Podolsky black hole and the no-hair theorem

R.R. Cuzinatto,[1,2] C.A.M. de Melo,[2,3] L.G. Medeiros,[3,4] B.M. Pimentel,[3] and P.J. Pompeia.[5]

[1] McGill University
[2] Federal University of Alfenas (Brazil)
[3] Instituto de Física Teórica, Sao Paulo State University (Brazil)
[4] Federal University of Rio Grande do Norte (Brazil)
[5] Instituto Tecnológico de Aeronáutica (Brazil)

e-mail: cuzinatto@physics.mcgill.ca; rodrigo.cuzinatto@unifal-g.edu.br

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Abstract

- Podolsky electrodynamics is generalized to curved spacetimes.
- The equations of motion are written for the case of a static spherically symmetric black hole (BH).
- BH exterior solutions are analyzed using Bekenstein's method.
- It is shown that the solutions split-up into two parts:
  (I) a non-homogeneous (asymptotically massless) regime;
  (II) a homogeneous (asymptotically massive) sector.
- The non-homogeneous exterior solutions to the BH are of the Maxwell's type leading to a Reissner-Nordström black hole.
- The only exterior solution (non-homogeneous or otherwise) consistent with the weak and null energy conditions is the Maxwell's one.
- Conclusion: the no-hair theorem is satisfied for Podolsky fields.

Outline

- I. Motivation
- II. Podolsky electrodynamics in curved spacetimes
- III. Podolsky BH exterior solution in the case $b = 0$
- IV. Podolsky BH exterior solution in the case $b \neq 0$
- V. Energy conditions and the no-hair theorem
- VI. Final comments

- The *no-hair theorem/conjecture* states that an exterior solution of a BH is completely characterized by its mass, electric charge and angular momentum.

- All other features of particles – the "*hair*" – have no contribution for the gravitational properties of the black hole.

- The no-hair theorem has been demonstrated for many cases but several results suggest that its validity is limited.

- Bekenstein analyzes BHs in the presence of the massive vector field of Proca electrodynamics.

I. Motivation

- Bekenstein's reasoning does not use an analytical explicit solution for Proca BH.
- Bekenstein shows that the Proca massive vectorial field cannot propagate outside the event horizon.
- On the other hand, Maxwell massless vectorial field does propagate beyond the horizon and gives BH its charge.
- Why can Maxwell field propagate outside the event horizon whilst Proca field can not?

- Difference between Proca field and Maxwell field:
  – Maxwell is massless and gauge invariant;
  – Proca is massive and non-gauge invariant.

- Thus, the previous question may be reformulated as: Can gauge invariance and mass/massless property be the keystones for the difference concerning field propagation in BH physics context?

- We address this question via Podolsky electrodynamics.

- Reasons for choosing Podolsky:
  (I) It is the only second order **gauge theory** for the U(1) group to preserve the linearity of the field equations;
  (II) the solution of the field equations shows Podolsky field splits in two modes: a **massive** mode and a massless one.

I. Motivation

- Podoslky Lagrangian exhibits Maxwell term plus a term scaling with the second derivative of the gauge field, leading to fourth-order field equations.

- The goal is to study the propagation of vector fields outside the event horizon for a Podolsky BH thereby determining if the **no-hair theorem** remains valid for Podolsky BH.

- Outline of the approach:
  – Generalize Podolsky electrodynamics to curved spacetime;
  – Investigate the properties of the exterior spherically symmetric solution using Bekenstein's approach;
  – Consider these properties under the scrutiny of the null and weak energy conditions.

Podolsky electrodynamics in flat spacetimes is derived from

$$\mathcal{L}_m^{\text{flat}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\mu F^{\mu\nu}\partial_\rho F^\rho_\nu\,, \tag{1}$$

where the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.
In a curved spacetime, $\mathcal{L}_m^{\text{flat}}$ is generalized to:

$$\mathcal{L}_m = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} + \frac{a^2}{2}\nabla_\beta F^{\alpha\beta}\nabla_\gamma F_\alpha{}^\gamma + \frac{b^2}{2}\nabla_\beta F^{\alpha\gamma}\nabla^\beta F_{\alpha\gamma}. \tag{2}$$

where the principle of general covariance $\eta_{\mu\nu} \to g_{\mu\nu}$ is assumed. The term scaling with a^2 is immediately understood from the minimal coupling prescription, $\partial_\mu \to \nabla_\mu$. The additional term scaling with b^2 is allowed by the requirement of gauge invariance under U(1) group.

