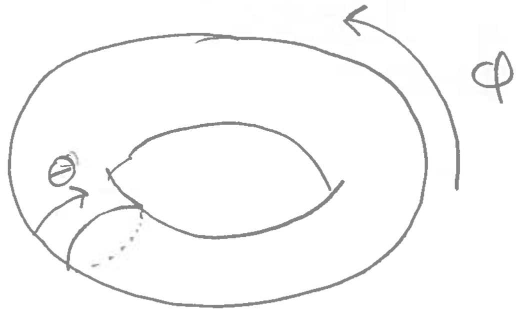
Q1:

Let's parametrize the torus with φ, θ.



φ and θ are angles so e.g. $\varphi = 0$ and $\varphi = 2\pi$ correspond to the same points.

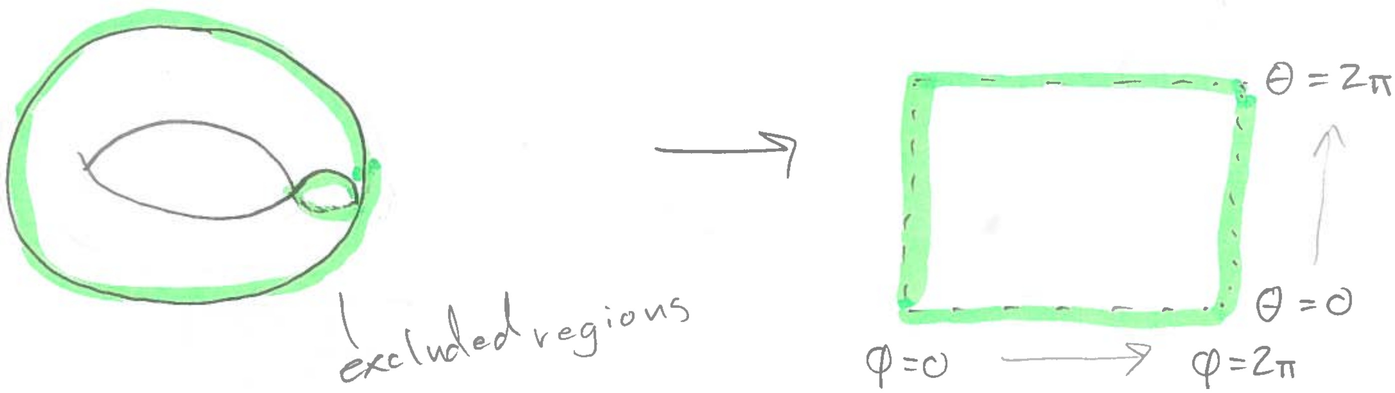
To get a one-to-one map from the torus onto an open subset of $\mathbb{R}^2$ we must exclude some points on the torus. We can e.g. choose

$$U_1 : T_2 \setminus \{(\theta, \varphi) \mid \theta \in]0, 2\pi[, \ \varphi \in]0, 2\pi[\} \longrightarrow \mathbb{R}^2,$$

torus

$$(\theta, \varphi) \longmapsto (\theta, \varphi).$$

i.e.

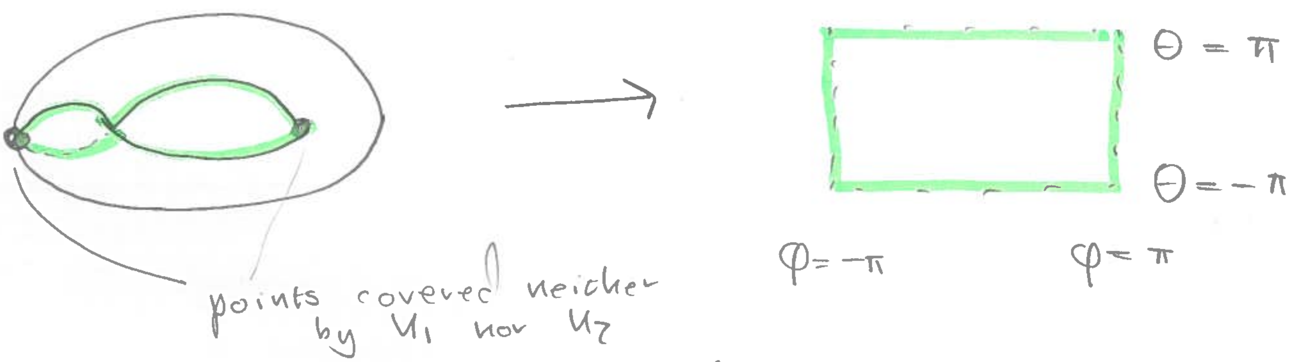


Then we need another patch for $\theta = 0$ and $\varphi = 0$

This can be chosen as

$$U_2 : \ T_2 \setminus \{(\theta, \varphi) \mid \theta \in]-\pi, \pi[, \ \varphi \in]-\pi, \pi[\} \to \mathbb{R}^2$$

