Complexity and Spacetime

1612.05439, 1706.03788 + to appear, with Alan Reynolds

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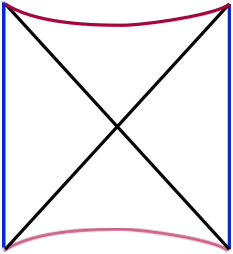
Seeing inside black holes

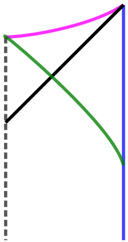
In a holographic theory, black holes are identified with thermal state, $\rho = e^{-\beta H}$.



Observables in right CFT only sensitive to region outside the horizon.

Seeing inside black holes

Collapse to a black hole is a pure state which thermalizes.



Observables sensitive to interior in principle, but accessible probes thermalise on scrambling time $t_* \sim \ln S$; dependence exponentially suppressed.
Two-sided observables in eternal black hole similarly suppressed.

Feature which does not saturate in this way: Complexity.

Complexity

N-qubit model: elementary gates are unitaries acting on $\mathcal{O}(1)$ qubits.

Circuit Complexity of a unitary U is minimum number of elementary gates required to construct U. Generically $\mathcal{O}(e^N)$.
Complexity of a state $|\psi\rangle$: given a reference state $|\psi_0\rangle$, $|\psi\rangle = U|\psi_0\rangle$.

Local Hamiltonian H couples adjacent qubits. Assuming no redundancy, $U(t) = e^{iHt}$ will have complexity $\mathcal{C} \sim t$.

Starting from a low-complexity state, local observables will thermalise in scrambling time $t_* \sim \ln N$, but complexity continues to grow until $t \sim e^N$.

Conjectured bound on speed of computation: Lloyd

$$\frac{d\mathcal{C}}{dt} \leq \frac{2E}{\pi\hbar}.$$

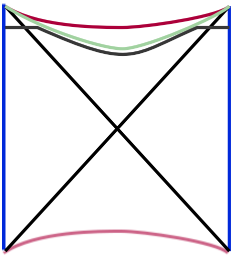
Complexity & Holography

Susskind; Brown, Roberts, Susskind, Swingle & Zhao

Two proposals:

- Complexity-Volume: For a boundary slice B_t, bulk surface Σ_t with $\partial\Sigma_t = B_t$,

$$\mathcal{C}_V \sim max_{\Sigma_t} \frac{V(\Sigma_t)}{G_N \ell_{AdS}}$$



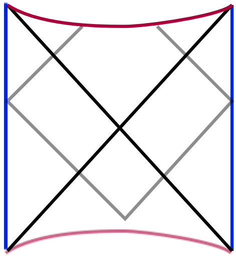
Growth in volume of Einstein-Rosen bridge reproduces expected linear growth of $\mathcal{C}$.

Complexity & Holography

Susskind; Brown, Roberts, Susskind, Swingle & Zhao

- Complexity-action: "Wheeler-de Witt patch" W is bulk causal domain of dependence of Σ_t,

$$\mathcal{C}_A = \frac{S_W}{\pi\hbar}.$$



- Reproduces linear growth
- No scale in relation
- Saturates conjectured bound on $d\mathcal{C}/dt$

Action

Lehner, Myers, Poisson & Sorkin

Require $\delta S = 0$ for variations which leave boundary geometry unchanged. Requires boundary terms in action:

$$S_{\mathcal{V}} = \int_{\mathcal{V}} (R - 2\Lambda)\sqrt{-g}\, dV + 2 \sum_{T_i, S_i} \int K \, d\Sigma - 2 \sum_{N_i} \int \kappa \, dS \, d\lambda + S_{joint},$$

κ measures non-affineness of λ on null generators, $k^\alpha \nabla_\alpha k^\beta = \kappa k^\beta$.
This expression is coordinate dependent! Can remove by adding

$$\Delta S = -2 \sum_{N_i} \int \Theta \ln |\ell \Theta| \, dS \, d\lambda,$$

$$\Theta = \frac{1}{\sqrt{\gamma}} \frac{\partial \sqrt{\gamma}}{\partial \lambda}.$$

Action divergences

Consider pure AdS in Poincare coordinates: the original action is

$$S_{\mathcal{V}} = \frac{\ell^{d-1} V_x}{\epsilon^{d-1}}[-4\ln(\epsilon/\ell) - 2\ln(\alpha\beta) - \frac{1}{d-1}],$$

assuming λ are affine parameters; α, β are normalization of λ.
Diverges like $V_B \ln V_B$; stronger than in CV.

