6. Lagrangian Mechanics

Goal: new way of thinking about physics

old way: Forces $\rightarrow$ EoM
new way: Action $\rightarrow$ EoM

$$\text{Action} \searrow \text{Forces} \nearrow \text{EoM}$$

Advantages:
- more fundamental
  closer to conservation laws
- involves only 1st derivatives
  $\rightarrow$ easier
- easier to use in other branches of physics
  E&M
  QFT
  GR

6.1 Euler-Lagrange Equations

System: particle with position x, conservative system

$T = \frac{1}{2}m\dot{x}^2$ kinetic energy 1-d

$V = V(x)$ potential energy

$$\underline{L \equiv T - V} \quad \text{Lagrangian}$$

$$L = \frac{1}{2}m\dot{x}^2 - V(x)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

Euler-Lagrange equation

$$(m\dot{x})^{\cdot} = -\frac{\partial V}{\partial x} = F$$

$$m\ddot{x}$$

$\downarrow$

Newton's 2nd law

System: particle in 3-d

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - V(x,y,z)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}$$

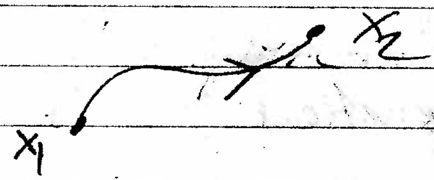
$$m\ddot{\vec{x}} = -\vec{\nabla} V$$

Ex Spring

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$\underset{\displaystyle m\ddot{x}}{\underset{\parallel}{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)}} = \frac{\partial L}{\partial x} = -kx$$

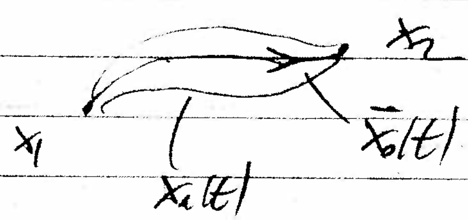
Principle of Stationary Action



Principle: Consider trajectories with fixed end points $\vec{x}(t_1) = \vec{x}_1$ and $\vec{x}(t_2) = t_2$. The physical path $\vec{x}_0(t)$ extremizes the action

$$S \equiv \int_{t_1}^{t_2} L(x,\dot{x},t)\,dt$$

among all paths $\vec{x}(t)$.



$\vec{x}_a(t) = \vec{x}_0(t) + a\vec{\beta}(t)$ family of curves let us extremize the action among the one-parameter family of curves $x_\alpha(t)$

want: $\left.\frac{\partial S}{\partial \alpha}\right|_{\alpha=0} = 0 \iff \vec{x}_0(t)$ extremizes action

$$\frac{\partial S}{\partial \alpha} = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x_i} \frac{\partial x_i}{\partial \alpha} + \frac{\partial L}{\partial \dot{x}_i} \frac{\partial \dot{x}_i}{\partial \alpha} \right)\Big|_{\alpha=0} dt$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} \right) \underbrace{\frac{\partial x_i}{\partial \alpha}\Big|_{\alpha=0}}_{\beta_i(t)} dt \stackrel{!}{=} 0 \quad \forall \beta_i(t)$$

$$\frac{\partial L}{\partial x_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}$$

Note: Explain Lemma of variational calculus

Names: "Least" action principle
stationary action principle
Hamilton principle

Note: 1) Q: Doesn't Hamilton Principle use something we do not know, namely final times?

A: yes and no,
given $\vec{x}(t_1), \dot{\vec{x}}(t_1)$ then t_2 for which $\vec{x}(t) = \vec{x}_2$ is determined
HP says: among all paths with $\vec{x}(t_2) = \vec{x}_2$ the physical one extremizes S

$\vec{x}(t_1), \dot{\vec{x}}(t_1) \longrightarrow \vec{x}(t_2), \dot{\vec{x}}(t_2)$