II. Podolsky electrodynamics in curved spacetimes

The dynamics is obtained from Einstein-Hilbert–Podolsky action

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[-R + 4\mathcal{L}_m \right], \tag{3}$$

From the variation with respect to A_μ we obtain Podolsky equations of generalized electromagnetism in curved spacetime:

$$\nabla_\nu \left[F^{\mu\nu} - \left(a^2 + 2b^2 \right) H^{\mu\nu} + 2b^2 S^{\mu\nu} \right] = 0, \tag{4}$$

where $H^{\mu\nu}$ and $S^{\mu\nu}$ are antisymmetric tensors defined as

$$\begin{aligned} H^{\mu\nu} &\equiv \nabla^\mu K^\nu - \nabla^\nu K^\mu, & (5) \\ S^{\mu\nu} &\equiv F^{\mu\sigma} R_\sigma{}^\nu - F^{\nu\sigma} R_\sigma{}^\mu + 2R^\mu{}_\sigma{}^\nu{}_\beta F^{\beta\sigma}, & (6) \end{aligned}$$

with

$$K^\mu \equiv \nabla_\gamma F^{\mu\gamma}. \tag{7}$$

II. Podolsky electrodynamics in curved spacetimes

Variation of S with respect to $g^{\mu\nu}$ leads to Einstein equations of gravity:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}, \tag{8}$$

where $T_{\mu\nu} = \left(T^M_{\mu\nu} + T^a_{\mu\nu} + T^b_{\mu\nu} \right)$ and

$$\begin{aligned}
T^M_{\mu\nu} &= \frac{1}{4\pi} \left[F_{\mu\sigma} F^{\sigma}{}_{\nu} + g_{\mu\nu} \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} \right], && (9) \\
T^a_{\mu\nu} &= \frac{a^2}{4\pi} \left[g_{\mu\nu} F_{\beta}{}^{\gamma} \nabla_{\gamma} K^{\beta} + \frac{g_{\mu\nu}}{2} K^{\beta} K_{\beta} \right. \\
&+ \left. 2F_{(\mu}{}^{\alpha} \nabla_{\nu)} K_{\alpha} - 2F_{(\mu}{}^{\alpha} \nabla_{\alpha} K_{\nu)} - K_{\mu} K_{\nu} \right], && (10) \\
T^b_{\mu\nu} &= \frac{b^2}{2\pi} \left[-\frac{1}{4} g_{\mu\nu} \nabla^{\beta} F^{\alpha\gamma} \nabla_{\beta} F_{\alpha\gamma} + F^{\gamma}_{(\mu} \nabla^{\beta} \nabla_{\beta} F_{\nu)\gamma} \right. \\
&+ \left. F_{\gamma(\mu} \nabla_{\beta} \nabla_{\nu)} F^{\beta\gamma} - \nabla_{\beta} \left(F_{\gamma}{}^{\beta} \nabla_{(\mu} F_{\nu)}{}^{\gamma} \right) \right]. && (11)
\end{aligned}$$

The notation $_{(\mu\nu)}$ indicates symmetrization with respect to indices $\mu\nu$.

Now we consider this system of field equations in the particular case of (static) spherical symmetry. The line element can be written as

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \ , \tag{12}$$

while the field strength is given by

$$F_{\mu\nu} = E(r) \left[\delta^1_\mu \delta^0_\nu - \delta^0_\mu \delta^1_\nu \right] . \tag{13}$$

In view of this parametrization, Podolsky equations (4) are:

$$E - \left(a^2 + 2b^2\right) \partial_1 K_0 + 2b^2 S_{10} = C \frac{e^{\frac{(\nu+\lambda)}{2}}}{r^2}, \tag{14}$$

where C is an arbitrary integration constant and

$$K_0 = \frac{e^{\frac{\nu-\lambda}{2}}}{r^2} \partial_1 \left(r^2 e^{-\frac{(\nu+\lambda)}{2}} E \right), \tag{15}$$

$$S_{10} = E e^{-\lambda} \left(\frac{\nu' - \lambda'}{r} \right) . \tag{16}$$

Prime denotes derivative with respect to r.

