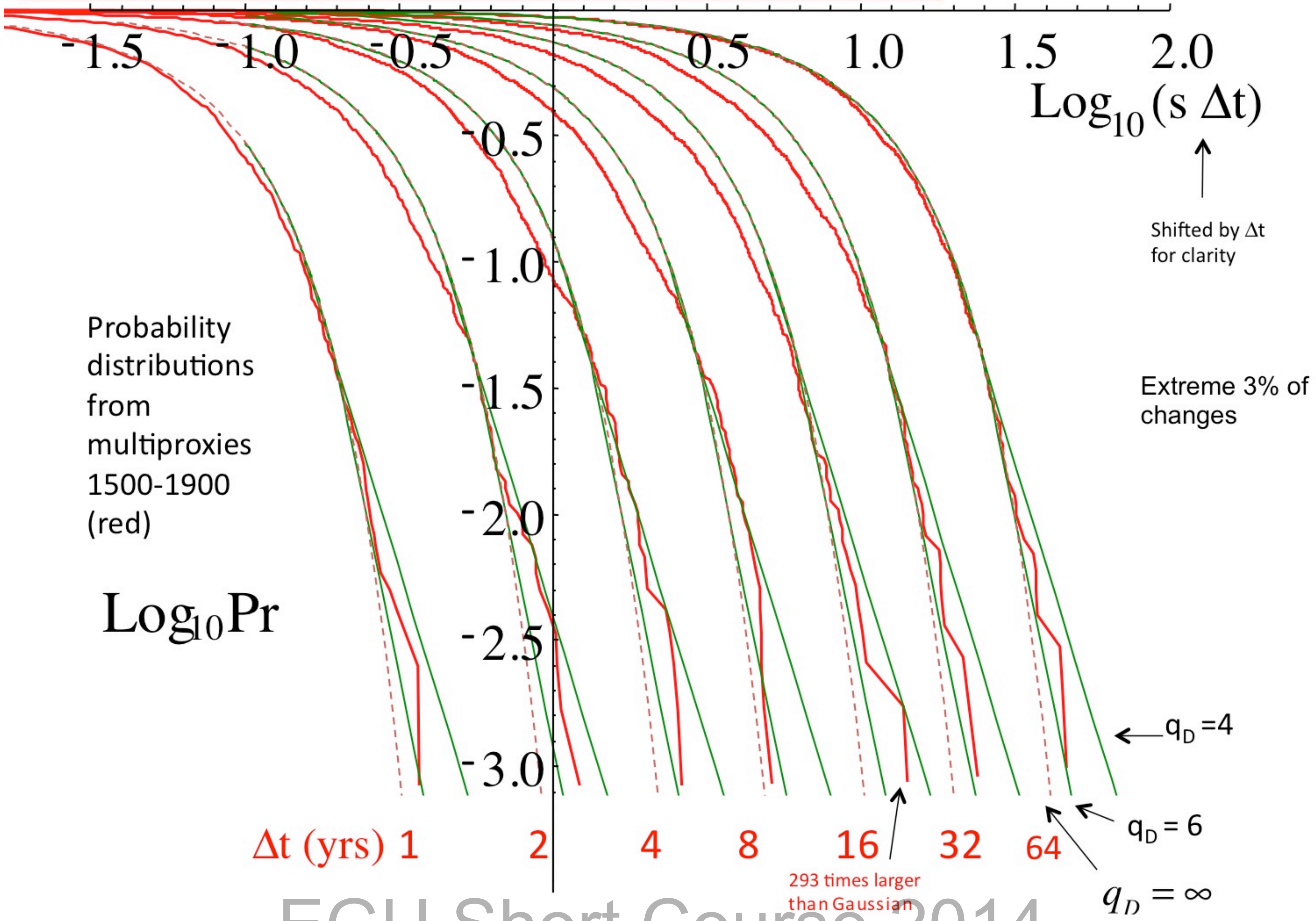
In time

sonic probe, 10 Hz

(Schmitt, S+L 1994)

Bracketing the temperature extremes with power laws

$s^{-4} > \Pr(\Delta T > s) > s^{-6}$



q_D estimates for various geophysical fields

Table 5.1a A summary of various estimates of the critical order of divergence of moments (q_D) for various atmospheric fields.

Field	Data source	Type	q_D	Reference
Horizontal wind	Sonic	10Hz, time	7.5	Schmitt *et al.*, 1994
	Sonic	10 Hz	7.3	Finn *et al.*, 2001
	Hot wire probe	Inertial range	7.7	Fig. 5.22, Radulescu *et al.*, 2002
	Hot wire probe	Dissipation range	5.4	Fig. 5.22, Radulescu *et al.*, 2002
	Anemometer	15 minutes	7	Tchiguirinskaia *et al.*, 2006
	Anemometer	Daily	7	Tchiguirinskaia *et al.*, 2006
	Aircraft, stratosphere	Horizontal, 40 m	5.7	Lovejoy and Schertzer, 2007
	Aircraft, troposphere	Horizontal, 280 m – 36 km	≈ 5	Fig. 5.10
	Aircraft, troposphere	Horizontal, 40 m – 20 km	≈ 7 ± 1	Chigirinskaya *et al.*, 1994
	Aircraft, troposphere	Horizontal, 100 m	≈ 5	Schertzer and Lovejoy, 1985
	Radiosonde	Vertical, 50 m	5	Schertzer and Lovejoy, 1985, Lazarev *et al.*, 1994
	Scaling gyroscopes cascade (SGC) model (Box 3.4)	Time	6.9 ± 0.2	Chigirinskaya and Schertzer, 1996
Potential temperature	Radiosonde	Vertical, 50 m	3.3	Schertzer and Lovejoy, 1985
Humidity	Aircraft, troposphere	Horizontal, 280 m – 36 km	≈ 5	Fig. 5.10
Temperature	Aircraft, troposphere	Horizontal, 280 m – 36 km	≈ 5	Fig. 5.10
	Hemispheric, global	Annual, monthly	≈ 5, 5	Lovejoy and Schertzer, 1986, and unpublished analysis respectively
	Daily, stations	Average over 53 stations in France, daily single station (Macon)	4.5, 4.5	Ladoy *et al.*, 1991
Paleotemperatures	Ice cores	350 years (time), 0.55 m, 1 m (depth)	5, 5	Lovejoy and Schertzer, 1986, Fig. 5.21 respectively
Geopotential anomalies	Reanalyses	500 mb, daily	2.7	Sardeshmukh and Sura, 2009
Vorticity anomalies	Reanalyses	300 mb, daily	1.7	Sardeshmukh and Sura, 2009
Visible radiances (ocean surface)	Remote sensing	7 m resolution MIES data	3.6	Lovejoy *et al.*, 2001
Passive scalar (SF_6)	Fast response SF_6 analyzer	1 Hz	4.7	Finn *et al.*, 2001
Vertical CO_2 flux (above a field)	Aircraft new ground	Horizontal ≈ 1 km resolution	5.3	Austin *et al.*, 1991
Seveso pollution	Ground concentrations	In-situ measurements	2.2	Salvadori *et al.*, 1993
Chernobyl fallout	Ground concentrations	In-situ measurements	1.7	Chigirinskaya *et al.*, 1998; Salvadori *et al.*, 1993
Density of meteorological stations	WMO surface network	Geographic location of stations	3.7 ± 0.1	Tessier *et al.*, 1994

Most exponents: range 3-5

L+S 2013

q_D estimates for various hydrological fields

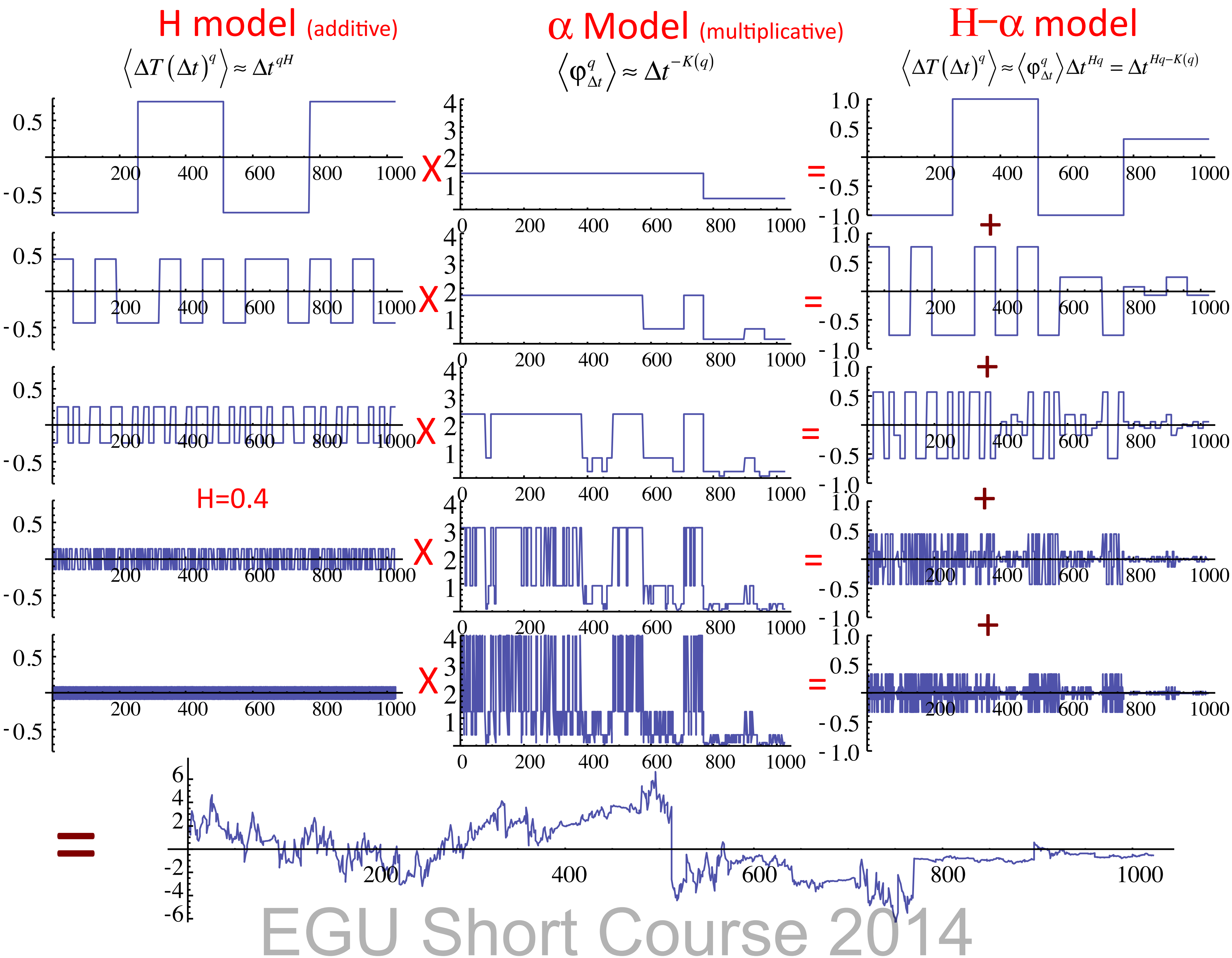
Table 5.1b A summary of various estimates of the critical order of divergence of moments (q_D) for various hydrological fields.

Field	Data source	Type	q_D	Reference
Radar reflectivity of rain	Radar reflectivity factor	1 km^3 resolution	1.1	Schertzer and Lovejoy, 1987
Rain rate	Gauges	Daily, Nimes	2.6	Ladoy *et al.*, 1991
	Gauges	Daily, time, France	≈ 3	Ladoy *et al.*, 1993
	Gauges	Daily, USA	1.7–3	Georgakakos *et al.*, 1994
	High-resolution gauges	8 minutes	≈ 2	Olsson, 1995
	High-resolution gauges	15 s	2.8–8.5	Harris *et al.*, 1996
	Gauges	Daily, time	3.6 ± 0.07	Tessier *et al.*, 1996
	Gauges	1–8 days	3.5	De Lima, 1998
	Gauges	Hourly, time	4.0	Kiely and Ivanova, 1999
	Gauges	Daily, four series from 18th century	3.78 ± 0.46	Hubert *et al.*, 2001
	Gauges	Hourly, time	≈ 3	Fig. 5.10c; Schertzer *et al.*, 2010
	Gauges	Hourly, time	≈ 3	Fig. 5.20b; Lovejoy *et al.*, 2012
	High-resolution gauges	15 s, averaged to 30 minutes	2.23	Verrier, 2011
Raindrop volumes	Stereophotography	10 m^3 sampling volume	5	Lovejoy and Schertzer, 2008
Liquid water at turbulent scales	Stereophotography	Total water in 40 cm cubes	3	Lovejoy and Schertzer, 2006b
Stream flow	River gauges (France)	Daily	3.2 ± 0.07	Tessier *et al.*, 1996
	River gauges (USA)	Daily	3.2 ± 0.07	Pandey *et al.*, 1998; Tessier *et al.*, 1996
	River gauges (France)	Daily	2.5–10	Schertzer *et al.*, 2006

Most exponents: ≈ 3

L+S 2013

Simulations


H model (additive)
$\langle \Delta T(\Delta t)^q \rangle \approx \Delta t^{qH}$
α Model (multiplicative)
$\langle \varphi_{\Delta t}^q \rangle \approx \Delta t^{-K(q)}$
H−α model
$\langle \Delta T(\Delta t)^q \rangle \approx \langle \varphi_{\Delta t}^q \rangle \Delta t^{Hq} = \Delta t^{Hq-K(q)}$
H=0.4
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Multiplicative processes

The process

$$\varepsilon_\lambda = e^{\Gamma_\lambda}$$

Multifractal process over scale range λ= Multiplicative cascade

The generator= Additive process

Dimension of space

Auxiliary variable α'

$$\frac{1}{\alpha'}+\frac{1}{\alpha}=1$$

$$\Gamma(\underline{r}) \propto C_1^{1/\alpha}\gamma_\alpha(\underline{r}) * |\underline{r}|_{[1,\lambda^{-1}]}^{-D/\alpha'} \longleftrightarrow \tilde{\Gamma}(\underline{k}) = \tilde{\gamma}_\lambda(\underline{k})|\underline{k}|^{-D/\alpha}$$

The codimension of the mean= Amplitude of the generator, sparseness of ε

i.i.d. Levy noise index α= subgenerator

Convolution with singularity over range 1 to λ^{-1}

Fourier space= power law filter

The statistics

$$\langle \varepsilon_\lambda^q \rangle = \lambda^{K(q)}$$

General multifractal statistics, convex K(q)

$$K(q) = \frac{C_1}{\alpha - 1}\left(q^\alpha - q\right)$$
$$q < q_D$$

Universal multifractals ("multiplicative central limit theorem")

$$\Pr(\varepsilon_\lambda > s) \approx s^{-q_D}$$
$$s >> 1$$

Extremes: "Fat tails"

Fractionally Integrated Flux (FIF) model (both additive and multiplicative)

S+L 1987

The process

$$I(\underline{r}) = \varepsilon_\lambda(\underline{r}) * |\underline{r}|^{-(D-H)} \longleftrightarrow \tilde{I}(\underline{k}) = \tilde{\varepsilon}_\lambda(\underline{k})|\underline{k}|^{-H}$$

Convolution= fractional integration order H

Fourier space= power law filter

The statistics

$$S_q(\underline{\Delta r}) = \left\langle \Delta I(\underline{\Delta r})^q \right\rangle = \left\langle \varepsilon_\lambda^q \right\rangle |\underline{\Delta r}|^{qH} = |\underline{\Delta r}|^{\xi(q)} \qquad \xi(q) = qH - K(q)$$

q^{th} order structure function

fluctuation

Note:

