1. Dynamics, Part I

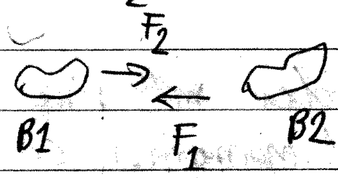
text pp 51–70
extra materials

1.1 Newton's Laws

N1 1^{st} law $v = \text{const}$ for body if $F=0$

N2 2^{nd} law $\frac{d}{dt}\vec{p} = \vec{F}$ Note – often $p = mv$

N3 3^{rd} law $\vec{F}_1 = -\vec{F}_2$



applets

Newton's laws not valid in all coord. systems ("frames")

Inertial frame : frame in which Newton's eq. are valid

Q: What is an inertial frame?

Newton's absolute space

HW1-1 a) What is absolute space?
b) Is concept of absolute space consistent with N3?
HW2 : F2 moving uniformly w.r.t. F2
F1 inertial → F2 inertial

Q: What is absolute space?

Comments

1. 1^{st} law universality
   every particle has the same inertial frame

2. 2^{nd} law universal response to any force — spring force, E&M force, gravit. force *

   $$\frac{a_1}{a_2} = \frac{m_2}{m_1}$$

   m inertial mass

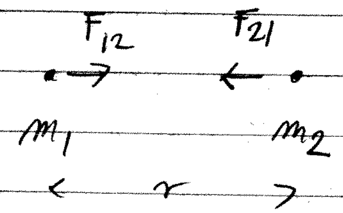
3. $\vec{F} = m\vec{a}$ is vector equation

4. system of point particles with fixed masses
   3^{rd} law → total momentum conserved

   $p_i = m_i v_i$

   i : index, refers to ith particle

Newtonian Gravity



$$F_{12} = G\frac{m_1 m_2}{r^2} = F_{21}$$

magnitude
direction

Newtonian gravitational force

$F_{12} = m_1 a_1$

$$a_1 = G\frac{m_2}{r^2}$$ gravitational mass
produces force field

sketch

Galileo: universality of gravity

sketch feather / stone

$$m_1^I a_1 = G \frac{m_1^G m_2^G}{r^2}$$

universality $\rightarrow$ $m^I = m^G$

Q: why?

A: leads to Einstein's GR

Q: why a mystery?

A: very different from E&M

$$F = q \frac{1}{r^2} \qquad q \text{ charge}$$

O	O
proton	neutron
charge 1	charge 0

same mass
different E field
different E forces

1.2 Dynamical Systems

Consider particle moving in 1 d

$F = ma$

q position

$\dot{q} = \frac{d}{dt} q$ velocity

$a = \frac{d}{dt}\dot{q} = \frac{d^2}{dt^2} q$ acceleration

$F = F(q, \dot{q}, t)$

(E1) $$\boxed{m \frac{d^2}{dt^2} q = F(q, \dot{q}, t)}$$ ODE second order dynamical system

Questions
- Who has not seen DE?
- Who thinks DE are easy?
- " " hard?
- Can we find exact solutions?

Ex spring q displacement

$F(q) = -kq$

$m\ddot{q} = -kq$

simple ODE: solution exponential
oscillatory
power law

do exp

Ansatz: $q(t) = e^{\alpha t}$

$m\alpha^2 = -k \quad \alpha^2 = -\underbrace{\frac{k}{m}}_{= \omega^2} \Rightarrow \alpha = \pm i\omega$

Ex: $F(q,t) = +k t^{-2} q$

$m\ddot{q} = +k t^{-2} q$

exponential: not a solution

power law: $q(t) = t^{\alpha}$

$m\alpha^2 = +k \qquad \alpha = \pm\left(\frac{k}{m}\right)^{1/2}$

Def: Dynamical system autonomous if $\frac{\partial F}{\partial t} = 0$

Q: Who has not seen partial derivatives?