In a *flat spacetime* the spherically symmetric solution to the electromagnetic sector, Eq. (14), depends on

$$A_\mu = (A_0, 0, 0, 0)$$

which is a function of $x^1 = r$ solely. It is the result by Podolsky:

$$A_0(r) = \frac{C_1}{r} - \frac{C_2 e^{-\frac{r}{r_p}}}{r}, \tag{17}$$

with $r_p^2 = a^2 + 2b^2$.

The potential bears the usual Maxwell term plus a Yukawa term in Minkowski spacetime.

The non-null components of Einstein Eq. (8) are given by

$$e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2} = 8\pi T_0^0, \tag{18}$$

$$-e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)+\frac{1}{r^2} = 8\pi T_1^1, \tag{19}$$

$$-\frac{1}{4r}e^{-\lambda}\left[\left(\nu'-\lambda'\right)\left(2+r\nu'\right)+2r\nu''\right] = 8\pi T_2^2, \tag{20}$$

where

$$T_0^0 = -\frac{g^{00}g^{11}}{8\pi}\left\{E\left[E-2\left(a^2+2b^2\right)\partial_1 K_0+4b^2 S_{10}\right] + \frac{a^2K_0^2}{g^{11}}+2b^2g^{11}\left[\left(\frac{K_0}{g^{11}}+\frac{2E}{r}\right)^2+\frac{2E^2}{r^2}\right]\right\}, \tag{21}$$

$$T_1^1 = -\frac{g^{00}g^{11}}{8\pi}\left\{E\left[E-2\left(a^2+2b^2\right)\partial_1 K_0+4b^2 S_{10}\right]\right.$$
$$\left.-\frac{a^2K_0^2}{g^{11}}-2b^2g^{11}\left[\left(\frac{K_0}{g^{11}}+\frac{2E}{r}\right)^2+\frac{2E^2}{r^2}\right]\right\}, \tag{22}$$
$$T_2^2 = \frac{g^{00}g^{11}}{8\pi}\left\{E^2-a^2\left[2E\partial_1 K_0-\frac{K_0^2}{g^{11}}\right]+\frac{2b^2K_0^2}{g^{11}}\right.$$
$$\left.-4b^2g^{11}\left[\left(\frac{K_0}{g^{11}}+\frac{2E}{r}\right)^2+\frac{ES_{10}}{2g^{11}}+\frac{2E^2}{r^2}\right]\right\}. \tag{23}$$

The solution to the set of Eqs. (14)-(23) give the line element of Podolsky BH. This solution is still to be found. We do not fulfill this task here, but the exact solution is not required to study the validity of the no-hear theorem. Instead, we follow Bekenstein's approach to address this point. This is done for both $b = 0$ and $b \neq 0$ cases.

III. Podolsky BH exterior solution in the case $b = 0$

By taking $b = 0$ in Podolsky field equation (4), contracting it with A_μ and integrating the result in the 4-volume exterior to BH, leads to:

$$I_1 + I_2 + I_3 = 0, \tag{24}$$

where

$$I_1 = \int \left\{ \sqrt{-g} \left(\partial_\nu A_\mu \right) F^{\mu\nu} - a^2 \left[K_\beta \partial_\alpha \left(\sqrt{-g} g^{\mu\beta} g^{\nu\alpha} \partial_\nu A_\mu \right) - W \right] \right\} d^4 x,$$

$$I_2 = a^2 \oint \sqrt{-g} g^{\mu\beta} g^{\nu\alpha} K_\beta \partial_\nu A_\mu dS_\alpha,$$

$$I_3 = -\oint \left[\sqrt{-g} A_\mu \left(F^{\mu\nu} - a^2 H^{\mu\nu} \right) \right] dS_\nu,$$

with

$$W = \sqrt{-g} g^{\mu\beta} g^{\nu\alpha} \left(\partial_\nu A_\mu \right) \left(\partial_\beta K_\alpha \right).$$