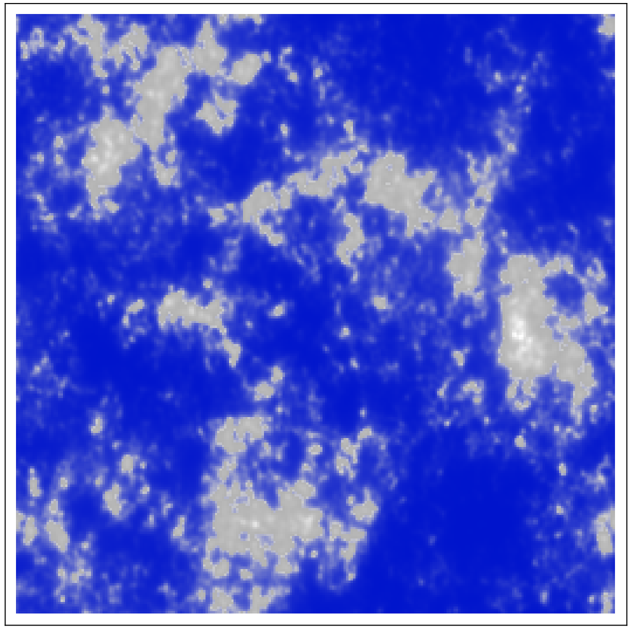
$\lambda = L/|\underline{\Delta r}|$

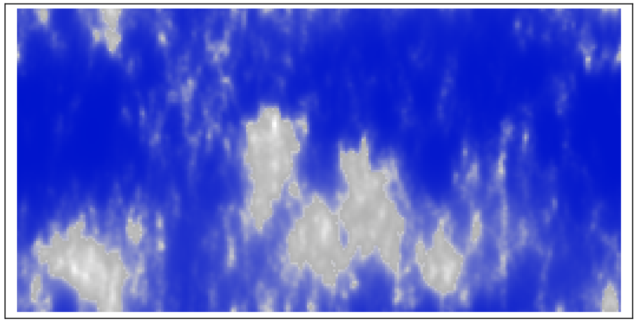
$\left\langle \varepsilon_\lambda^q \right\rangle = \lambda^{K(q)}$

structure function exponent

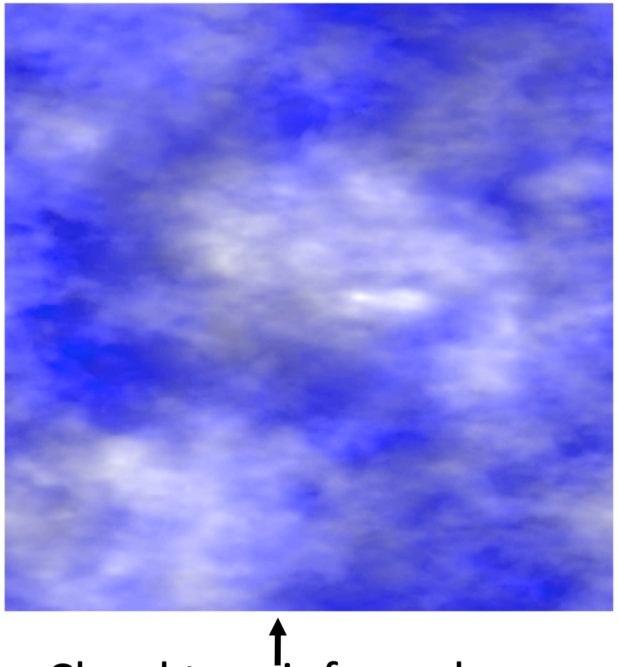
FIF modeling: clouds and radiative transfer

Cloud liquid water (top)



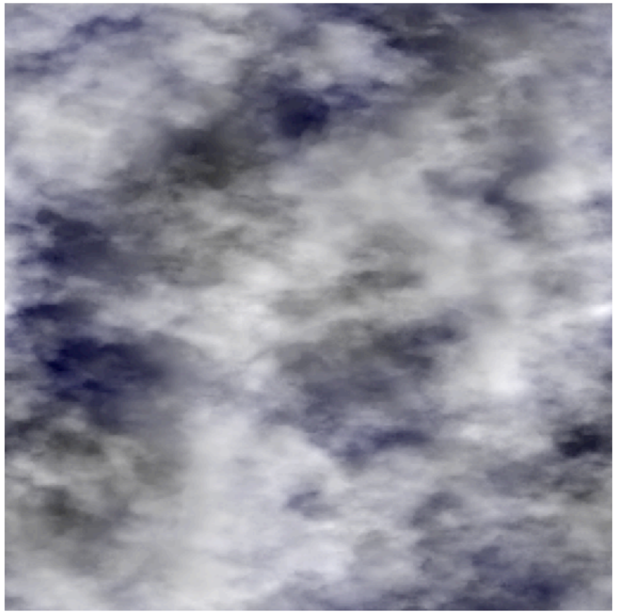


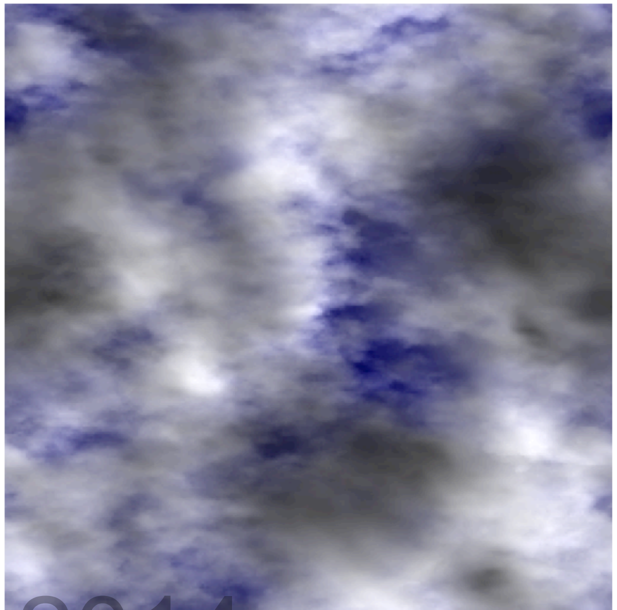
Cloud liquid water (side)



Cloud top, infra red

Cloud top visible





Cloud bottom visible

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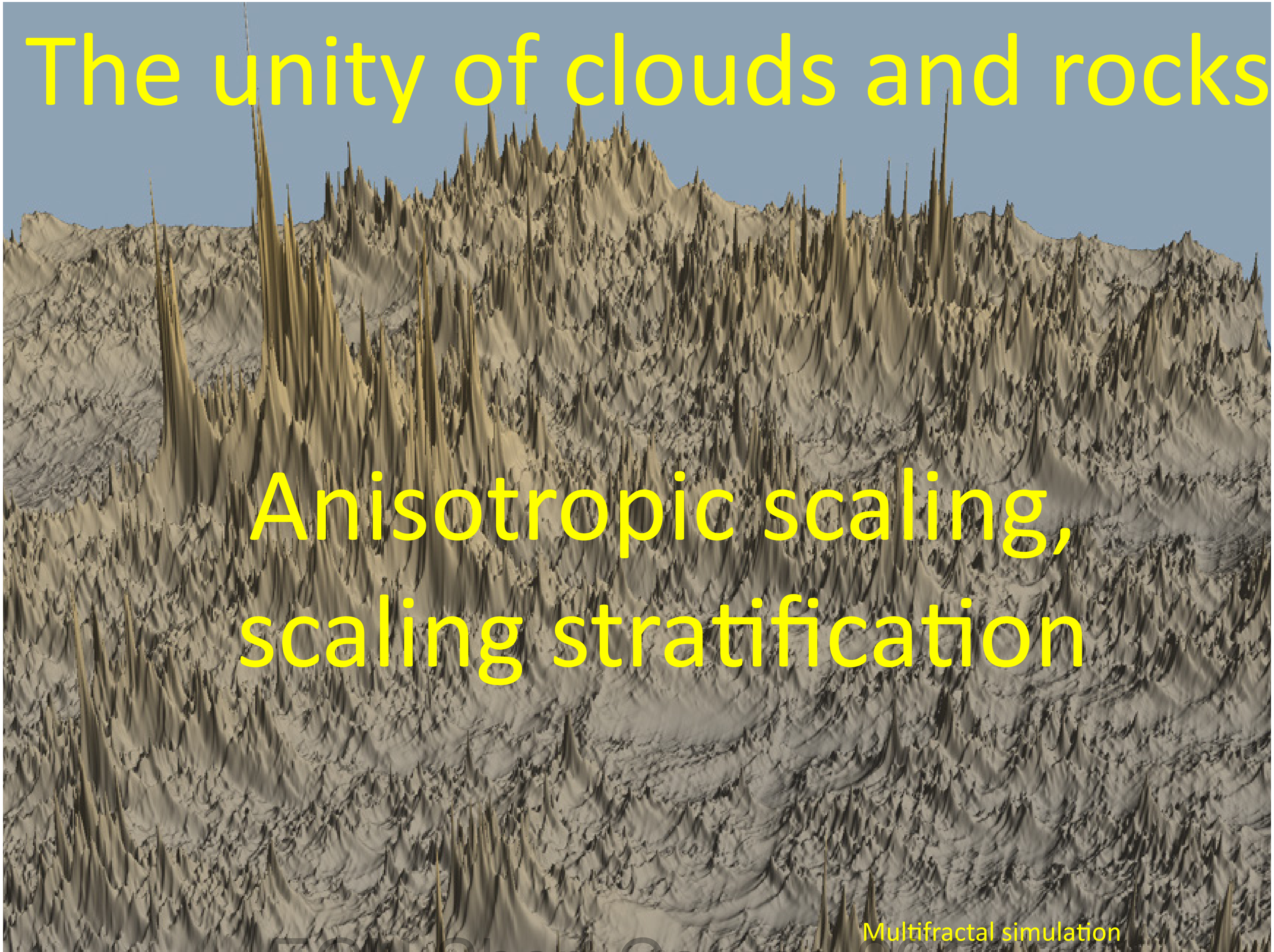
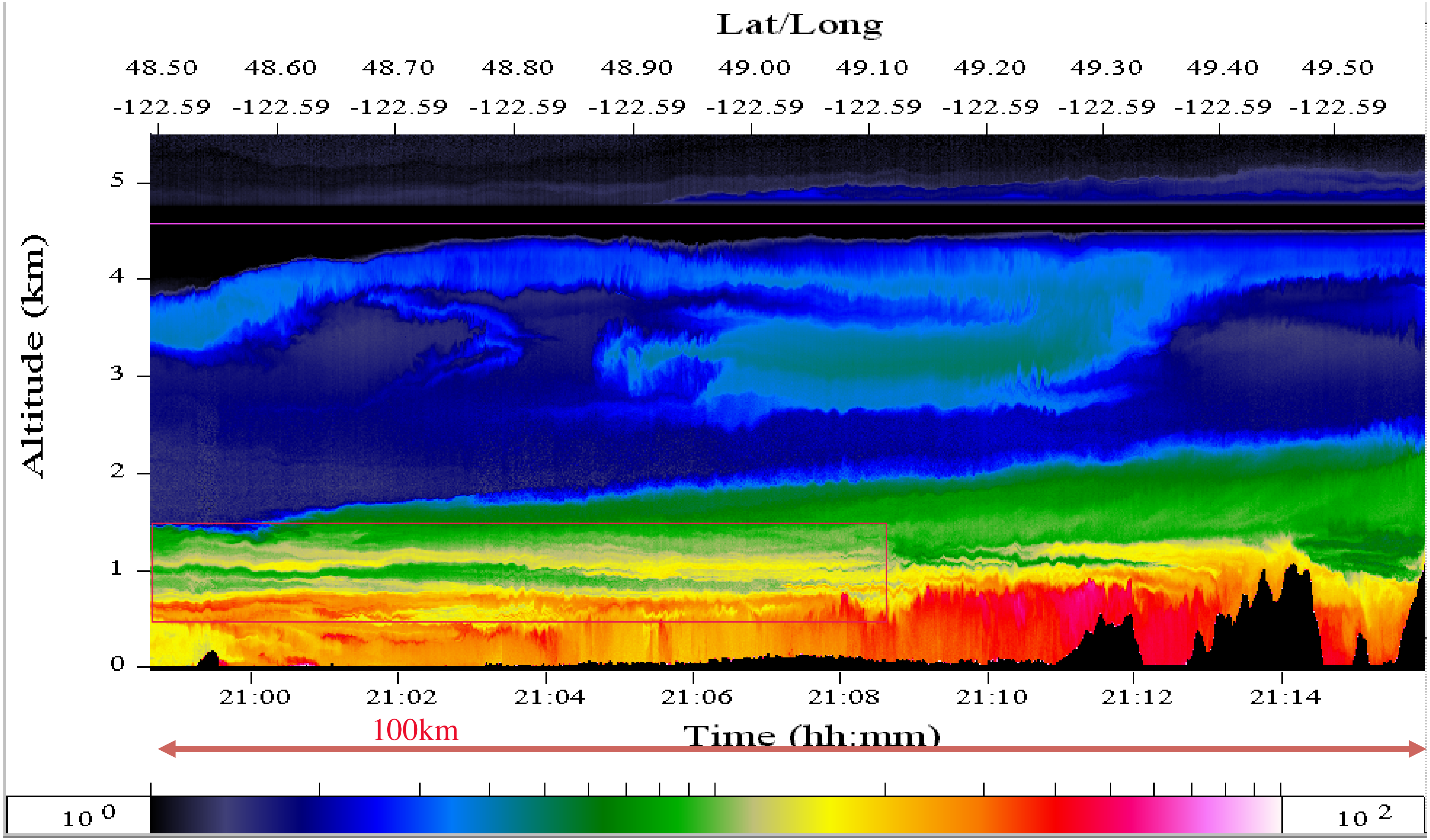
The unity of clouds and rocks

Anisotropic scaling, scaling stratification

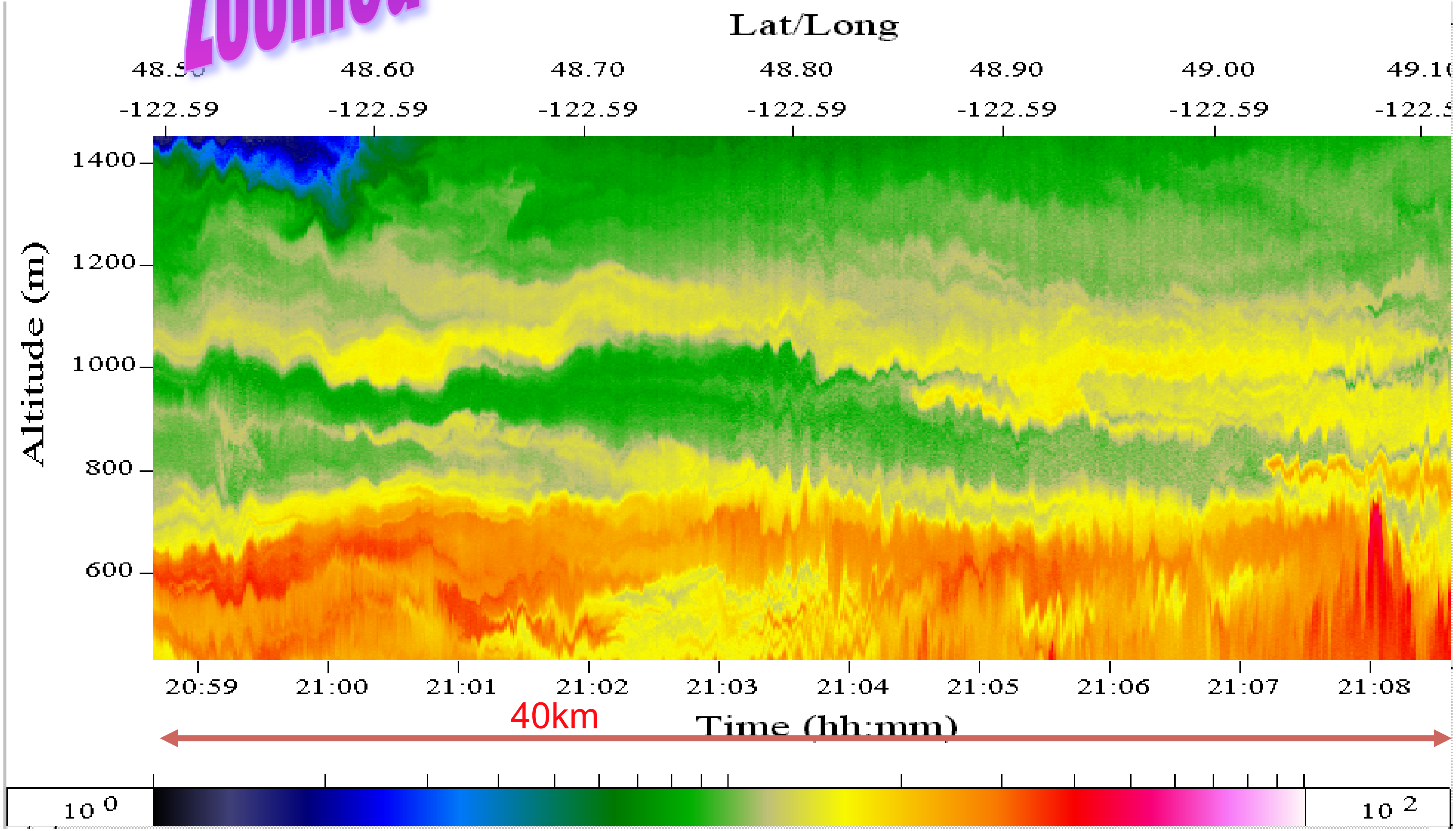


AERIAL Lidar Data

(courtesy of K. Strawbridge)