Def: $p = m\dot{q}$ momentum

Second order DE (E1) can be written as a pair of coupled first order DE

$$\boxed{\begin{aligned} \dot{q} &= \frac{p}{m} \\ \dot{p} &= F(q,p,t) \end{aligned}}$$

$x = \begin{pmatrix} q \\ p \end{pmatrix}$	phase space	$\in \mathbb{R}^2$	$\in \mathbb{R}^{2n}$	
q	configuration space	$\in \mathbb{R}$	$\in \mathbb{R}^n$	q_i
p	momentum space	$\in \mathbb{R}$	$\in \mathbb{R}^n$	p_i

$i = 1, \dots n$

$n = 3$ 1 particle in 3d
3 particles in 1d

$n > 3$ n particles in 1d
$n/3$ " " 3d

Consider n particles in 1-d of masses m_i $i=1,\dots n$

$\underline{F} = (F_1, \dots, F_n)$

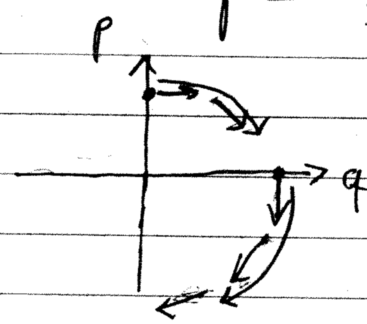
↑ force on i'th particle

(E2) $$\boxed{\begin{aligned} \dot{q}_i &= p_i/m_i \\ \dot{p}_i &= F_i(q_1 \dots q_n, p_1 \dots p_n, t) \end{aligned}}$$

Eg: grav. force between 2 particles

Flow in Phase space

Ex spring $\dot{q} = p/m$
$\dot{p} = -kq$

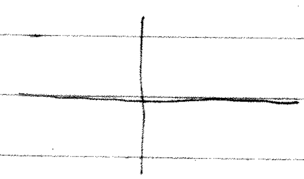


start at rest

start at equilibrium with push

✓ resulting orbits

Ex $m = k = 1$ $\dot{q} = p$
$\dot{p} = -q$



insert

procedure: • take grid of points in phase space
• at each pt. draw tangent vector
• trajectory from any starting pt. follows a trajectory which at all times is parallel to tg. vectors

Note: $k = m$: circles
$k \neq m$ ellipses

Consider initial time s, initial phase space point $\underline{y}$
↓ evolve (E2) until time $t > s$

$\underline{x} = \phi_{ts}(\underline{y})$

$\phi_{ts} : \mathbb{R}^2 \to \mathbb{R}^2$ flow of dynamical system

Note: If F is autonomous then $\phi_{ts} = \phi_{t-s\,0} \equiv \phi_{t-s}$

Local existence
Global uniqueness } of solution of (E2)

Thm If $F : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous & differentiable
then $\forall s \in \mathbb{R}$
$\forall \underline{y}_0 \in \mathbb{R}^2$
$\exists$ nbhd J of s in $\mathbb{R}$
nbhd U of $\underline{y}_0$ in $\mathbb{R}^2$
such that flow exists & is unique locally
in time (J) and phase space (U)
$\underline{x} \equiv \phi_{t-s}(\underline{y})$ exists $\forall t \in J$ $\forall \underline{y} \in U$

and i) $\phi_0(\underline{y}) = \underline{y}$

ii) $\frac{d}{dt}\phi_{t-s}(\underline{y}) = F(\phi_{t-s}(\underline{y}))$

iii) $\phi_{t-s}(\underline{y})$ is C^1 in t & $\underline{y}$

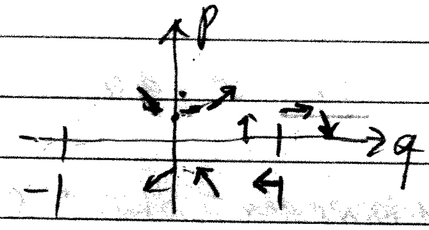
Note: Theorem generalizes to many d.o.f. systems

Ex Spring $\dot q = p$
$\dot p = -q$ (ps sketch)

Ex. "Double well" $F(q) = -4q(q^2-1)$

spring: $F = 0$ at $q = 0$

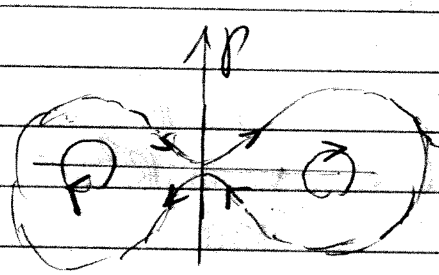
here: $F=0$ at $q = 0, \pm 1$



about $\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

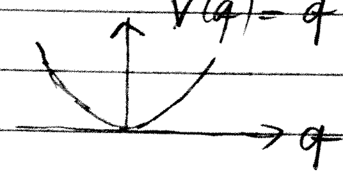
about $\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $F \simeq -(q^2-1)$

about $\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ by mirror symmetry



Interpretation: gravity on surface of earth

a) $V(q) = q^2$

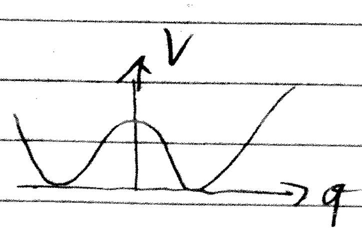


$F = -\frac{\partial V}{\partial q}$

$\updownarrow$

spring

b)