III. Podolsky BH exterior solution in the case $b = 0$

Under spherical symmetry the exterior 4-volume is limited by the horizon r_H, by r_∞ ($r \to \infty$) and by the past and future infinite times $t \to \pm\infty$. In this situation, $A_\mu = (A_0, 0, 0, 0)$ and depends only on $x^1 = r$. Consequently,

$$
\begin{aligned}
I_1 &= \int \sqrt{-g}\, g^{00} \left[-g^{11} E^2 + (aK_0)^2 \right] d^4x && (25) \\
I_2 &= a^2 \left[\int_{\Omega_{r_H}} + \int_{\Omega_{r_\infty}} \right] \sqrt{-g}\, g^{00} g^{11} K_0 E dS_1 && (26) \\
I_3 &= \left[\int_{\Omega_{r_H}} + \int_{\Omega_{r_\infty}} \right] \frac{\sqrt{-g} A_0}{g_{00} g_{11}} \left[E - a^2 \partial_1 K_0 \right] dS_1 && (27)
\end{aligned}
$$

where Ω_{r_H} (Ω_{r_∞}) means that the integral is performed on the surface of constant $r = r_H$ ($r = r_\infty$).

III. Podolsky BH exterior solution in the case $b = 0$

Let us analyze the properties of I_2 and I_3 at both r_H and r_∞.

When $r \to \infty$ the spacetime becomes flat (Minkowski): $\sqrt{-g}dS_1 \approx r^2 dS$. Moreover, Podolsky field equation in flat spacetime is solved by

$$A_0 \approx \frac{C_1}{r} - \frac{C_2 e^{-\frac{r}{a}}}{r}$$

Using this in Eqs. (26) and (27) shows the integrals over Ω_{r_∞} appearing in I_2 and I_3 are null.

The case for the integrals over Ω_{r_H} is more complicated.

First, we recall that the trace of the energy-momentum tensor for $b = 0$ is:

$$T = \frac{a^2}{4\pi} g^{00} (K_0)^2 .$$

This is a scalar with physical meaning – it is associated with the energy of the system –, hence it must be finite on the horizon. On the other hand, $g^{00}(r_H) \to \infty$. Consequently, $(K_0)^2$ must approach zero at least at the same rate as g_{00} in order to guarantee a finite value for T on the horizon. In this case,

$$I_2 \sim a^2 \int_{\Omega_{r_H}} E\, r^2 \sin\theta \sqrt{-\frac{g_{00}}{g^{11}} g^{11} g^{00}} \sqrt{g_{00}}\, dS_1 \sim 0, \tag{28}$$

due to the facts that the electric field is finite on $r = r_H$ and $g^{11}(r_H) = 0$. We conclude that the integral I_2 is null on the horizon.

III. Podolsky BH exterior solution in the case $b = 0$

The analysis of I_3 begins by taking $b = 0$ in Podolsky field equation (14) for electromagnetism on a spherically symmetric background:

$$E - a^2 \partial_1 K_0 = C \frac{e^{\frac{\nu+\lambda}{2}}}{r^2} \tag{29}$$

Replacing this back into Eq. (27) for integral I_3:

$$I_3 = -C \int_{\Omega_{r_H}} A_0 \sin\theta \, dS_1 . \tag{30}$$

Eq. (29) is a second order differential equation for the field E. It may be homogeneous or non-homogeneous according to values of C.

The homogeneous solution $E_{(h)}$ corresponds to set $C = 0$. This choice renders $I_3 = 0$ automatically.

III. Podolsky BH exterior solution in the case $b = 0$

We have just proved that both I_3 vanish for the homogeneous solutions $E_{(h)}$. Previously, it was shown that $I_2 = 0$. Then, from Eq.(24), $I_1 + I_2 + I_3 = 0$, we conclude

$$I_1 = \int \sqrt{-g} \left[-g^{11} g^{00} E_{(h)}^2 + g^{00} \left(a K_{0(h)} \right)^2 \right] d^4 x = 0. \qquad (31)$$

Since $g_{00} > 0$ and $g_{11} < 0$ in the region exterior to the horizon, each term in the square-brackets of I_1 is positive-definite. Hence, the only possible solution to Eq. (31) is:

$$E_{(h)} = K_{0(h)} = 0 \qquad \text{for} \qquad r \geq r_H . \qquad (32)$$

III. Podolsky BH exterior solution in the case $b = 0$

Eq. (32), $E_{(h)} = 0$, should also be true in the asymptotic regime $r \gg r_H$.