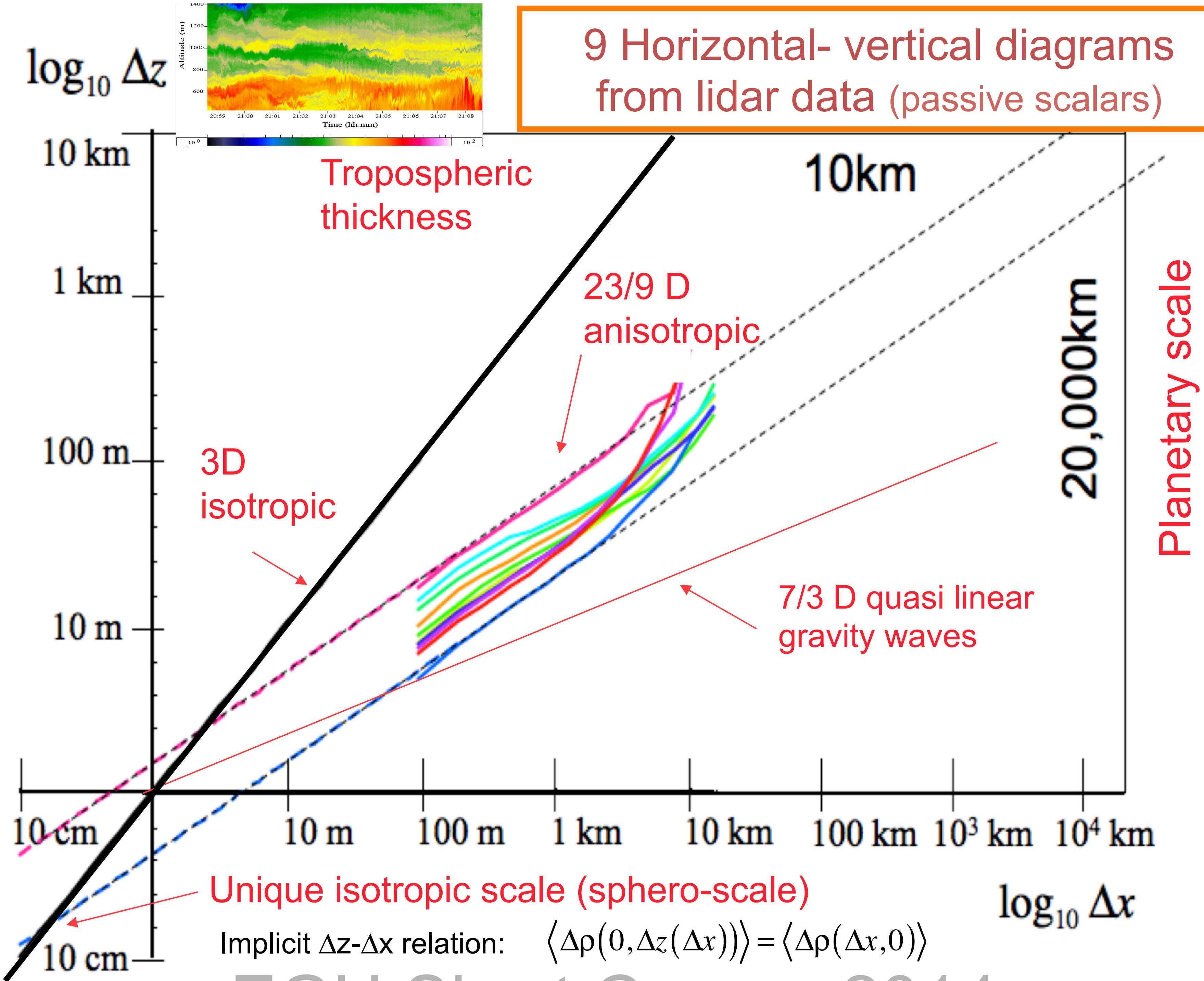


Zoomed



Lat/Long
48.60
48.70
48.80
48.90
49.00
-122.59
Altitude (m)
1400
1200
1000
800
600
20:59
21:00
21:01
21:02
21:03
21:04
21:05
21:06
21:07
21:08
Time (hh:mm)
40km
10^0
10^2

9 Horizontal- vertical diagrams from lidar data (passive scalars)



Implicit Δz-Δx relation: $\langle \Delta\rho(0,\Delta z(\Delta x)) \rangle = \langle \Delta\rho(\Delta x,0) \rangle$

The physical scale function and differential scaling

$$\left|\underline{\Delta r}\right| \rightarrow \left\|\underline{\Delta r}\right\|$$

Usual distance (=vector norm)

Scale function (scale notion)

Scale symmetry $\left\|\lambda^{-G}\underline{r}\right\| = \lambda^{-1}\left\|\underline{r}\right\|$

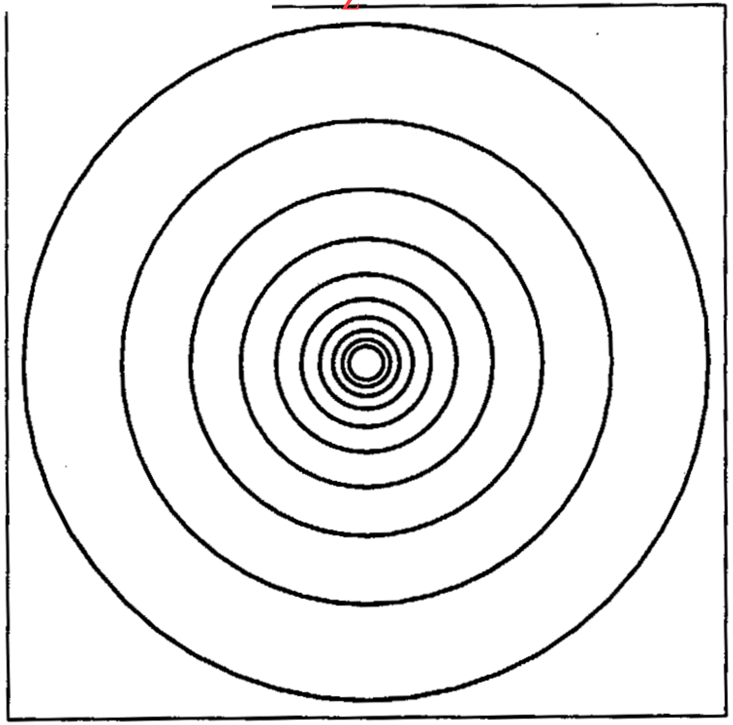
"canonical" scale function:

$$\left\|(\Delta x,\Delta z)\right\| = l_s\left(\left(\frac{\Delta x}{l_s}\right)^2 + \left(\frac{\Delta z}{l_s}\right)^{2/H_z}\right)^{1/2}$$

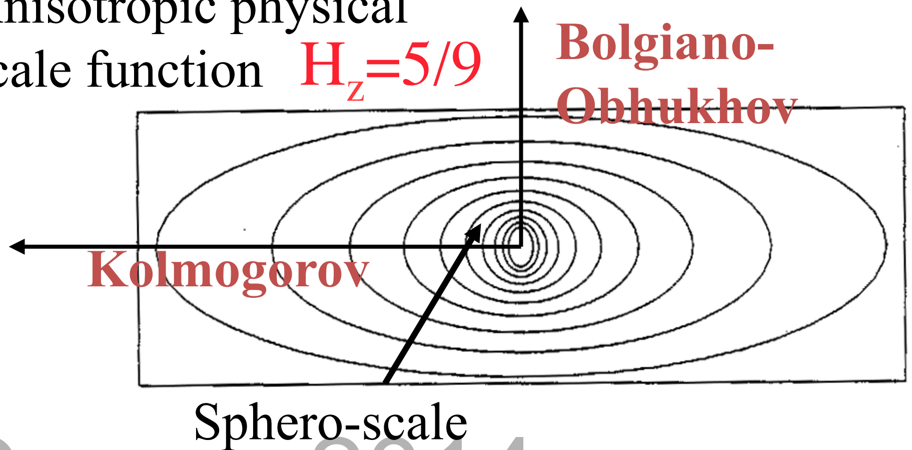
$$G = \begin{pmatrix} 1 & 0 \\ 0 & H_z \end{pmatrix}$$

Vertical sections

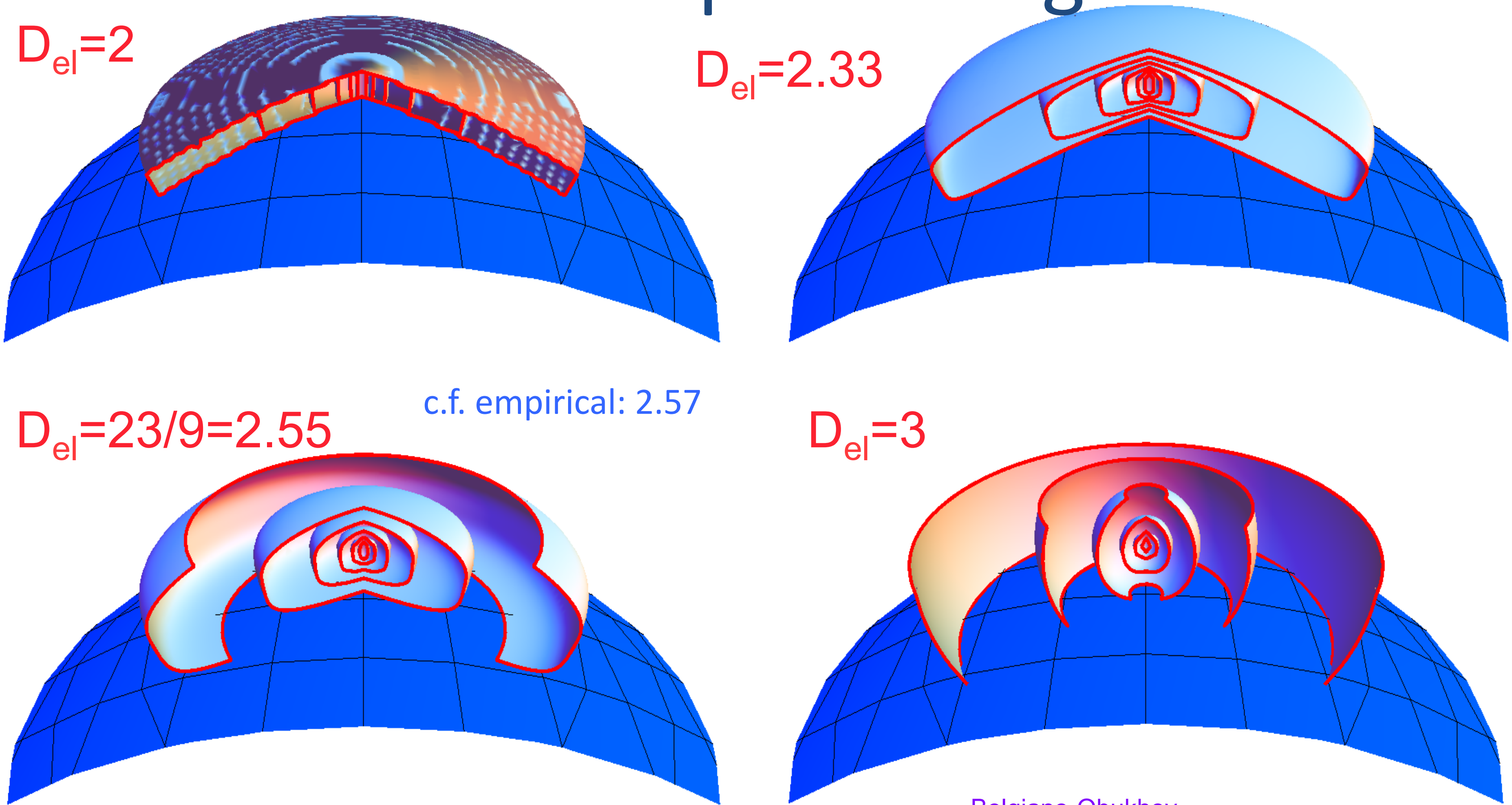
Isotropic function $H_z=1$



Anisotropic physical scale function $H_z=5/9$



Anisotropic Scaling



The 23/9D model:

Bolgiano-Obukhov

$$\Delta v(\Delta x) = \varepsilon^{1/3}\Delta x^{1/3}; \quad \Delta v(\Delta z) = \phi^{1/5}\Delta z^{3/5}$$

$H_z=(1/3)/(3/5)=5/9$

Kolmogorov

Volume $\approx L \times L \times L^{Hz} \approx L^{Del}$

$D_{el}=2+H_z=23/9$

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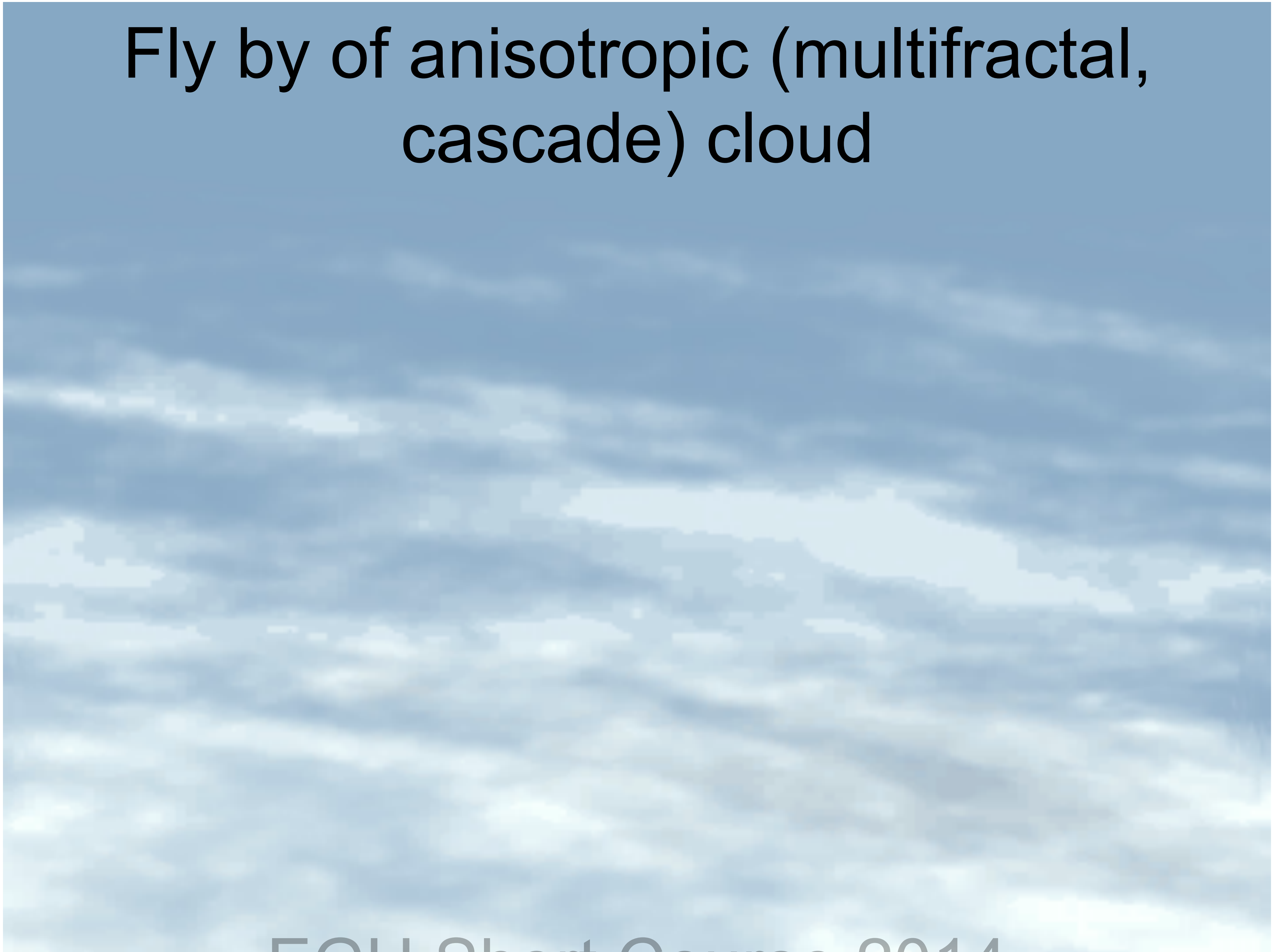
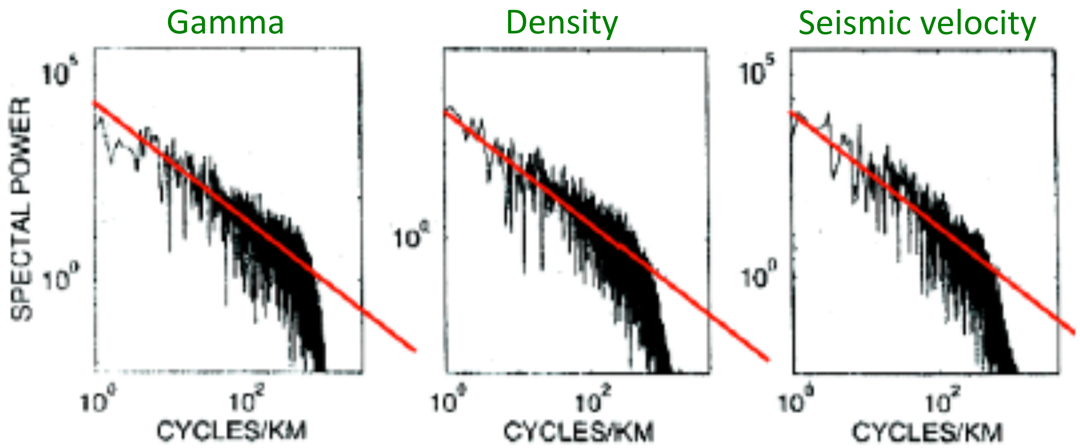
Fly by of anisotropic (multifractal, cascade) cloud