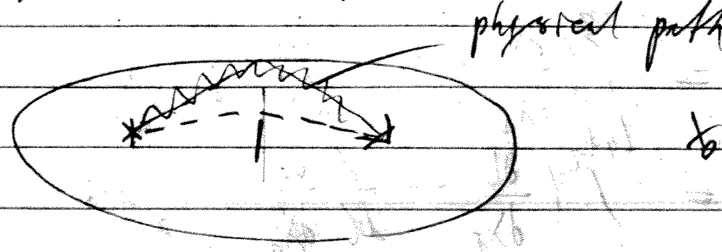
$\vec{x}(t_1), \vec{x}(t_2) \longrightarrow \dot{\vec{x}}(t_1), \dot{\vec{x}}(t_2)$

2) $\vec{x}_0(t)$ usually a minimum
sometimes saddle pt.
never maximum

ii) $\Delta S = \frac{1}{2}\int_{t_1}^{t_2}\left(m\dot{\xi}^2 - k\xi^2\right)dt > 0$

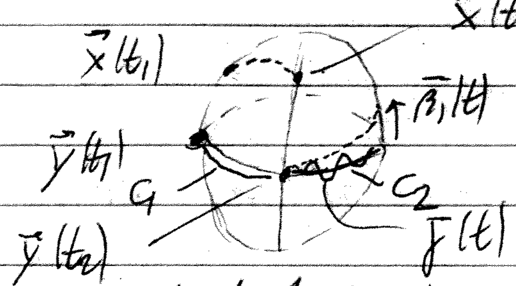
for harmonic oscillator $\qquad x_0(t)$ minimum

iii) for light: S = light travel time



$x_0(t)$ saddle

Ex: Particle on a sphere



$S = \int \frac{1}{2}\dot{\vec{x}}^2(t)\,dt$

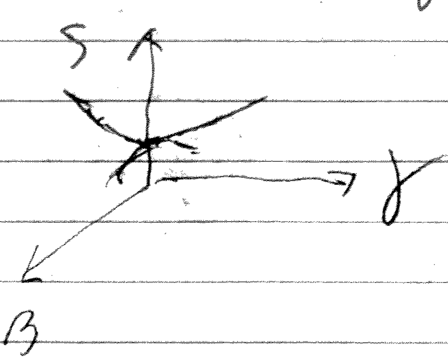
shortest length $\to$ lowest $\dot{x}^2 \to$ smallest S

$\vec{y}_1(t) \to \vec{y}_2(t)$ $\qquad C_1$ global minimum
$\qquad C_2$ saddle

C_2: $\vec{y}_{C_2}(t)$

$\vec{x}(t) = \vec{y}_{C_2}(t) + a\vec{\beta}_1(t)$ shorter

$\vec{x}(t) = \vec{y}_{C_2}(t) + a\vec{\gamma}_1(t)$ longer



saddle

Def: Given space $\mathcal{M}$
Curve on $\mathcal{M}$ which extremizes length between two points: geodesic

Ex: $\mathcal{M} = S^2$ geodesic: great circle

Change of Coordinates

$x_1, \dots x_m$ first set of coords. e.g. Cartesian x, y, z
$q_1, \dots q_m$ second set " " e.g. polar r, θ, φ
$L(x, \dot{x})$ fct. in phase space
indep. of coordinates used

$$S = \int_{t_1}^{t_2} L(x(t), \dot{x}(t))\, dt$$ indep. of coords. used

Corollary: action principle is coordinate invariant

Goal: EoM in q_i coords.

step 1: $x_i(q_j)$, $\dot{x}_i(q_j, \dot{q}_j)$

step 2: $$S = \int L(x(q), \dot{x}(q, \dot{q}))\, dt = \int \tilde{L}(q_j, \dot{q}_j)\, dt$$

step 3: $$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{q}_i} = \frac{\partial \tilde{L}}{\partial q_i}$$

Ex: $x = r\cos\theta\cos\varphi$
$y = r\cos\theta\sin\varphi$
$z = r\sin\theta$

$$\dot{x}^2+\dot{y}^2+\dot{z}^2 = \dot{r}^2 + r^2\dot{\vartheta}^2 + r^2\cos^2\vartheta\,\dot{\varphi}^2$$