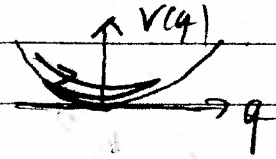


$V = (q-1)^2(q+1)^2$

$\updownarrow$

double well

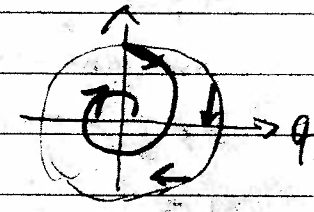
Ex system with friction



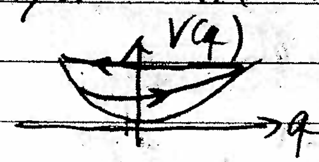
- sketch time evolution

$$\ddot{q} = -\Gamma \dot{q} - q \qquad 0 < \Gamma << 1$$

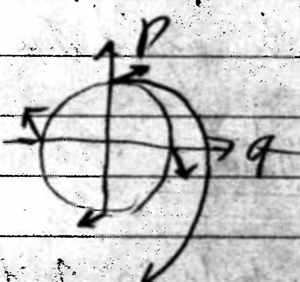
$$\dot{q} = p$$
$$\dot{p} = -\Gamma p - q$$



Ex system with antifriction



$$\ddot{q} = +\Gamma \dot{q} - q \qquad 0 < \Gamma << 1$$
$$\dot{q} = p$$
$$\dot{p} = \Gamma p - q$$



Stability

Def : $\underline{x}$ fixed pt. if $\Phi_{t,s}(\underline{x}) = \underline{x} \quad \forall t,s$

Ex : spring : $\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is fixed pt.

double well : $\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, ($

are fixed pt

unstable

stable

Mathematical description of stability:

$\underline{y}$ phase space vector
$|\underline{y}|$ length of vector in $\mathbb{R}^2$

Def: A point $\underline{y}_0$ in phase space is a <u>stable fixed point</u> under the flow $\Phi_{t,s}$ if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that if } |\underline{y} - \underline{y}_0| < \delta$$

$$|\Phi_{t,s}(\underline{y}) - \underline{y}_0| < \varepsilon \quad \forall t,s \quad t > s$$

Ex spring $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ stable

Ex double well $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ unstable

N.B. give the arguments

Potential

a) particle in 1 d, position q

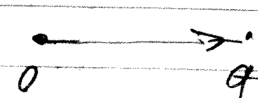
$F(q)$

$v = \dot{q}$

$$m \frac{d}{dt} \dot{q} = F$$

$$\| $$

$$m \frac{d}{dq} \dot{q} \frac{dq}{dt} = m v \frac{dv}{dq}$$



consider motion from 0 to q

$$m \int dq \, v \frac{dv}{dq} = \int F(q) \, dq$$

$$\|$$

$$\frac{1}{2} m \int \frac{d}{dq}(v^2) = \frac{1}{2} m v^2 \Big|_0^q = E_k(q) - E_k(q_0)$$

Ex: spring $F = -kx$ $V = \frac{1}{2}kx^2$

$E_k = \frac{1}{2}mv^2$ kinetic energy

$V(q) = -\int_0^q F(q')\,dq'$ potential energy relative to $q=0$

Note: $F(q) = -\frac{\partial V}{\partial q}$

Note: $F = F(q) \Rightarrow E_{total} \doteq E_k + V$ conserved

b) n particles with central force (in 3d)

$\underline{q}_i$: $i = 1, \ldots n$

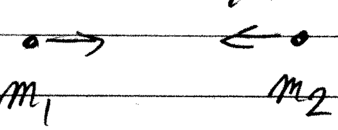
m_i mass of particle i

$$m_i \ddot{\underline{q}}_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \underline{F}_{ij}(\underline{q}_i - \underline{q}_j) \equiv \underline{F}_i$$

$$\underline{F}_{ij}(\underline{q}_i, \underline{q}_j) = \frac{\underline{q}_i - \underline{q}_j}{|\underline{q}_i - \underline{q}_j|} f_{ij}(|\underline{q}_i - \underline{q}_j|) \;\&\; f_{ij} = f_{ji}$$

central force system

Eg: Newtonian gravity



$$f_{12} = f_{21} = -G\frac{m_1 m_2}{r^2} \qquad r = |\underline{q}_1 - \underline{q}_2|$$