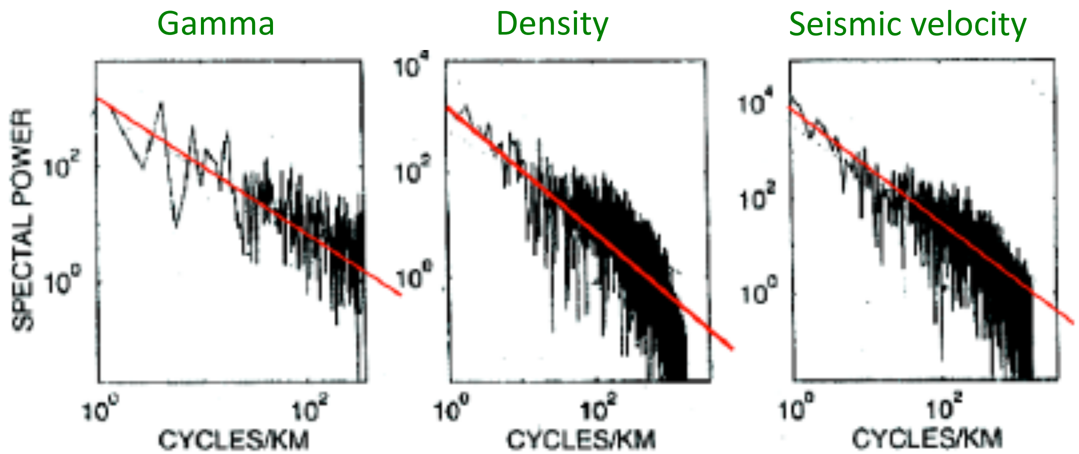


Horizontal versus vertical borehole rock densities



Horizontal boreholes (β_h =1.4)

$H_z = (\beta_h - 1)/(\beta_v - 1) = 2$



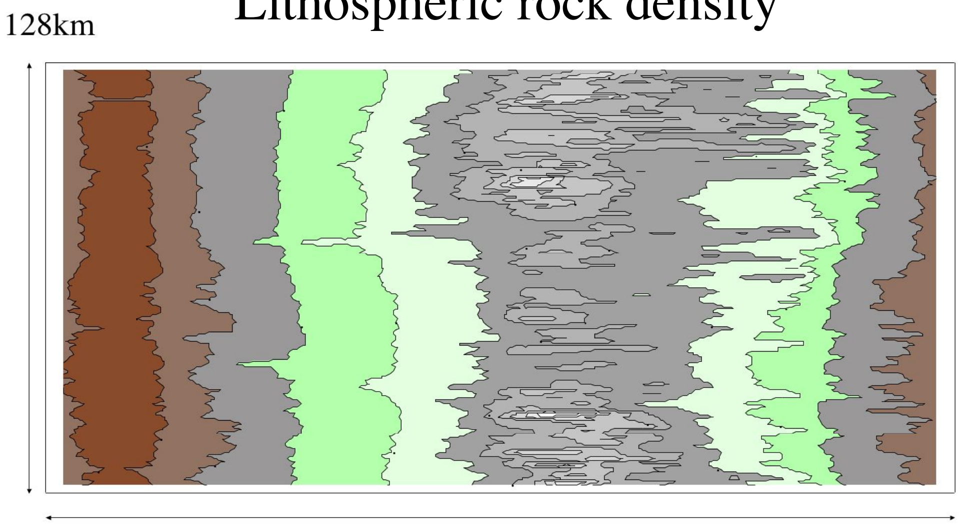
Vertical boreholes (β_v =1.2)

Lovejoy and Schertzer 2007 (adapted from Leary 1997)

Stratified Multifractal Crust, Mantle rock density simulation

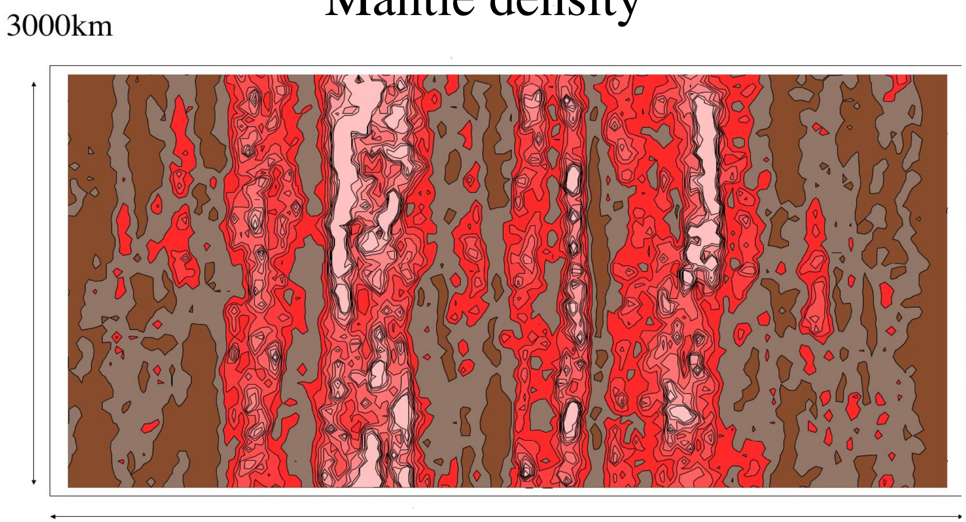
Vertical cross-sections $D_{el}=3$

Lithospheric rock density



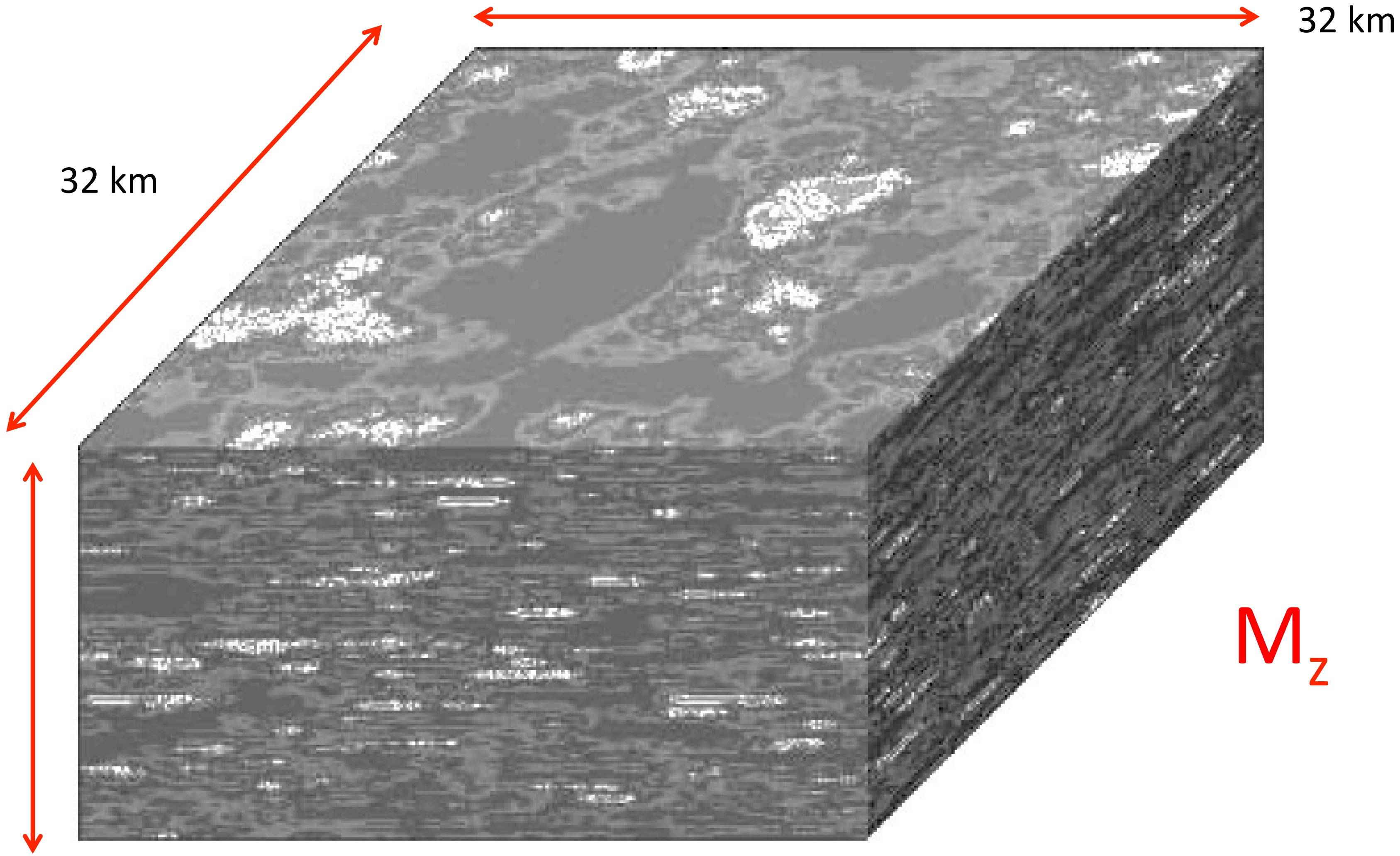
Sphero-scale l_s=256km, with 1 pixel = 1km.

Mantle density



Sphero scale = 1 pixel. Each pixel is 50 km, sphero-scale = 25km. Hot (low density) plumes shown as white/red (this is a model for either density or temperature fluctuations (the two being proportional; we assume constant expansion coefficient). These are for fluctuations with respect to the mean vertical profile

Simulated magnetization field for horizontally isotropic crustal magnetization



Parameters: are H_z =1.7, s = 4, H = 0.2, α =1.98, C_1 = 0.08, l_s = 2500 km,

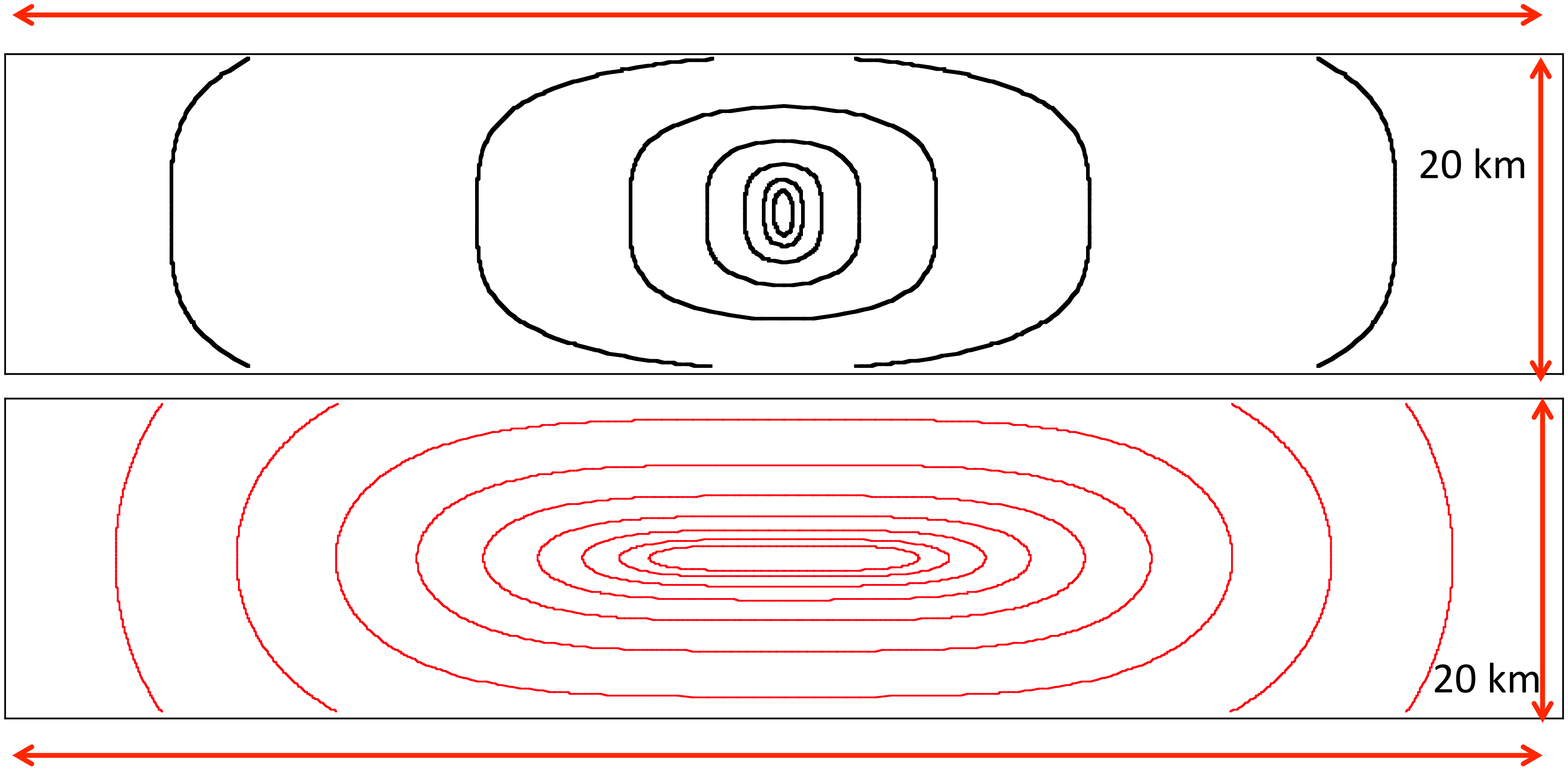
The unity of geosciences: clouds and rocks

2000 km

Parameters: $H_z = 5/9$, $L_s = 0.1$ km

Cloud density

Rock density



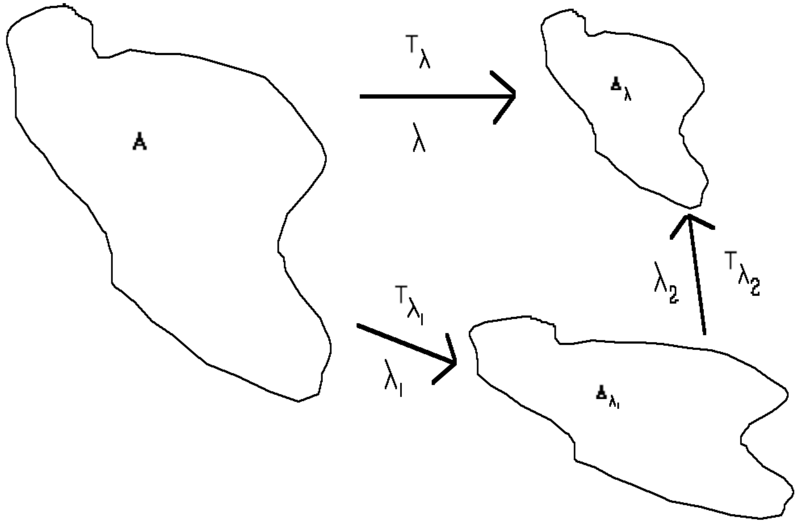
2000 km

Parameters: $H_z = 2.$, $L_s = 40000$ km

aspect ratios = 1/5

Generalized Scale Invariance

The scale changing operator T_λ which transforms the scale of vectors by scale ratio λ