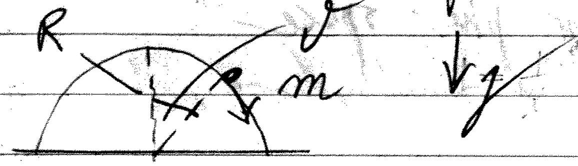
$$L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\vartheta}^2 + r^2\cos^2\vartheta\,\dot{\varphi}^2\right)$$

ELE $\quad \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r}$

$$\ddot{r} = r\dot{\vartheta}^2 + r\cos^2\vartheta\,\dot{\varphi}^2$$

<u>Forces of constraint</u>

<u>Ex</u> particle on a semi-hoop



1st analysis: only ϑ $\quad L = \frac{1}{2}mR^2\dot{\vartheta}^2 - mgR\cos\vartheta$

$$mR^2\ddot{\vartheta} = + mgR\sin\vartheta$$

$$\ddot{\vartheta} = \frac{g}{R}\sin\vartheta \simeq \frac{g}{R}\vartheta$$

(small ϑ

2nd analysis: r, ϑ as variables

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\vartheta}^2 - mgr\cos\vartheta - V(r)$$

$$m\,(r^2\dot{\vartheta})^{\cdot} = mgr\sin\vartheta$$

$$m\ddot{r} = -V'(r) + mr\dot{\vartheta}^2 - mg\cos\vartheta$$

if $r = R$ then require constraint force

$$-V'(R) = mg\cos\vartheta - mR\dot{\vartheta}^2$$

$F''(R)$ upward force

$F(R) > 0$ ball stays on hoop

$F(R) = 0$ ball leaves hoop (sketch)

$$g\cos\theta = R\dot{\theta}^2$$

<u>Method</u>
- start with reduced system
- add direction $\parallel$ constraint force with $V(r)$
- consider EOM in $\parallel$ direction
- Find V' such that constraint is satisfied

<u>6.2 Symmetries and Conservation Laws</u>

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

<u>Thm</u>: If L is independent of q_i then

$$\frac{\partial L}{\partial \dot{q}_i} = p_i \quad \text{is conserved}$$

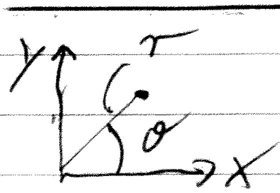
<u>Note</u>: q_i is then called a <u>cyclic coord</u>.

<u>Ex</u>: <u>Translation invariance in x direction</u> $\partial L/\partial x = 0$

$$\frac{\partial L}{\partial \dot{x}} = p_x \quad \text{conserved}$$

<u>momentum conservation</u>

<u>Ex</u>: <u>Rotational invariance</u> in cylindrical symm.



$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - V(r)$$

$$\frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$ angular momentum

angular momentum conservation

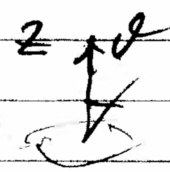
$$\frac{\partial L}{\partial z} = 0 \quad \text{if } V = V(r) \Rightarrow m\dot{z} = 0$$

linear momentum conservation (again)

Note: $\frac{\partial L}{\partial r} \neq 0$ even if $V = 0$.

Ex: Rotational invariance in spherical coords.

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\vartheta}^2 + r^2\sin^2\vartheta\,\dot{\varphi}^2) - V(r, \vartheta)$$



$$\frac{\partial L}{\partial \varphi} = 0 \Rightarrow r^2 \sin^2\vartheta\,\dot{\varphi} = \text{const}$$

angular momentum conservation

Def: $H = \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$ Hamiltonian

Ex. $L = \frac{1}{2}m\dot{q}^2 - V(q)$

$H = \frac{1}{2}m\dot{q}^2 + V(q) = E$

Thm. $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$

Pf: $$\frac{dH}{dt} = \sum_{i=1}^{N}\left[\cancel{\frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i} + \underbrace{\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right)\dot{q}_i - \frac{\partial L}{\partial q_i}\dot{q}_i}_{=0 \text{ by ELE}} - \cancel{\frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i}\right] - \frac{\partial L}{\partial t}$$