Theorem. For an n-particle system with central forces $\underline{F}_{ij}$ there exists a potential function $V(\underline{q}_1, \ldots \underline{q}_n)$ such that

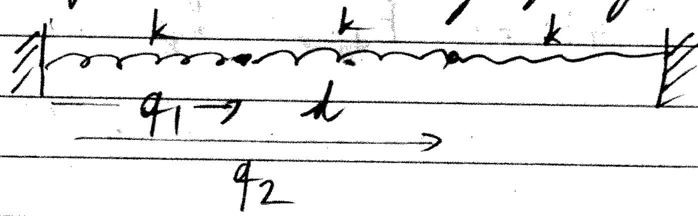
$$\underline{F}_i = -\frac{\partial V}{\partial \underline{q}_i}$$

Proof by construction

$$V = \sum_{1 \le i < j \le n} V_{ij} \qquad V_{ij}(\underline{q}) = -\int_{|\underline{q}_0|}^{|q|} dr\, f_{ij}(r)$$

$$-\frac{\partial}{\partial \underline{q}_i} V_{ij}(|q_i - q_j|) = f_{ij}(|q_i - q_j|)\, \frac{\underline{q}_i - \underline{q}_j}{|\underline{q}_i - \underline{q}_j|}$$

Ex 2 particles connected by springs



* $V(q_1, q_2) = \frac{1}{2} k q_1^2 + \frac{1}{2} k (q_2 - q_1)^2 + \frac{1}{2} k (d - q_2)^2$
* $m_1 \ddot{q}_1 = -\frac{\partial}{\partial q_1} V = -k q_1 + k(q_2 - q_1)$

  $m_2 \ddot{q}_2 = -\frac{\partial}{\partial q_2} V = -k(q_2 - q_1) + k(d - q_2)$
* stable fixed point

$$\left.\begin{aligned} q_2 &= 2q_1 \\ 2q_2 - q_1 &= d \end{aligned}\right\} \; 3q_1 = d \quad q_1 = \tfrac{1}{3} d \quad q_2 = \tfrac{2}{3} d$$

Linearization

Ex double well $F(q) = -4q(q^2 - 1) = -4q(q+1)(q-1)$

sketch of ps

focus on x near $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$q = \tilde{q} + 1 \qquad |\tilde{q}| \ll 1$$

$$F(q) = -4(1 + \tilde{q})\, \tilde{q}\, (2 + \tilde{q}) = -8\tilde{q} + O(\tilde{q}^2)$$

$$\ddot{q} = \ddot{\tilde{q}} = -8\tilde{q}$$

solution: $\tilde{q}(t) = A\sin(\sqrt{8}\,t) + B\cos(\sqrt{8}\,t)$

$|A|, |B| \ll 1$

<u>Procedure</u>: * find fixed point $\begin{pmatrix} q_0 \\ 0 \end{pmatrix}$

* expand about fixed point $\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} q_0 + \tilde{q} \\ p \end{pmatrix}$

* Taylor expand force to linear order in $\tilde{q}$ about q_0

<u>Ex</u> double well, expand about unstable fixed pt. $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} \tilde{q} \\ p \end{pmatrix} = \begin{pmatrix} q \\ p \end{pmatrix} \quad |q| \ll 1$

$F(q) = +4q + \mathcal{O}(q^2)$

$\ddot{q} = 4q$

$q(t) = Ae^{2t} + Be^{-2t}$

$|A|, |B| \ll 1$

linearization breaks down once $|2t| \gg 1$

<u>Hooke's law</u> Any dynamical system can be linearized about a stable fixed point and solution of linearized equations is good approximation for all times.

<u>Procedure</u>: at the level of potential

* find fixed point $\begin{pmatrix} q_0 \\ 0 \end{pmatrix}$

* expand about fixed pt., $\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} q_0 + \tilde{q} \\ p \end{pmatrix}$

* Taylor expand V to 2nd order in $\tilde{q}$

Ex: $V(q) = (q-1)^2 (q+1)^2$

$q_0 = 1 \qquad q = 1 + \tilde{q}$

$V(q) = \tilde{q}^2 (2+\tilde{q})^2 = 4\tilde{q}^2 + O(\tilde{q}^3)$

$F(\tilde{q}) = -\dfrac{\partial V}{\partial \tilde{q}} = -8\tilde{q}$

$\ddot{\tilde{q}} = -8\tilde{q}$ as before