T_λ is the rule relating the statistical properties at one scale to another and involves only the scale ratio. This implies that T_λ has certain properties. In particular, if and only if $\lambda_1\lambda_2 = \lambda$, then:

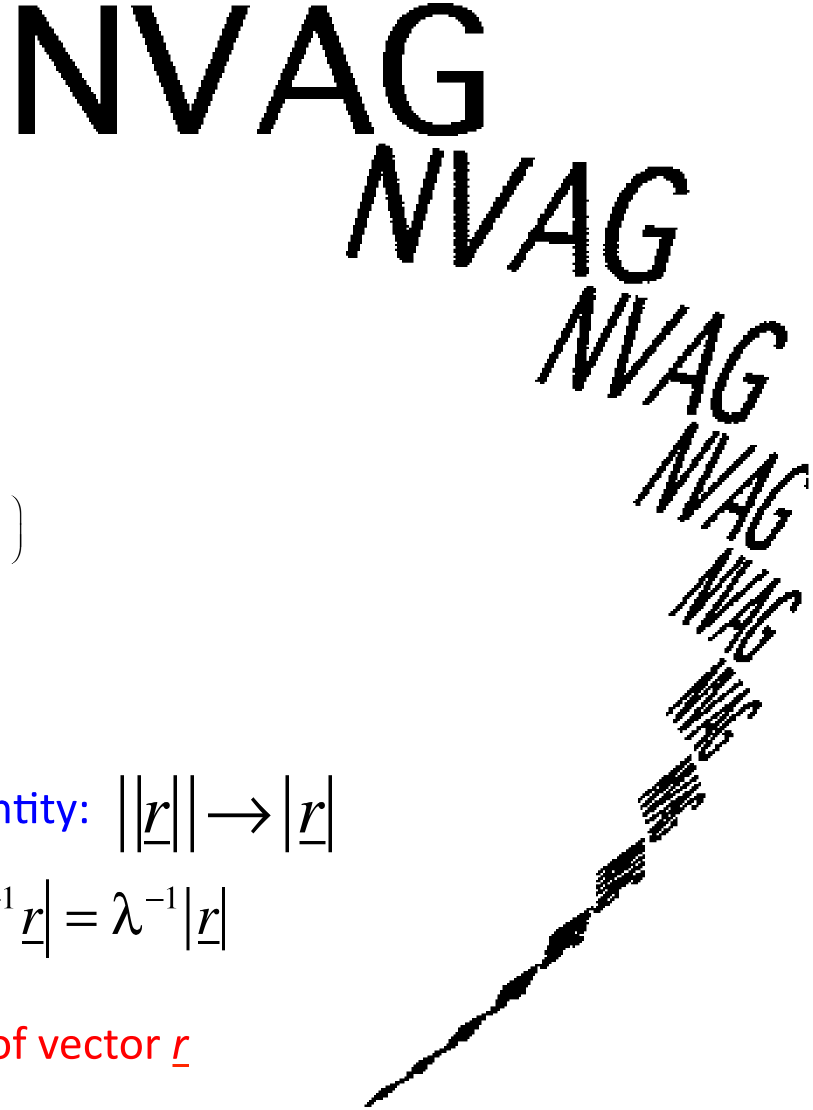
$$B_\lambda = T_\lambda B_1 = T_{\lambda_1\lambda_2} B_1 = T_{\lambda_1} B_{\lambda_2} = T_{\lambda_2} B_{\lambda_1}$$

it is also commutative $\quad T_\lambda = T_{\lambda_2} T_{\lambda_1} = T_{\lambda_1} T_{\lambda_2}$

This implies that T_λ is a one parameter multiplicative group with parameter λ: $\boxed{T_\lambda = \lambda^{-G}}$

One parameter Lie group, G= generator

Example of anisotropic "Blow down"



$$T_\lambda = \lambda^{-G}$$

A generalized blow-down with increasing   of the acronym "NVAG". If $G = I$, we would have obtained a standard reduction, with all the copies uniformly reduced converging to the centre of the reduction. Here the parameters are $G = \begin{pmatrix} 1.3 & -1.3 \\ 0.3 & 0.7 \end{pmatrix}$

and each successive reduction is by 28%.

Scale function equation

$$\left\|\lambda^{-G}\underline{r}\right\| = \lambda^{-1}\left\|\underline{r}\right\|$$

generator

Scale ratio

Scale function: size of vector $\underline{r}$

G=identity: $\left\|\underline{r}\right\| \to \left|\underline{r}\right|$

$$\left|\lambda^{-1}\underline{r}\right| = \lambda^{-1}\left|\underline{r}\right|$$

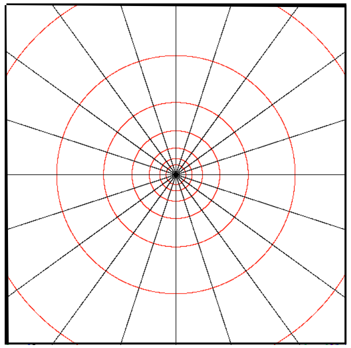
Scale functions in linear GSI (position independent)

Isotropic (self similar)

$$T_\lambda = \lambda^{-G}$$

Scale functions

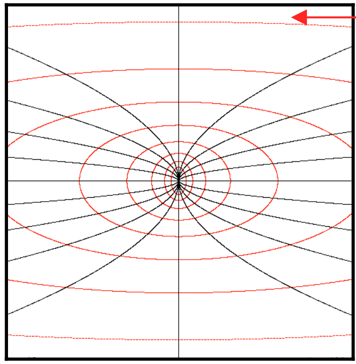
$$\left\|\lambda^{-G}\underline{r}\right\| = \lambda^{-1}\left\|\underline{r}\right\|$$



$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

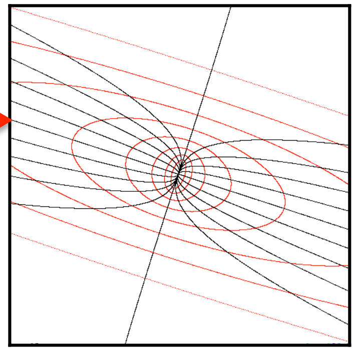
Scale isolines in red $\left\|\underline{r}\right\|$=contstant

Self-affine



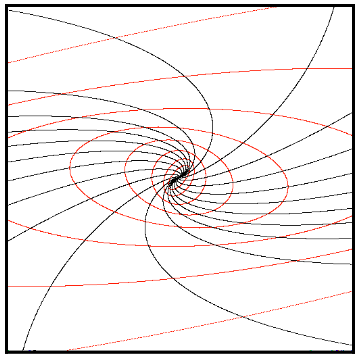
$$G = \begin{pmatrix} 1.35 & 0 \\ 0 & 0.65 \end{pmatrix}$$

Stratification dominant (real eigenvalues)



$$G = \begin{pmatrix} 1.35 & 0.25 \\ 0.25 & 0.65 \end{pmatrix}$$

Rotation dominant (complex eigenvalues)



$$G = \begin{pmatrix} 1.35 & -0.45 \\ 0.85 & 0.65 \end{pmatrix}$$

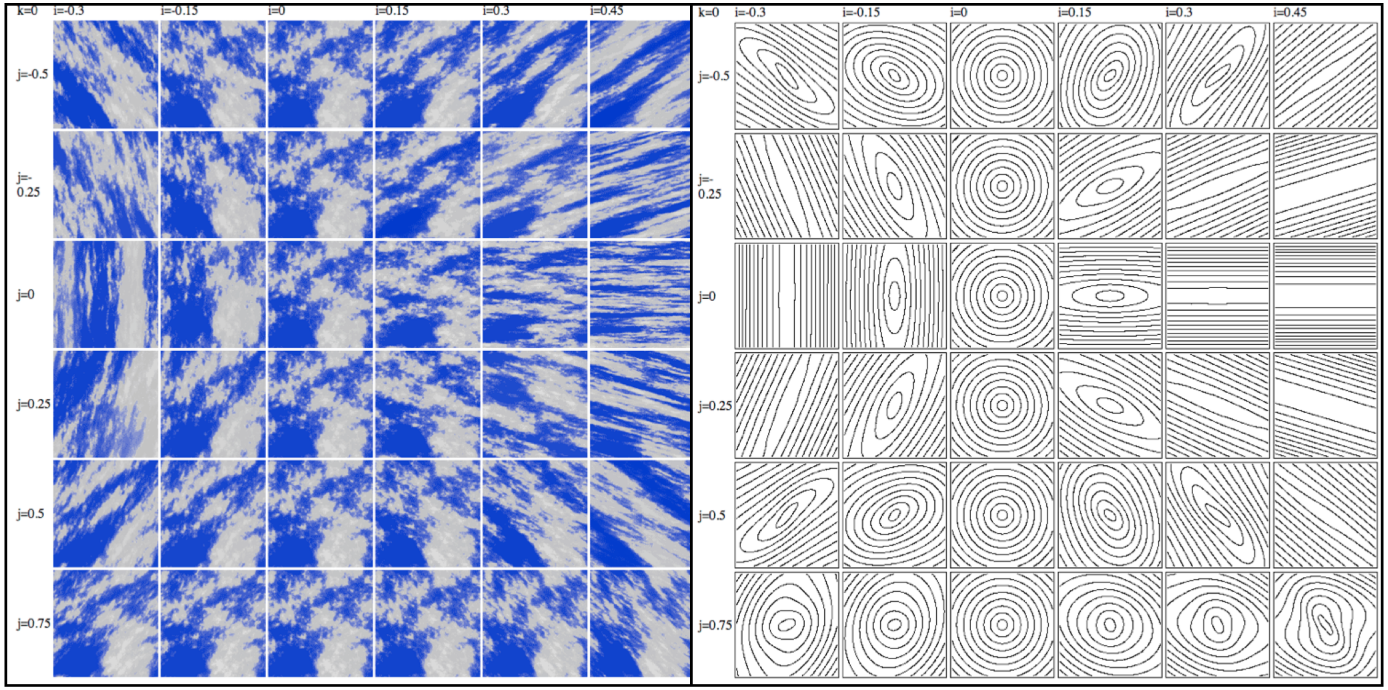
Roundish unit ball

k =0: we vary *r* (denoted *i*) from -0.3, -0.15, …0.45 left to right and *e* (denoted *j*) from -0.5, -0.25, …0.75 top to bottom. On the right we show the contours of the corresponding scale functions.

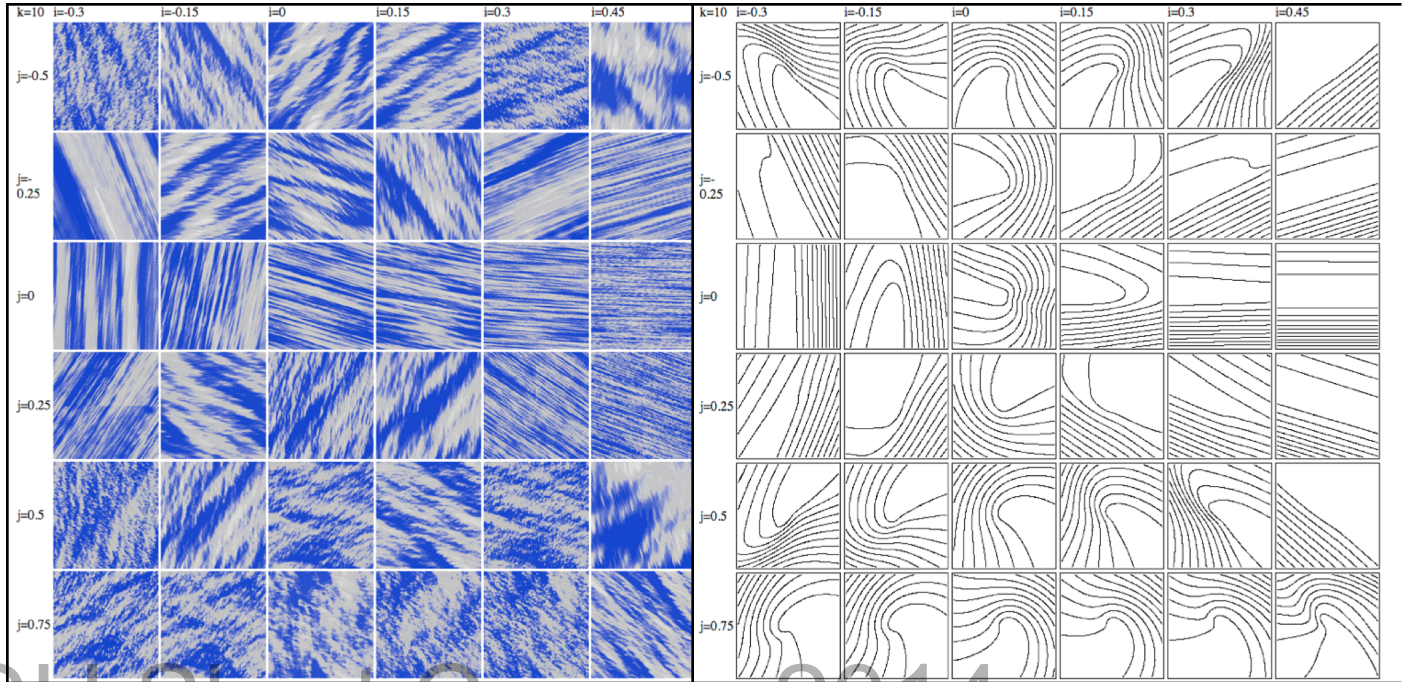
$$G = \begin{pmatrix} 1 & r-e \\ r+e & 1 \end{pmatrix}$$

r

e



Highly anisotropic unit ball: *k* =10



Polar coordinate scale function for unit ball

$$\|\underline{r}\| = r\Theta(\theta'') = 1$$

with

$$\Theta(\theta'') = 1 + \frac{1-2^{-k}}{1+2^{-k}}\cos\theta''$$

Hence:

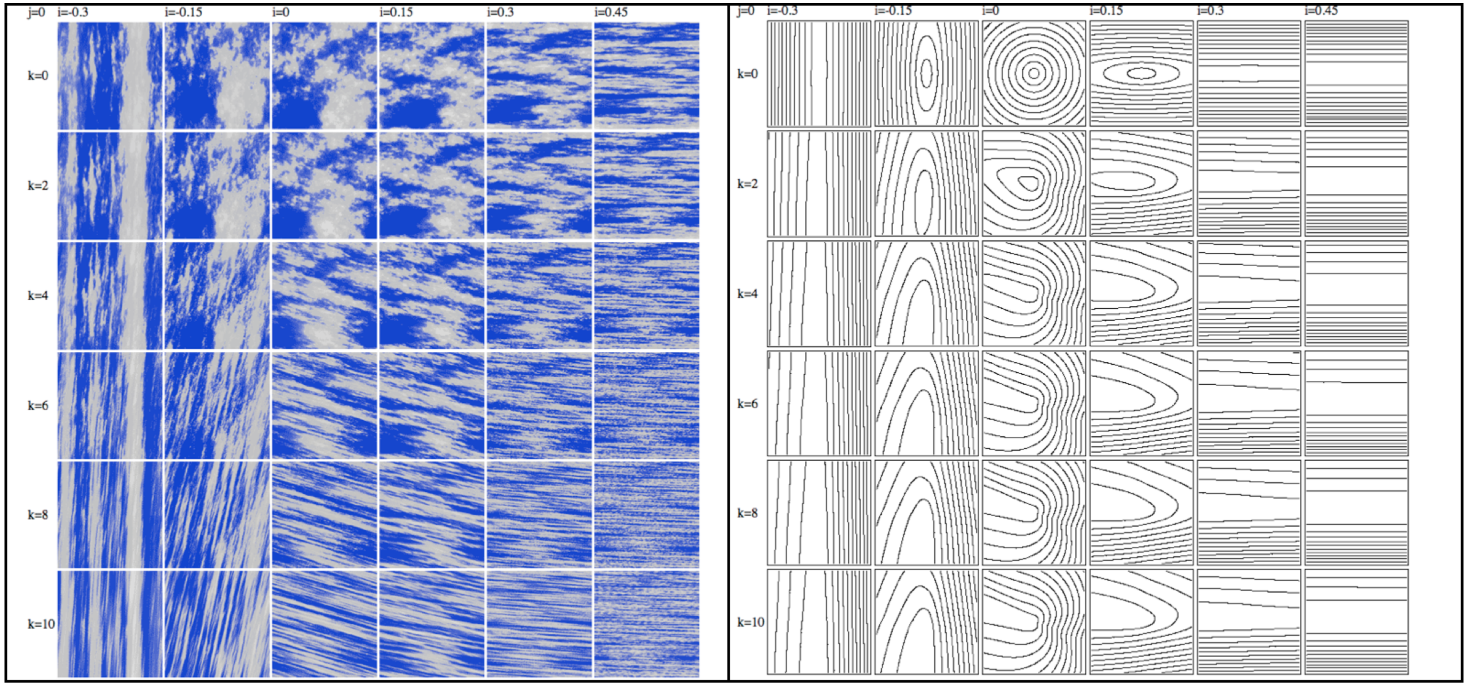
$$\max(\Theta(\theta''))/\min(\Theta(\theta'')) = 2^k$$