$\frac{\partial L}{\partial t} = 0 \Rightarrow$ energy conservation

time translation invariance

Ex where this is not true

<u>Thm</u>: (Noether) For each symmetry of L there is a conserved quantity

<u>Symmetry</u>: $q_i \to q_i + \varepsilon k_i(q)$
leaves L <u>invariant</u> $\left.\frac{dL}{d\varepsilon}\right|_{\varepsilon=0} = 0$

$$0 = \frac{dL}{d\varepsilon} = \sum_i \left(\frac{\partial L}{\partial q_i}\frac{dq_i}{d\varepsilon} + \frac{\partial L}{\partial \dot q_i}\frac{d\dot q_i}{d\varepsilon} \right)$$

$$= \sum_i \left(\frac{\partial L}{\partial q_i} k_i + \frac{\partial L}{\partial \dot q_i} \dot k_i \right)$$

$$\overset{ELE}{=} \sum_i \left[\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right)\right) k_i + \frac{\partial L}{\partial \dot q_i} \dot k_i \right]$$

$$= \frac{d}{dt}\left(\sum_i \left(\frac{\partial L}{\partial \dot q_i} k_i \right)\right)$$

$$\Rightarrow \underbrace{\sum_i \frac{\partial L}{\partial \dot q_i} k_i}_{P(q,\dot q)} = \text{const} \qquad \underline{\text{conserved quantity}}$$

<u>Ex</u>: <u>translation invariance</u> $q_i \to q_i + \varepsilon V_i$

$$I(q,\dot q) = \sum_i \frac{\partial L}{\partial \dot q_i} V_i$$

momentum in direction $\vec V$

<u>Ex</u>

$$L = \frac{1}{2}(\dot x^2 + \dot y^2) - \frac{k}{2}(x^2+y^2)$$

$x \to x + \varepsilon y$ $\qquad k_x = y$

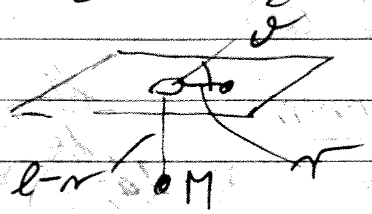
$y \to y - \varepsilon x$ $\qquad k_y = -x$

$$\left.\frac{\partial L}{\partial \varepsilon}\right|_{\varepsilon=0} = 0$$

$P = \dot x y - \dot y x$ $\qquad$ z component of ang. momentum

6.3 Applications

Ex $L = \frac{1}{2} M\dot{r}^2 + \frac{1}{2} m(\dot{r}^2 + r^2\dot{\vartheta}^2) + Mg(l - r)$



Goal: • fixed point
• small oscillations about fixed point

step 1 $(mr^2\dot{\vartheta})' = 0$

$(M+m)\ddot{r} = mr\dot{\vartheta}^2 - Mg = \frac{L^2}{mr^3} - Mg$

fixed point: $r_0^3 = \frac{L^2}{mMg}$; $r_0 = \frac{Mg}{m\dot{\vartheta}^2}$ (*)

step 2 a) $r = r_0 + \delta r$

$\dot{\vartheta} = \dot{\vartheta} + \delta\dot{\vartheta}$

expand L to quadratic order in $\delta r, \delta\dot{\vartheta}$

$\frac{d}{dt}\frac{\partial L}{\partial \delta\dot{r}} = \frac{\partial L}{\partial \delta r}$