This regime is consistent with the flat spacetime limit, in which case the electromagnetic field equation (29) gives:

$$E_{(h)} \simeq -C_1 \frac{e^{-\frac{r}{a}}}{r^2} \left(1 + \frac{r}{a}\right).$$

This is the Podoslky electric field solution that would constitute a "hair" to the BH. However, Eq. (32) demands $C_1 = 0$, thus imposing the **no-hair** requirement: no Podolsky field exists outside the horizon.

Notice, however, that the demonstration fails for the non-homogeneous solutions $E_{(Nh)}$ – corresponding to $C \neq 0$ in Eq. (29). This is because $I_3 \sim C$ is different from zero in this case. This case is considered bellow.

III. Podolsky BH exterior solution in the case $b = 0$

The non-homogeneous solution to Podolsky field equation (29) is:

$$E_{(Nh)} = C \frac{e^{\frac{\nu+\lambda}{2}}}{r^2}, \tag{33}$$

from which we verify that $K_{0(Nh)} = 0$ by using Eq.(15),

$$K_0 = \frac{e^{\frac{\nu-\lambda}{2}}}{r^2} \partial_1 \left(r^2 e^{-\frac{(\nu+\lambda)}{2}} E \right)$$

If we replace this result in Einstein equations – Eqs. (18,19) – we obtain Reissner-Nordström solution; in this case, constant C is the electric charge. Once charge is not considered "hair" of a BH, there is no hair associated to the non-homogeneous solution $E_{(Nh)}$ too.

We conclude the exterior solution of the Einstein-Podolsky BH for $b = 0$ is independent of parameter a. This corroborates the no-hair theorem.

IV. Podolsky BH exterior solution in the case $b \neq 0$

Analogously to previous section, we contract Podoslsky field equation (4) with A_μ and integrate over the region exterior to the BH's horizon. This leads to

$$I_{1b} + I_{2b} + I_{3b} = 0 \tag{34}$$

where

$$I_{1b} = \int \sqrt{-g} g^{00} \left[-g^{11} E^2 + \left(a^2 + 2b^2\right) \left(K_0\right)^2 + 2b^2 \left(g^{11} E\right)^2 \left(\frac{\nu' - \lambda'}{r} \right) \right] d^4 x, \tag{35}$$

$$I_{2b} = \left(a^2 + 2b^2\right) \int_{r_H} \sqrt{-g} g^{00} g^{11} K_0 E dS_1, \tag{36}$$

$$I_{3b} = \int_{\Omega_{r_H}} \sqrt{-g} g^{00} g^{11} A_0 \left[E - \left(a^2 + 2b^2\right) \partial_1 K_0 + 2b^2 S_{10} \right] dS_1. \tag{37}$$

We have already imposed spherical symmetry and the fact that integrals over Ω_{r_∞} vanish.

IV. Podolsky BH exterior solution in the case $b \neq 0$

Subtracting Einstein equations (19) and (18) implies in:

$$\lambda' + \nu' = 2rg^{00}g_{11}\left[a^2\left(K_0\right)^2 + 2b^2\left[\left(K_0 + \frac{2g^{11}E}{r}\right)^2 + \frac{2\left(g^{11}E\right)^2}{r^2}\right]\right]. \tag{38}$$

This result is then used to rewrite Eq. (35) for I_{1b} as

$$\begin{aligned} I_{1b} = &\int \sqrt{-g}\,g^{00}\left[-g^{11}E^2 + \left(a^2 + 2b^2\right)\left(K_0\right)^2 + \frac{4b^2}{r}\frac{g_{00}'}{g_{00}}\left(g^{11}E\right)^2\right]d^4x \\ &- 4b^2\int \sqrt{-g}\,g^{11}\left(g^{00}E\right)^2 r\left[a^2\left(K_0\right)^2 + 2b^2\left[\left(K_0 + \frac{2g^{11}E}{r}\right)^2 + \frac{2\left(g^{11}E\right)^2}{r^2}\right]\right]d^4x. \end{aligned} \tag{39}$$

If we assume $g_{00}' \geq 0$ in the region exterior to the horizon then each term of I_{1b} is positive-definite. From a physical point of view, this hypothesis is the only acceptable one once $g_{00}' < 0$ is associated to repulsive gravity.