e = 0

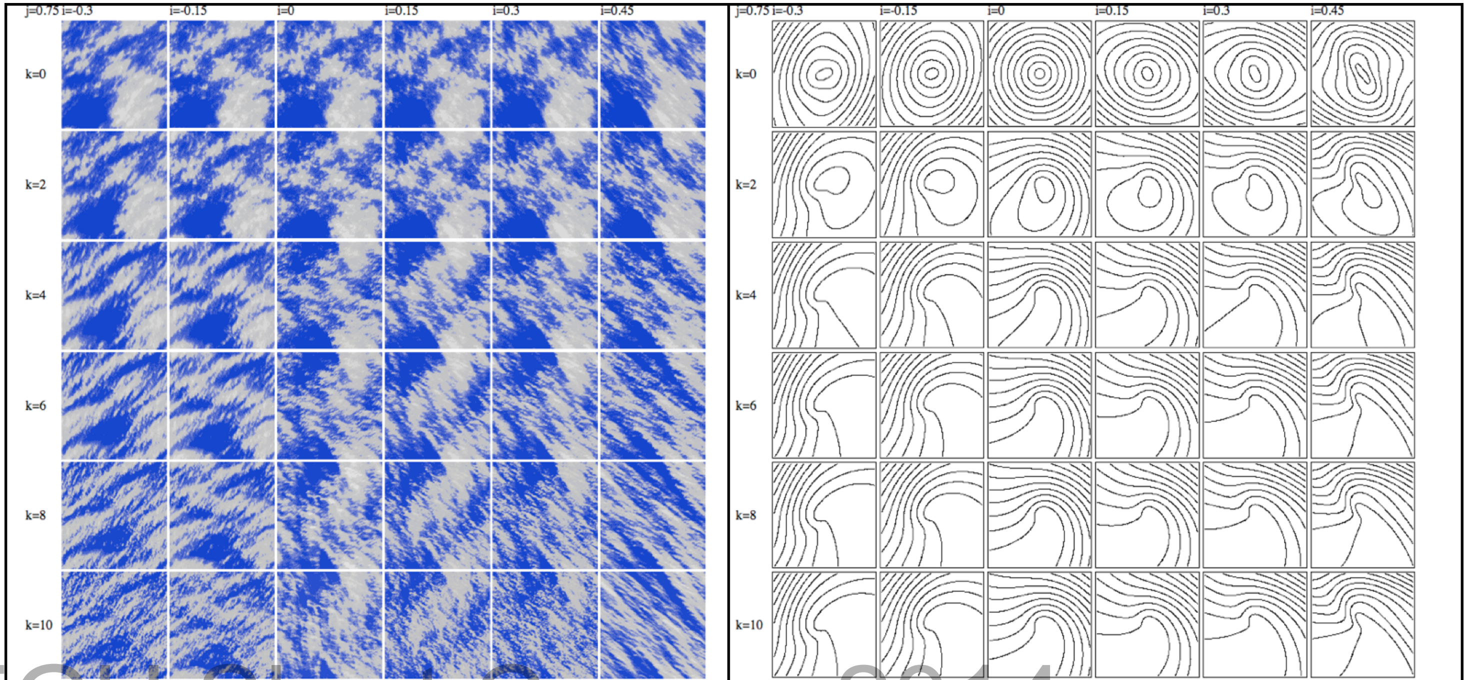
r is increased from -0.3, -0.15, …0.45 left to right, from top to bottom, *k* is increased from 0, 2, 4,..10.

r

k



e = 0.75

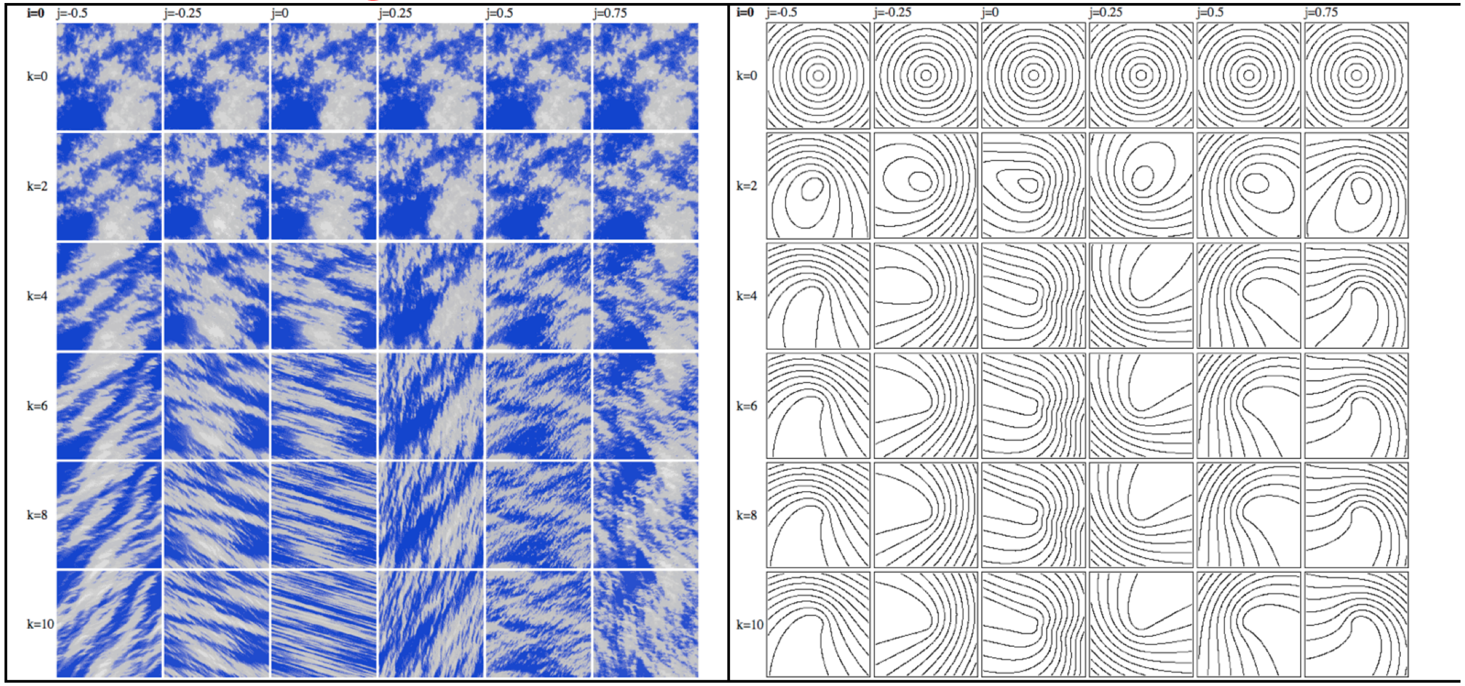


e

$r = 0$

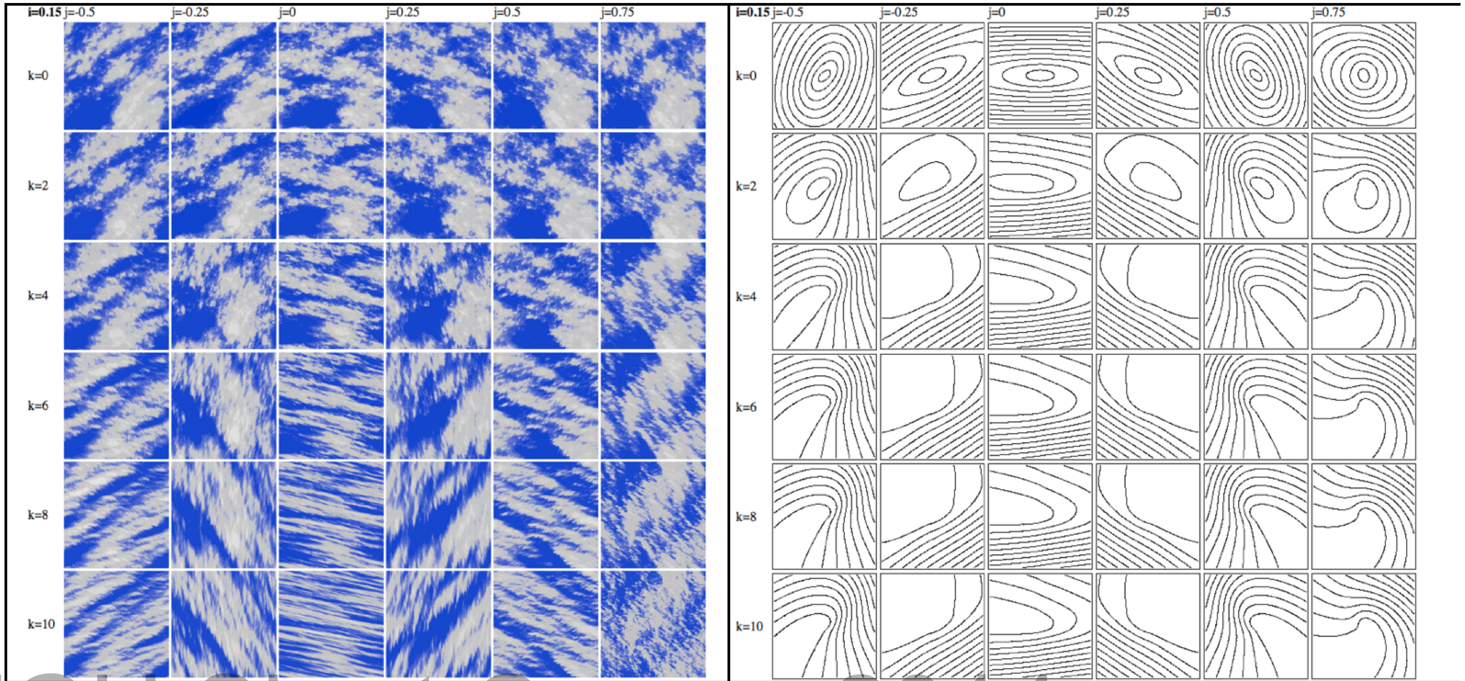
e left to right is: -0.5, -0.25, …0.75.

k



$r = 0.15$

In all rows, from top to bottom, *k* is increased (0, 2, 4,..10),



Changing G

http://www.physics.mcgill.ca/~gang/multifrac/index.htm

multifractal explorer

all for circular sphero-scale

$$G = \begin{pmatrix} 1-i & -j \\ j & 1+i \end{pmatrix}$$

| introduction | multifractals | clouds | topography | misc | movies | glossary | publications |

| isotropic | self-affine| GSI |

GANG

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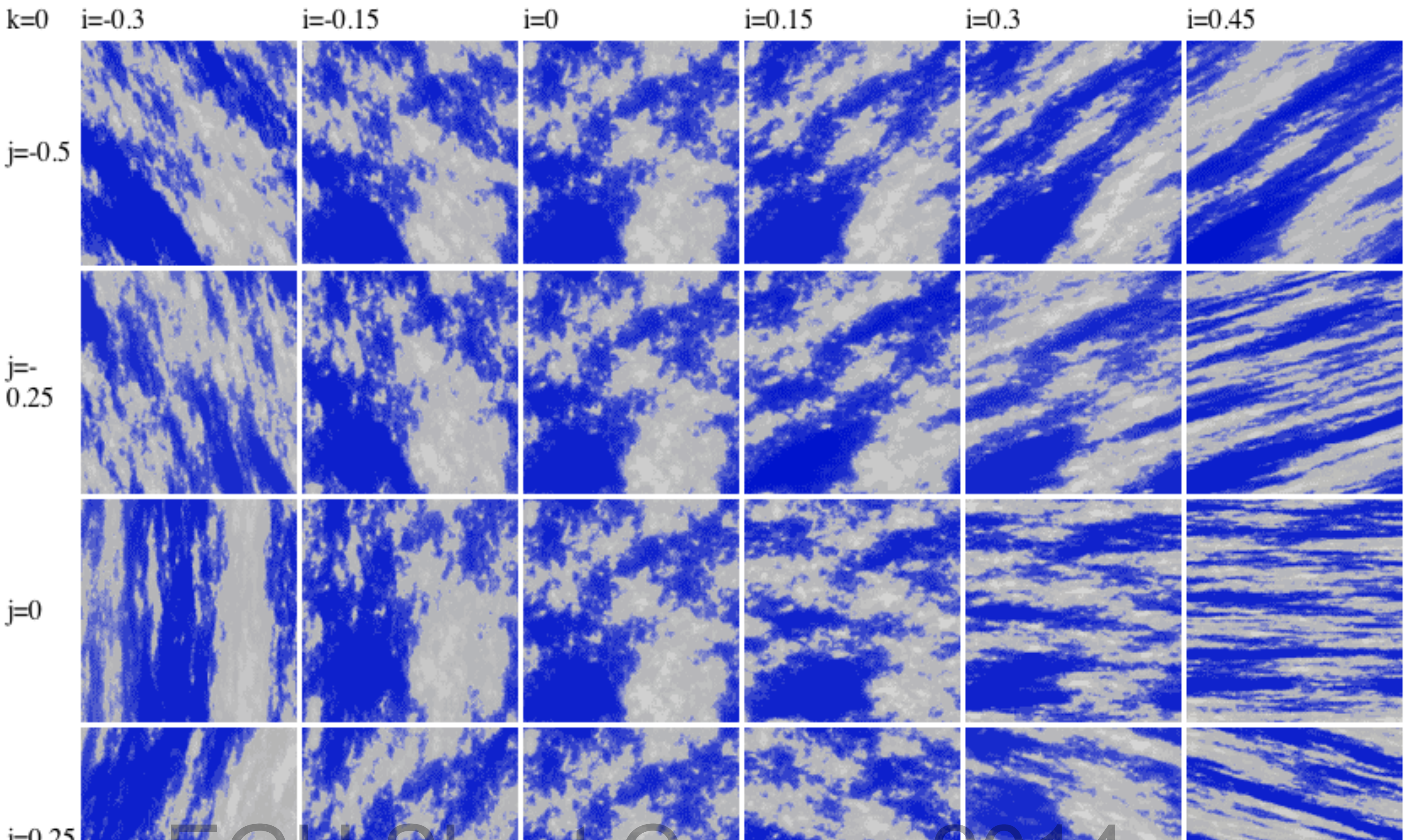
people

projects

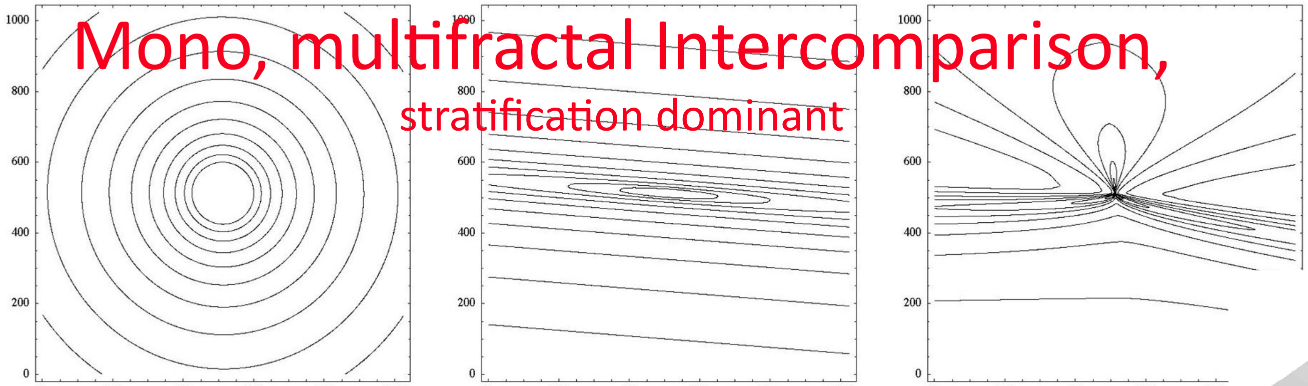
simulations | scale functions



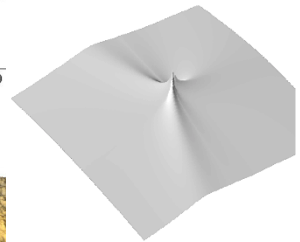
Mono, multifractal Intercomparison,

stratification dominant

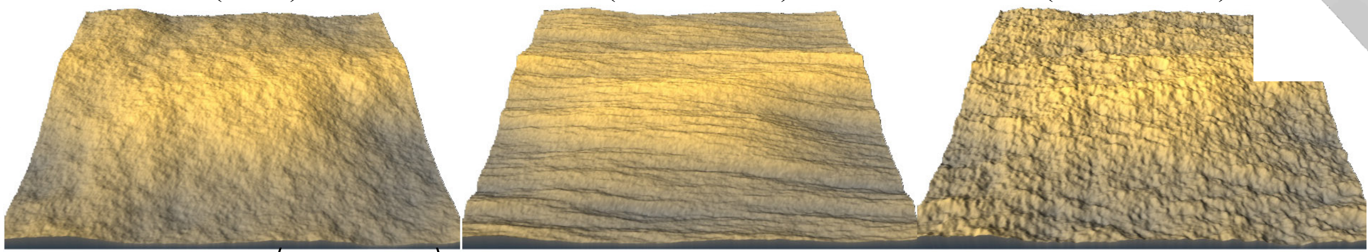
Contours of the s
functions



$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $G = \begin{pmatrix} 0.8 & -0.05 \\ 0.05 & 1.2 \end{pmatrix}$ $G = \begin{pmatrix} 0.8 & -0.05 \\ 0.05 & 1.2 \end{pmatrix}$