$\frac{d}{dt}\frac{\partial L}{\partial \delta\dot{\vartheta}} = 0$

diagonalize

↓

complicated

b) use L conservation — direct insert in EoM

$\dot{\vartheta} m r^2 = L = \text{const} = L_0$

$L = \frac{1}{2}(M+m)\dot{r}^2 + \frac{1}{2}\frac{L^2}{mr^2} + Mg(l-r)$

$= \frac{1}{2}(M+m)\dot{r}_0^2 + (M+m)\dot{r}_0\delta\dot{r} + \frac{1}{2}(M+m)\delta\dot{r}^2$

$$+\frac{1}{2}\frac{L^2}{m r_0^2}\left(1-\frac{2\delta r}{r_0}+3\frac{\delta r^2}{r_0^2}\right)-Mg\,\delta r$$

constant terms: irrelev.
linear terms cancel

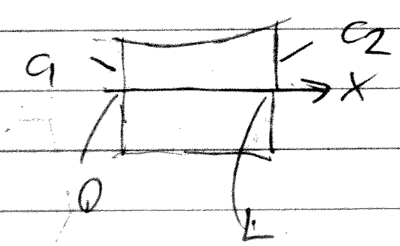
$$(M+m)\,\delta r'' = -\frac{3L^2}{m r_0^4}\,\delta r$$

$$\delta r'' + \frac{3L^2}{m(M+m)r_0^4}\,\delta r = 0$$

$$W = \left(\frac{3L^2}{m(M+m)}\right)^{1/2}$$

Recipe:
- Consider all conservation laws
- Insert into EoM to reduce DoF
- find fixed point
- expand EoM to $\mathcal{O}(\varepsilon^2)$ about fixed point

Ex. Minimal surface of revolution



$$A = \int_0^L dx\, \underbrace{y\sqrt{1+y'^2}}_{\mathcal{L}}\, 2\pi$$

$\delta A = 0$ among all $y(x)$

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial y}$$

$$\left(\frac{1}{\sqrt{1+y'^2}}\,y y'\right)' = \sqrt{1+y'^2}$$

$$\frac{y'^2}{\sqrt{}} - \frac{y y'^2 y''}{\sqrt{}^3} + \frac{y y''}{\sqrt{}} = \sqrt{}$$

$$y y''(1+y'^2-y'^2) + y'^2(1+y'^2) = (1+y'^2)^2$$

$$y y'' = 1+y'^2 \quad \text{nonlinear !!}$$

$y = \cosh x$ is a solution

$$\frac{y'y''}{1+y'^2} = \frac{y'}{y} \quad \int dx$$

$$\frac{1}{2}\ln(1+y'^2) = \ln y + C \quad \exp$$

$$1+y'^2 = By^2 \qquad B = e^{2c}$$

$$y(x) = \frac{1}{\sqrt{B}} \cosh\left[\sqrt{B}(x+d)\right]$$

sketch

Ex Motion of Test Particles

$$L = \frac{1}{2}\left(\dot{t}^2 - \dot{\vec{x}}^2\right) = \frac{1}{2}\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \qquad \dot{} = \frac{\partial}{\partial\sigma}$$

$$x^\mu = \begin{pmatrix} t \\ x_i \end{pmatrix} \qquad \eta_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$$S = \int L\, d\sigma$$

$$\delta S = \int d\sigma\, \eta_{\mu\nu}\dot{x}^\mu \delta\dot{x}^\nu = -\int d\sigma\, \eta_{\mu\nu}\ddot{x}^\mu \delta x^\nu \stackrel{!}{=} 0$$

$$\Rightarrow \ddot{x}^\mu = 0$$

Ex Motion of Test Particles in curved S-T

$$\eta_{\mu\nu} \rightarrow g_{\mu\nu}(\underline{x}, t)$$

$$\delta S = \int d\sigma \left(g_{\mu\nu}\dot{x}^\mu \delta\dot{x}^\nu + \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\dot{x}^\mu \dot{x}^\nu \delta x^\alpha \right)$$

$$= -\int d\sigma \left[(g_{\mu\nu}\dot{x}^\mu)^{\cdot} - \frac{1}{2}\frac{\partial g_{\alpha\mu}}{\partial x^\nu}\dot{x}^\mu \dot{x}^\alpha \right]\delta x^\nu = 0$$

$$\Rightarrow (g_{\mu\nu}\dot{x}^\mu)^{\cdot} - \frac{1}{2}\frac{\partial g_{\alpha\mu}}{\partial x^\nu}\dot{x}^\mu \dot{x}^\alpha = 0$$

geodesic eq. in curved ST

Def: geodesic