Removing coordinate dependence also removes this stronger divergence:

$$S = S_{\mathcal{V}} + \Delta S = \frac{4\ell^{d-1} V_x}{\epsilon^{d-1}} \ln(d-1).$$

(Note Jefferson & Myers have found $V_B \ln V_B$ divergences in FT calculations for some choices of reference state.)
For more general asymptotically AdS solutions, subleading divergences determined by local geometry of boundary.

Complexity in AdS Soliton

Consider a FT on T^{d-1}, with antiperiodic bc for fermions on one S^1. Dual of ground state is AdS soliton,

$$ds^2 = \frac{r^2}{\ell^2}\left[-dt^2 + \left(1 - \frac{r_+^d}{r^d}\right) d\chi^2 + d\vec{x}^2\right] + \left(1 - \frac{r_+^d}{r^d}\right)^{-1} \frac{\ell^2}{r^2} dr^2.$$

Period $\Delta\chi = \frac{4\pi\ell^2}{dr_+}$. This has a negative energy, corresponding to Casimir energy in FT,

$$E = -\frac{V_x \Delta\chi r_+^d}{\ell^{d+1}}.$$

Spacetime ends at $r = r_+$; effective IR cutoff induced by bc.

Complexity in AdS Soliton

Complexity:

Homogeneity $\Rightarrow$ holographic $\mathcal{C} = V_x \Delta\chi c(r_+)$.

CV: maximal surface at constant t,

$$\mathcal{C}_V \propto V_x \Delta\chi \frac{r_{UV}^{d-1} - r_+^{d-1}}{\ell^{d-1}}$$

Increasing IR scale r_+ decreases complexity.

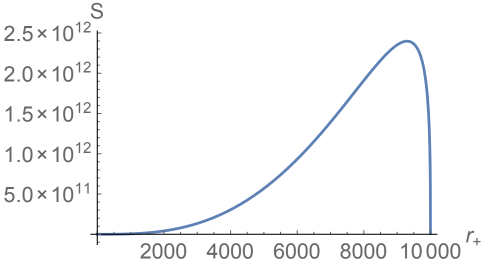
Complexity in AdS Soliton

CA: for $r_+ \ll r_{UV}$, expand in power series. Symmetry fixes

$$\mathcal{C}_A = V_x \Delta\chi \left[\frac{4\ln(d-1)}{\ell^{d-1}} r_{UV}^{d-1} + l_0 r_+^{d-1} + \dots \right]$$

for some coefficient l_0.

Remove leading divergence by adding $S_{ct} = -4\ln(d-1)\int_\Sigma \sqrt{h} dS$.



Complexity in AdS Soliton

- Increasing IR scale r_+ increases complexity initially; vanishes as IR scale approaches UV cutoff.
- Change action? $\delta S = 0$ for variations fixed on boundary only fixes action up to boundary terms depending on geometry of boundary.
  $\star$ Boundary of Wheeler-de Witt patch extends into interior, can modify finite terms.
- Comparison to FT: Jefferson & Myers calculated $\mathcal{C}$ for free scalar on a toroidal lattice. Extend to fermions, look at change under change in bc.

de Sitter boundary

Consider a field theory on de Sitter space,

$$ds_\delta^2 = \frac{1}{H^2\eta^2}(-d\eta^2 + d\vec{x}^2);$$

simple laboratory for time dependence.
Bulk AdS geometries with de Sitter boundaries easy to construct.
Slicing of pure AdS:

$$ds^2 = \ell^2(d\rho^2 + \frac{\sinh^2\rho}{\eta^2}(-d\eta^2 + d\vec{x}^2)).$$

Related to Poincare coordinates by $z = -\frac{\eta}{\sinh\rho}$, $t = \eta\coth\rho$.

de Sitter times circle boundary

de Sitter $\times S^1$: two possible bulk solutions.