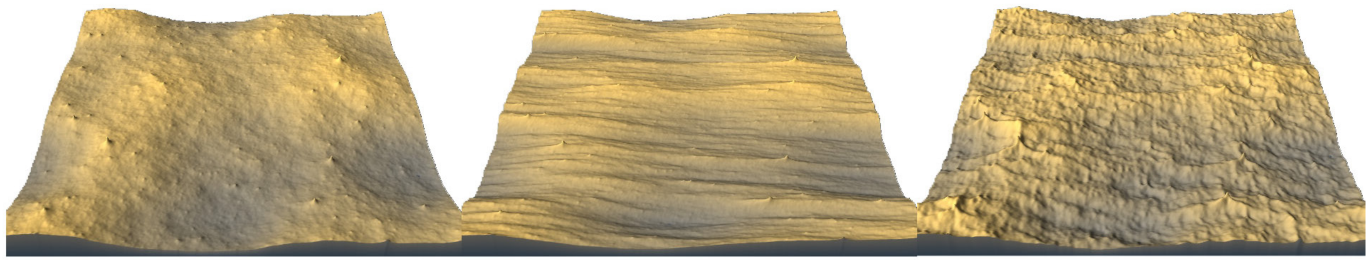


Fractional
Brownian motion,
H=0.7



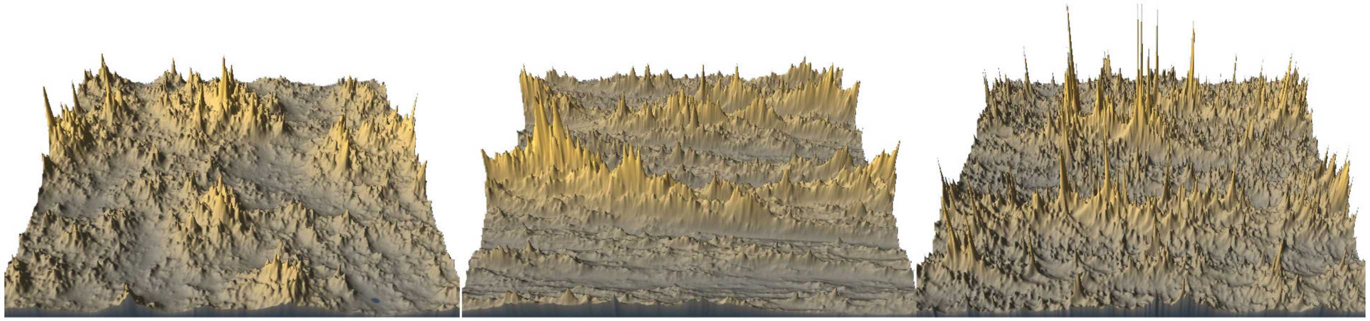
$$\left\langle \Delta h(\Delta r)^q \right\rangle \approx \Delta r^{qH-K(q)} \qquad K(q)=0$$

Fractional Levy
motion,
H=0.7, α =1.8



Multifractal FIF
H=0.7, α =1.8,
C_1=0.12

$$K(q)=\frac{C_1}{\alpha-1}\left(q^{\alpha}-q\right)$$



isotropic Anisotropic no trivial anisotropy Anisotropic with trivial anisotropy

Mono, multifractal Intercomparison

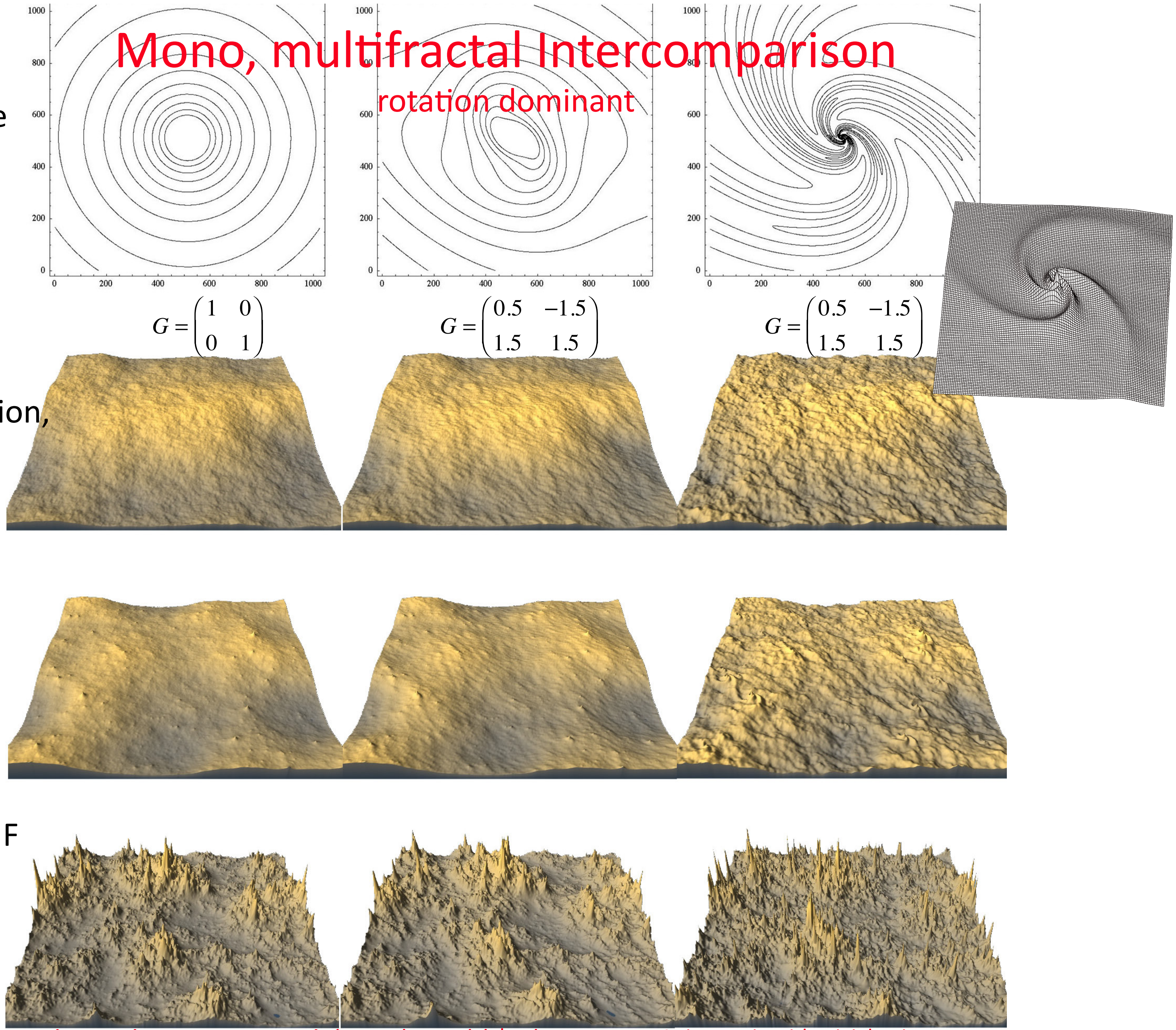
rotation dominant

Contours of the scale functions

Fractional Brownian motion, H=0.7

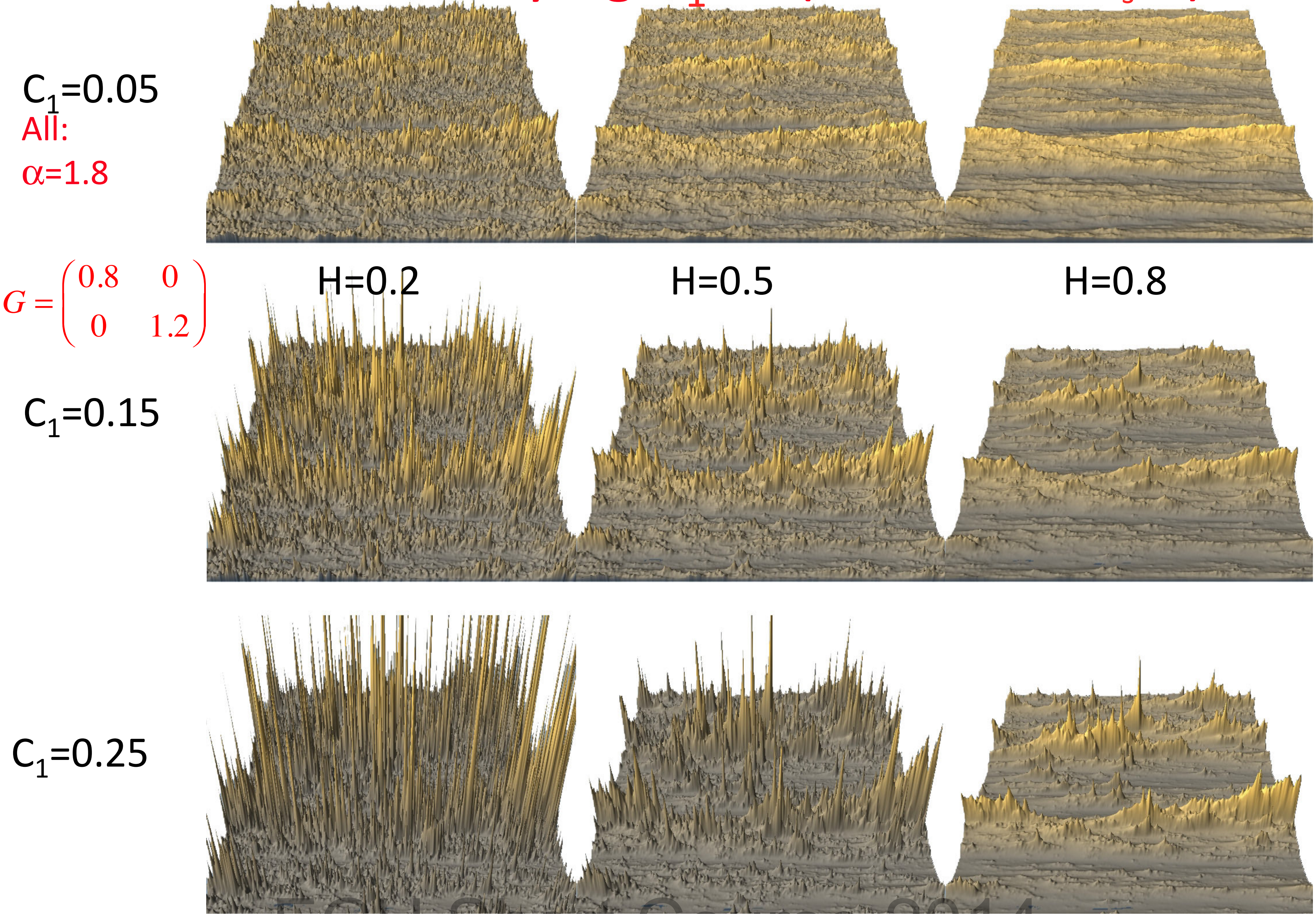
Fractional Levy motion, H=0.7, α=1.8

Multifractal, FIF H=0.7, α =1.8, C_1=0.12

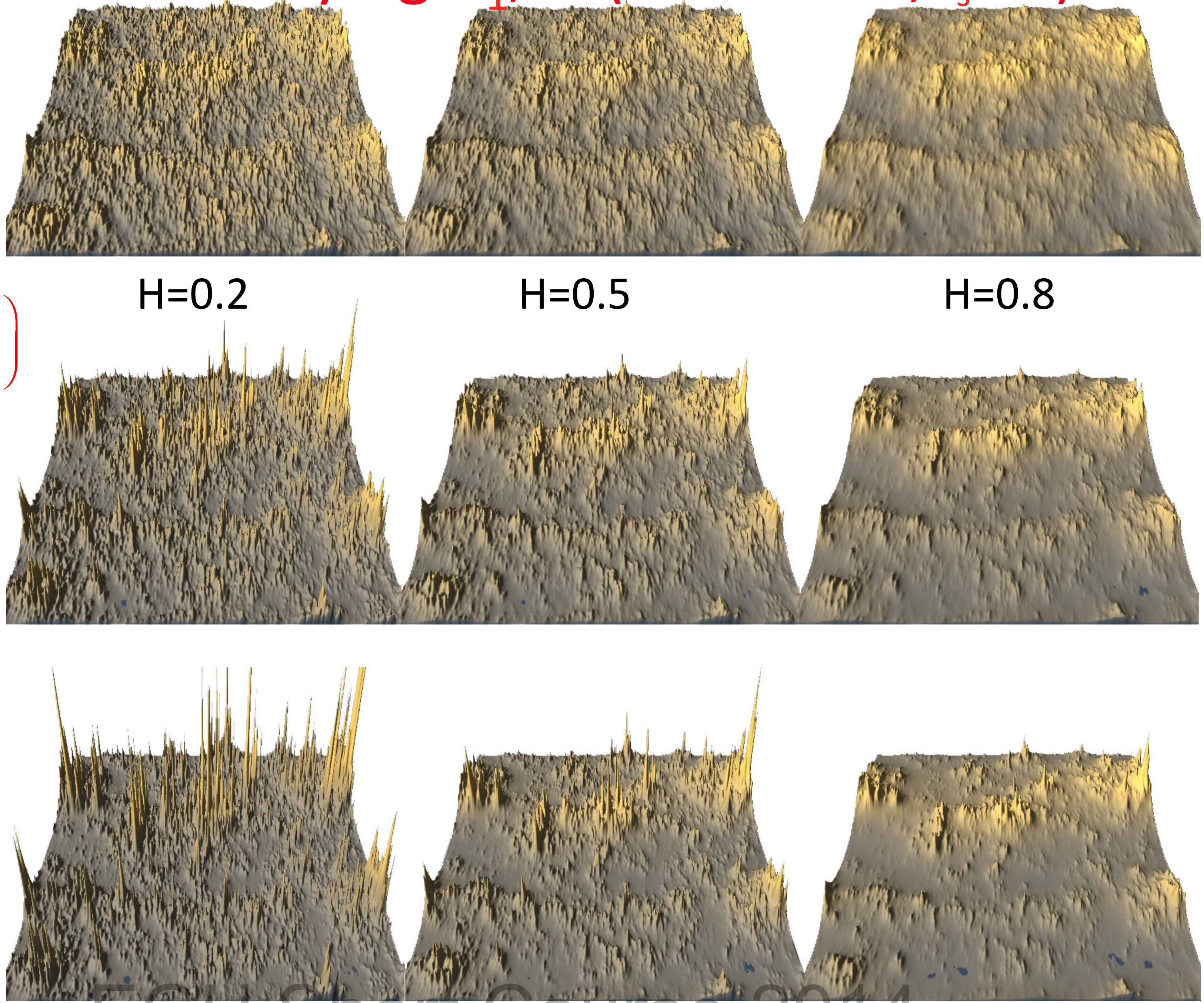


isotropic | Anisotropic no trivial anisotropy | Anisotropic with trivial anisotropy

Effect of varying C_1, H (self-affine, l_s=1)