IV. Podolsky BH exterior solution in the case $b \neq 0$

Indeed, if there exists a sub-region $r_1 < r < r_2$ exterior to r_H where $g'_{00} < 0$, then particles moving radially with low velocities would experience a repulsive force given by

$$\frac{d^2 r}{dt^2} \simeq -c^2 \Gamma^1_{00} \Rightarrow \frac{d^2 r}{dt^2} \simeq c^2 g^{11} g'_{00}.$$

Thus, we would have a region where the particle is impelled to move away from the origin.

An additional non-physical effect appearing if $g'_{00} < 0$ is the gravitational blue-shift of a electromagnetic wave emitted at r_1 and detected at $r_2 > r_1$.

Now that we have characterized I_{1b}, the next step is to show under which conditions the integrals I_{2b} and I_{3b} are null.

IV. Podolsky BH exterior solution in the case $b \neq 0$

The trace of the energy-momentum tensor with $b \neq 0$,

$$
\begin{aligned}
T = & \frac{\left(a^2 - 2b^2\right)}{4\pi} g^{00} \left(K_0\right)^2 \\
& + \frac{b^2}{\pi} g^{00} \left[\left(g^{11} E\right)^2 \left[\frac{3}{2} \frac{\nu' - \lambda'}{r} - \frac{4}{r^2} \right] + g^{11} E \left(\partial_1 K_0 - \frac{4 K_0}{r} \right) \right], \quad (40)
\end{aligned}
$$

follows from Eqs. (21)-(22).

In order to keep T finite at r_H, K_0 must tend to zero at least as $\sqrt{g_{00}}$. Then, from Eqs. (36) and (28) for I_{2b} and I_2, we conclude:

$$I_{2b} \sim I_2 \sim 0.$$

Podolsky field equation (14) may be used to cast integral I_{3b} in the form:

$$I_{3b} = -C \int_{\Omega_{r_H}} A_0 \sin\theta dS_1 .$$

IV. Podolsky BH exterior solution in the case $b \neq 0$

By arguments identical to those in the previous section, the homogeneous solutions $E_{(h)}$ of Eq. (14) ($C = 0$) imply $I_{3b} = 0$.

Thus, Eq. (34), $I_{1b} + I_{2b} + I_{3b} = 0$ implies in $I_{1b} = 0$. Under the hypothesis $g'_{00} \geq 0$ at $r \geq r_H$ it follows again that

$$E_{(h)} = 0 \qquad \text{for} \qquad r \geq r_H. \tag{41}$$

Hence, the only solution of Eq.(14) that could possibly be non-null is the non-homogeneous solution $E_{(Nh)}$, whose asymptotic behavior ($r \gg r_H$) is of the type C/r^2. But this is again Maxwell solution of a massless photon, in which case the no-hair theorem is satisfied.

This asymptotic behavior does not guarantees that $E_{(Nh)}$ is still hairless towards the horizon. Next section gives an argument contrary to the existence of a non-null $E_{(Nh)}$ different from Maxwell's.

V. Energy conditions and the no-hair theorem

Null energy condition (NEC) and weak energy condition (WEC) can be used to show that the only non-trivial solution exterior to the horizon is the non-homogeneous solution $E_{(Nh)}$ obtained with $b = 0$, i.e. Mawxell solution leading to Reisner-Nordström BH: no hair is allowed.

The energy-momentum tensor $T_{\mu\nu}$ respects the null (weak) energy condition if the inequality

$$T_{\mu\nu} k^{\mu} k^{\nu} \geq 0 \tag{42}$$

holds for every null (timelike) vector k^{μ}. For the particular case of a diagonal T^{μ}_{ν}, the energy conditions are simply

$$\rho + p_i \geq 0 \quad \text{with} \quad i = 1, 2, 3, \tag{43}$$

where $\rho \equiv T^0_0$ is the energy density and $p_1 \equiv -T^1_1$, $p_2 \equiv -T^2_2$ e $p_3 \equiv -T^3_3$ are the principal pressures. The WEC is satisfied if, besides Eq. (43), we have:

$$\rho \geq 0. \tag{44}$$

V. Energy conditions and the no-hair theorem

Eqs. (14), (21) and (22) for Podolsky field equation, T_0^0 and T_1^1 make it possible to rewrite the Eqs. (43,44) for NEC and WEC as