- Bubble

$$ds^2 = f(r)d\chi^2 + f(r)^{-1}dr^2 + \frac{r^2}{\eta^2}(-d\eta^2 + d\vec{x}^2)$$

with $f(r) = 1 + \frac{r^2}{\ell^2} - \frac{r_0^{d-2}}{r^{d-2}}$. Gapped solution: cut off at $r = r_+$, $f(r_+) = 0$. $\Delta\chi = \frac{4\pi\ell^2 r_+}{(dr_+^2 + (d-2)\ell^2)}$: two bubbles for given $\Delta\chi$.

- Locally AdS: set $r_0 = 0$, $r = \ell \sinh\rho$,

$$ds^2 = \cosh^2\rho d\chi^2 + \ell^2[d\rho^2 + \frac{\sinh^2\rho}{\eta^2}(-d\eta^2 + d\vec{x}^2))]$$

Horizon at $\rho = 0$. Ungapped solution.

de Sitter complexity

Holographically, $\mathcal{C} \propto V_x$. Symmetry $\Rightarrow \mathcal{C} \propto \frac{V_x}{\eta^{d-2}}$.
For de Sitter$\times S^1$ cases, write

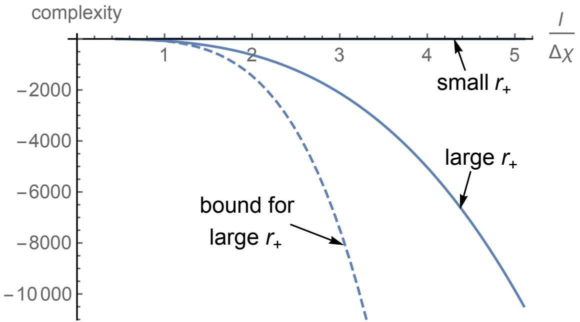
$$\mathcal{C} = \frac{V_x \Delta\chi \ell^{d-2}}{|\eta|^{d-2}} c.$$

c complexity density. Constant state-dependent factor. For locally AdS, indep of $\Delta\chi$. For bubble, depends on $\Delta\chi$ through r_+.
Complexity growth bound $\Rightarrow$

$$c \leq \frac{2\ell}{(d-2)\pi\hbar}\rho.$$

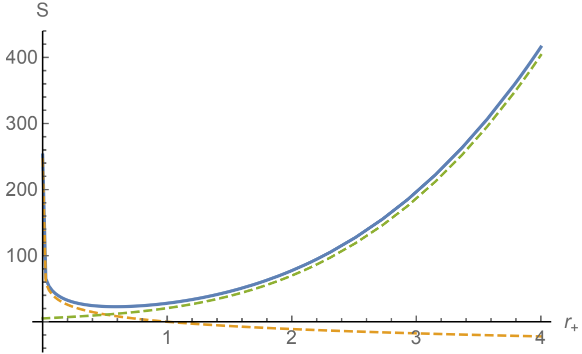
Difference between locally AdS and bubble $\Delta\rho = -\frac{r_+^2(\ell^2+r_+^2)}{\ell^5}$.

Volume results



Analytically, for large bubbles, $\Delta c \propto r_+^{d-1}$; slower than r_+^d behaviour of $\Delta\rho$. Reasonable from complexity point of view; volume in units of IR cutoff.

Action results



Action *increases* for both large and small bubbles. $\sim \ln r_+$ for small bubbles, $\sim r_+^{d-1}$ for large bubbles.

Discussion

- Conjectured relation of complexity to bulk geometry could provide a probe naturally sensitive to black hole interior.
- Not yet related to holographic dictionary; action calculation intrinsically Lorentzian
- Action calculation has coordinate dependence, gives stronger divergence. Related to choice of reference state? Or change action.
- Consider further examples: AdS soliton, de Sitter boundary
- CA gives surprising results. Complexity grows as IR scale increases?
- Further change of action? Comparison to FT?