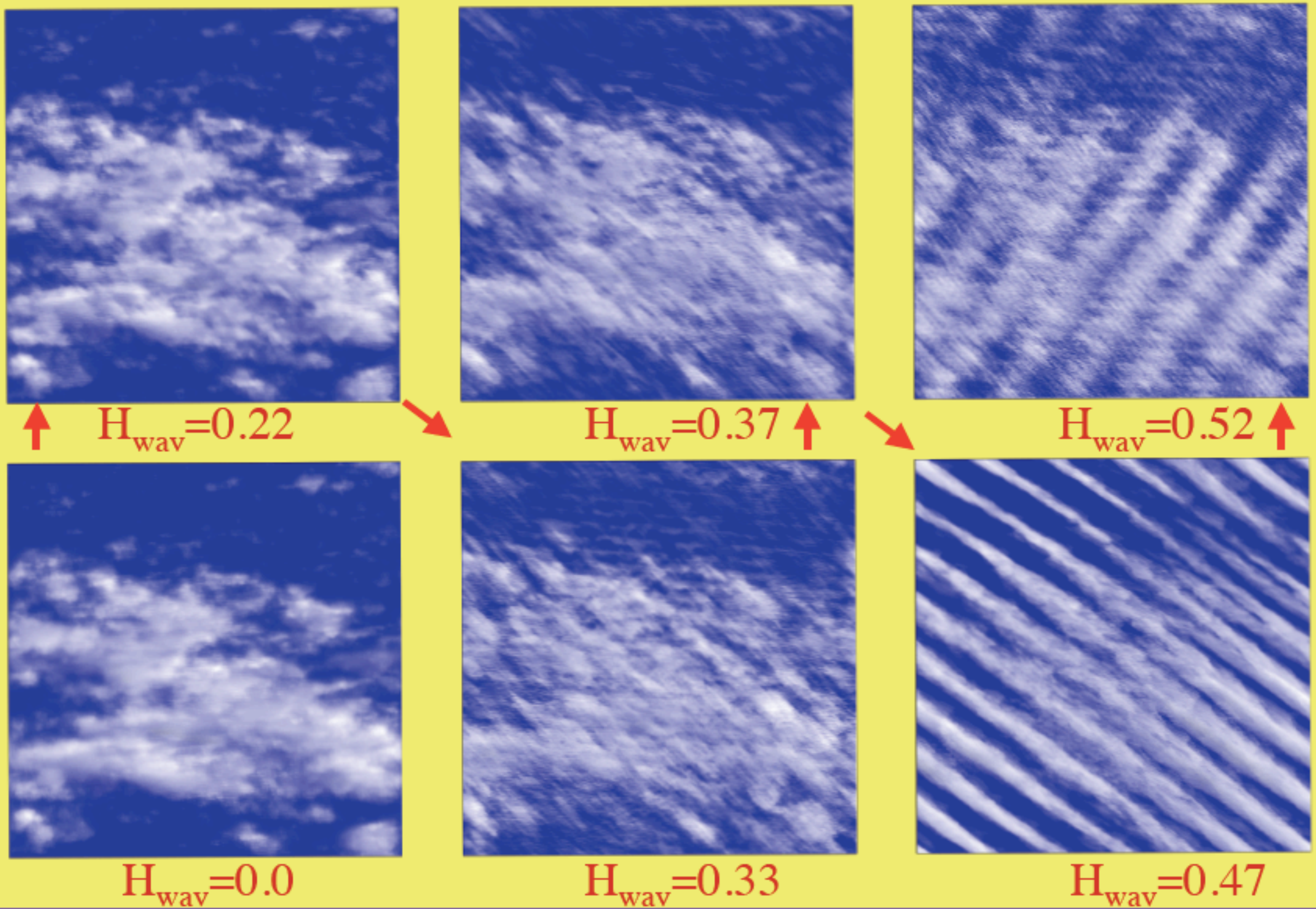


Effect of varying C_1, H (self-affine, l_s=64)



Extension from space to space-time (including waves)

Cascades from localized to increasingly unlocalized structures: $H_{wav} = 1/3 - H_{tur}$



Predictability and stochastic forecasting

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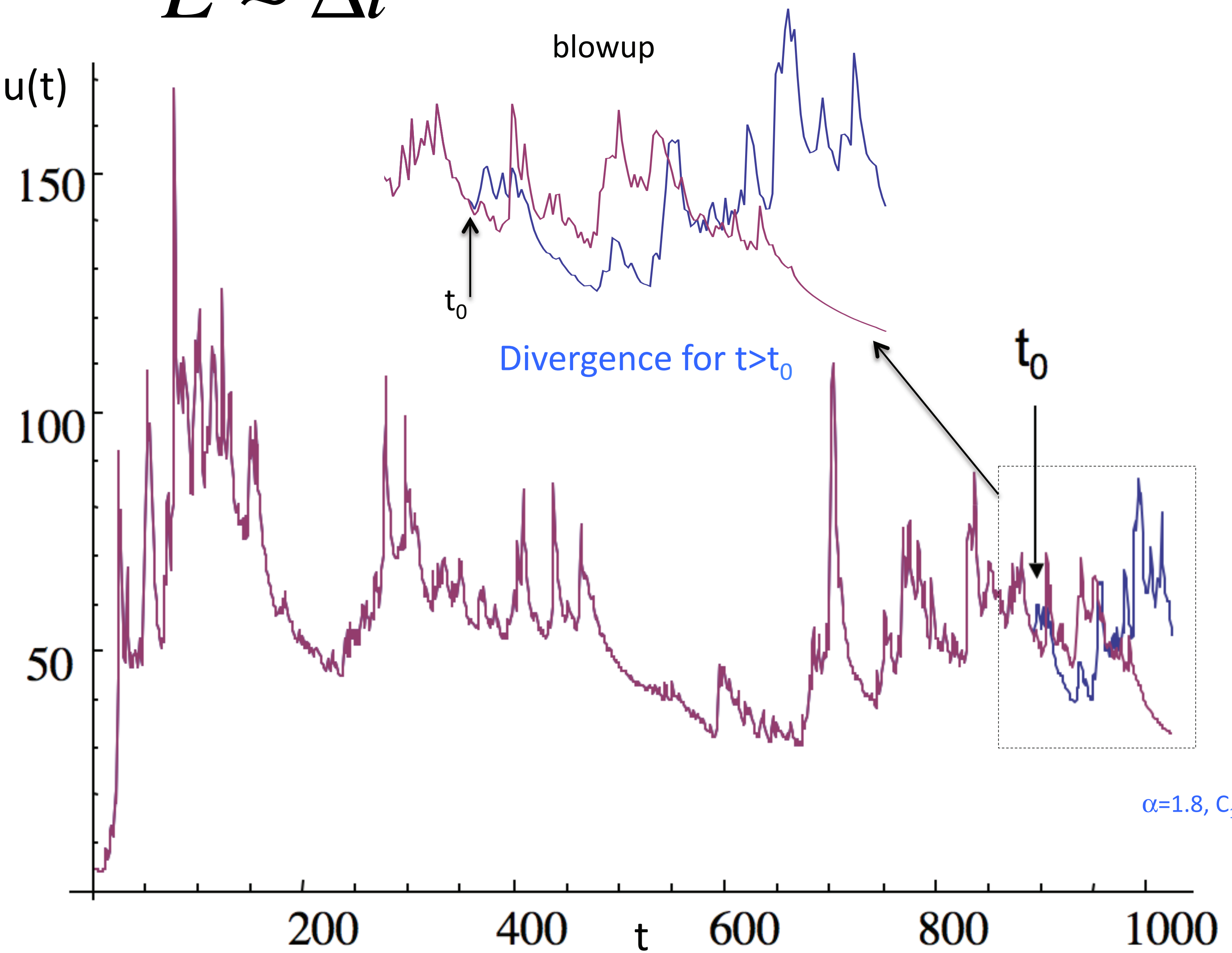
Predictability limits algebraic:

Prediction error: $E \approx \Delta t^{H}$

Lyapunov exponent $\rightarrow \infty$



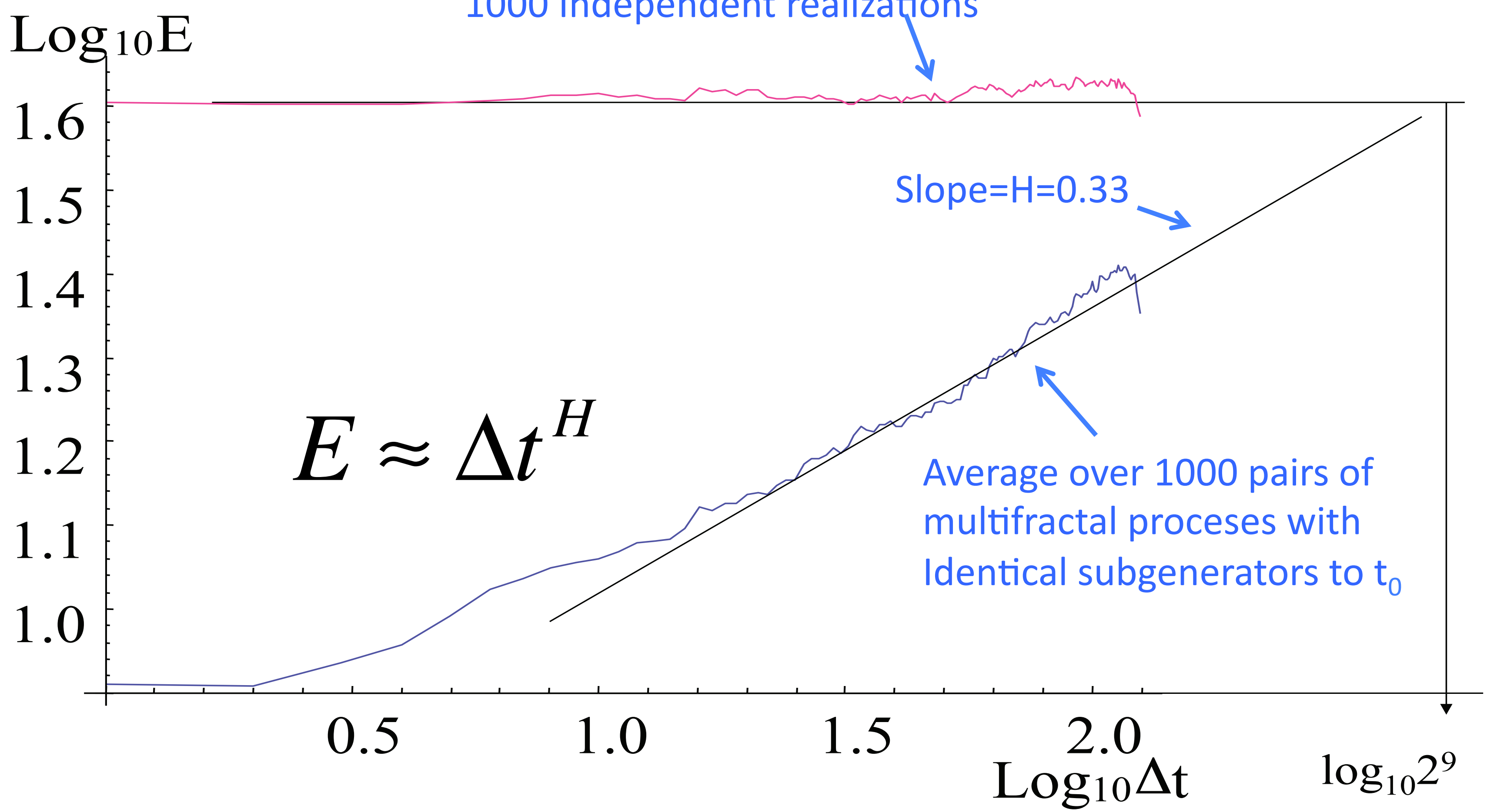
α=1.8, C_1=0.1, H=0.33

Two multifractal proceses with Identical subgenerators to t_0

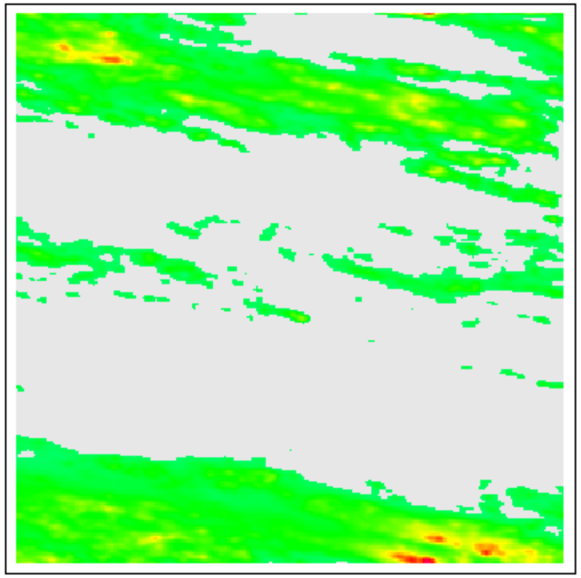
Algebraic divergence of realizations

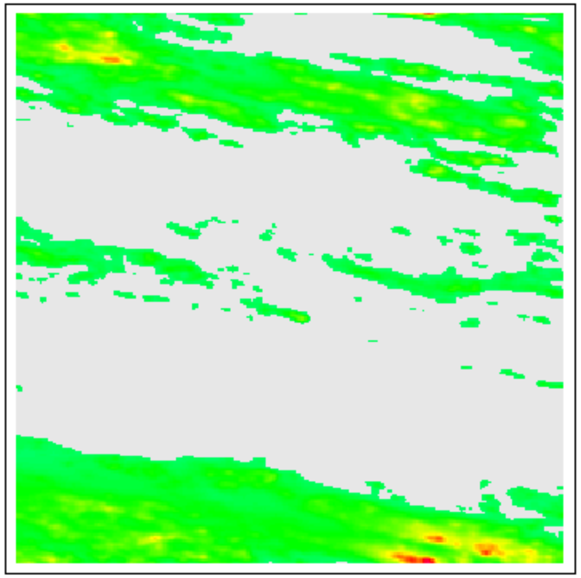
Lyapunov exponent $\rightarrow \infty$

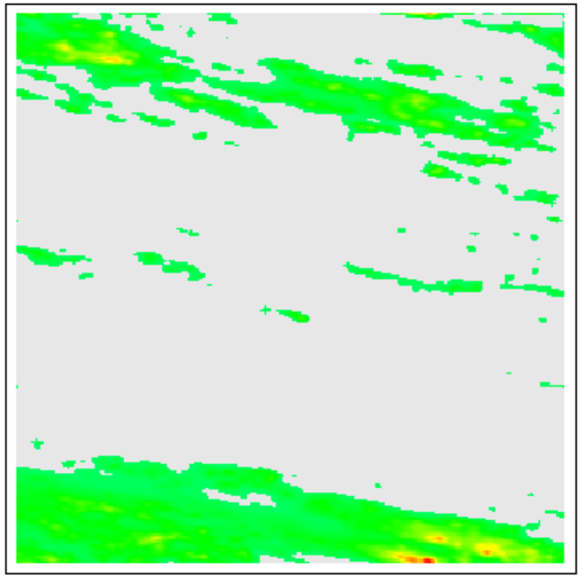
$$E = 2\left\langle \frac{I_2(t_0+\Delta t) - I_1(t_0+\Delta t)}{I_2(t_0+\Delta t) + I_1(t_0+\Delta t)} \right\rangle$$



Space-time Cascades, stochastic nowcasting (rain)







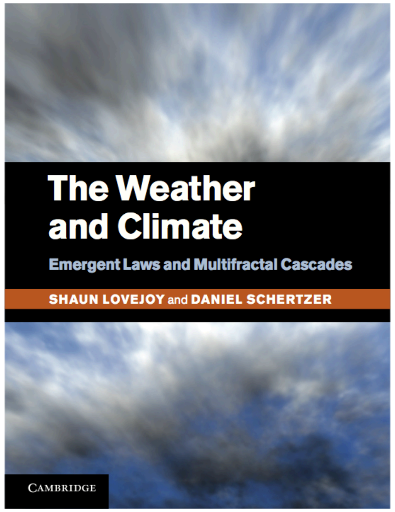
Realization A

Realization B

Forecast based on first 16 time steps

(all same initially)

Conclusions



1. High level stochastic turbulence laws emerge from (deterministic) continuum mechanics at strong nonlinearity

2. Regimes: Weather, macroweather, climate

3. Analysis techniques: Haar fluctuations: accurate yet simple to interpret

4. Generalize classical laws: a) : Intermittency using cascades
b) wide range of scales using anisotropic scaling, stratification

5. Unity of the geosciences: anisotropic scaling, multifractality