$$\rho = -\frac{g^{00}g^{11}}{8\pi}E\left[\frac{2Ce^{\frac{(\nu+\lambda)}{2}}}{r^2} - E\right] - \frac{a^2}{8\pi}g^{00}\left(K_0\right)^2 - \frac{b^2}{4\pi}g^{00}\left[\left(K_0 + \frac{2g^{11}E}{r}\right)^2 + \frac{2\left(g^{11}E\right)^2}{r^2}\right] \geq 0, \tag{45}$$

$$\rho + p_1 = -\frac{a^2}{4\pi}g^{00}\left(K_0\right)^2 - \frac{b^2}{2\pi}g^{00}\left[\left(K_0 + \frac{2g^{11}E}{r}\right)^2 + \frac{2\left(g^{11}E\right)^2}{r^2}\right] \geq 0. \tag{46}$$

Eq. (46) is satisfied in a region exterior to the horizon under two situations only, namely:

1. $K_0 = b = 0$, which leads to a Maxwell-like solution cf. Eq. (15);
2. $K_0 = E = 0$, which implies a null field at $r \geq r_H$.

V. Energy conditions and the no-hair theorem

In the first case, we also have $\rho + p_2 \geq 0$ and the energy density is given by

$$\rho = -\frac{g^{00}g^{11}}{8\pi}E^2,$$

which is positive-definite. Hence, for $b = 0$ we conclude that the only solution compatible with both NEC and WEC is $E = E_{(Nh)}$ given by Eq.(33). This leads to Reisner-Nordström solution.

In the second situation (where $b \neq 0$), the only solution satisfying NEC and WEC is the trivial solution $E = 0$. This result disfavors the existence of a non-null solution $E_{(Nh)}$ in the region exterior to the horizon.

VI. Final comments

- We have discussed black holes in the presence of a matter field given by Podolsky electrodynamics.
- The study is composed of three main parts:

  (I) the generalization of Podolsky electrodynamics to curved spacetime;

  (II) the analysis of a static spherically symmetric solutions exterior to Podolsky BH horizon;

  (III) the scrutiny of the solutions in the light of the null and weak energy conditions.

VI. Final comments

- The generalization of Podolsky electrodynamics to curved spacetimes give rise to two possible types of Lagrangian:

  (I) The first one is obtained by performing the minimal coupling prescription in Eq. (1), which implies $b = 0$ in Eq.(2).

  (II) The second possible Lagrangian is built from Utiyama's approach, meaning $b \neq 0$ in Eq. (2). This was shown to be equivalent to the first Lagrangian up to non-minimally coupled terms depending on the contraction of the Riemann tensor and the field strength.

- This study has its importance not only at the classical level but also in the quantum context, where Podolsky theory could help to control ultraviolet 1-loop divergences that are present in the Einstein-Maxwell case.

V. Final comments

- The exterior solutions were analyzed for two distinct cases, namely those obtained by taking $b = 0$ and $b \neq 0$ in the equations of motion.
- The only non-trivial solution for the electromagnetic field when $b = 0$ was show to be Maxwell's solution which leads to Reissner-Nordström BH.
- For the case where $b \neq 0$, we have verified that the homogeneous (asymptotically massive) solutions $E_{(h)}$ are null in the region exterior to the BH horizon under the physical hypothesis $g'_{00} \geq 0$.
- Podolsky Electrodynamics preserves $U(1)$ gauge invariance. Therefore, the absence of propagation of one of the Podolsky modes in the region exterior to the horizon is directly associated to the fact that this is a massive mode; the lack of a Podolsky propagating mode is not related to the theory's gauge invariance.

V. Final comments

- In the last part, we verified that the only exterior solution consistent with the weak and null energy conditions is Maxwell's solution, i.e. $E_{(Nh)}$ with $b = 0$.

- Therefore, any possible non-Maxwellian solution (a solution with hair – e.g. $E_{(Nh)}$ with $b \neq 0$) necessarily violates NEC and WEC.

- The conclusion is: under reasonable physical hypotheses, the static spherically symmetric Podolsky BH satisfies the no-hair theorem.

Acknowledments

THANK YOU!