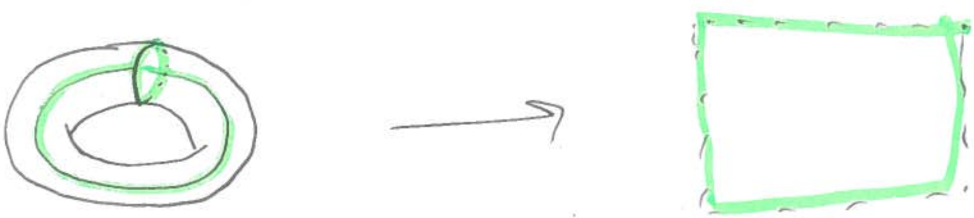
$$(\theta, \varphi) \mapsto (\theta, \varphi).$$



We have still not covered the points $\varphi = 0, \ \theta = \pi$ and $\varphi = \pi, \ \theta = 0$. Thus we need a third patch

$$U_3 : \ T_2 \setminus \{(\theta, \varphi) \mid \theta \in]\tfrac{\pi}{2}, \tfrac{5\pi}{2}[, \ \varphi \in]\tfrac{\pi}{2}, \tfrac{5\pi}{2}[\} \to \mathbb{R}$$

$$(\theta, \varphi) \mapsto (\theta, \varphi)$$



It's clear that coordinate transformations in the overlap of the patches, like $U_2 \circ U_1^{-1}$, are continuous since they're only translation in $\mathbb{R}^2$.

Q2:

Solution method 1:

We can derive this generalized geodesic equation directly from the action

$$S = -\int \sqrt{-f}\, d\sigma$$

where $f = g_{\mu\nu} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}$. Then

$$\delta S = -\frac{1}{2}\int \frac{\delta f}{\sqrt{-f}}\, d\sigma .$$

Since σ is not an affine parameter we can't assume that $f = -1$.

As in the book by Carroll (p. 107) we get that

$$\delta f = 2 g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{d}{d\sigma}(\delta x^\nu) + \partial_\lambda g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}\delta x^\lambda$$

Integrating the first term by parts gives

$$\delta S = \frac{1}{2}\int \frac{d}{d\sigma}\left(\frac{2 g_{\mu\nu}\frac{dx^\mu}{d\sigma}}{\sqrt{-f}}\right)\delta x^\nu d\sigma - \frac{1}{2}\int \partial_\lambda g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}\delta x^\lambda$$

The derivation continues in the same way as in Carroll except that now we get a new term

$$\frac{1}{2}\int 2 g_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{d\,(-f)^{-1/2}}{d\sigma} \delta x^\nu$$

$$= \frac{1}{2}\int g_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{\frac{df}{d\sigma}}{(-f)^{3/2}} \delta x^\nu.$$

Thus

$$\delta S = \int \frac{1}{\sqrt{-f}} \Big[g_{\mu x} \frac{d^2x^\mu}{d\sigma^2} + \frac{1}{2}\big(\partial_\mu g_{\nu x} + \partial_\nu g_{x\mu} - \partial_x g_{\mu\nu}\big) \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} - g_{\mu x}\frac{dx^\mu}{d\sigma} \frac{\frac{df}{d\sigma}}{2f} \Big] \delta x^x \, d\sigma$$

The equation is therefore

$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu x} \frac{dx^\nu}{d\sigma}\frac{dx^x}{d\sigma} = \frac{\frac{d}{d\sigma}\left(\frac{dx^x}{d\sigma}\frac{dx_x}{d\sigma}\right)}{2\,\frac{dx^x}{d\sigma}\frac{dx_x}{d\sigma}} \frac{dx^\mu}{d\sigma}$$

If σ is an affine parameter $\frac{d}{d\sigma}\left(\frac{dx^x}{d\sigma}\frac{dx_x}{d\sigma}\right) = \frac{d(-1)}{d\sigma} = 0$ and we get the usual geodesic equation.

<u>Solution method 2:</u>

Let λ be an affine parameter and σ a general parameter. We have that

$$\frac{d}{d\lambda} = \frac{d\sigma}{d\lambda}\frac{d}{d\sigma} \quad \text{and}$$

$$\frac{d^2}{d\lambda^2} = \frac{d}{d\lambda}\left(\frac{d\sigma}{d\lambda}\frac{d}{d\sigma}\right) = \frac{d^2\sigma}{d\lambda^2}\frac{d}{d\sigma} + \left(\frac{d\sigma}{d\lambda}\right)^2 \frac{d^2}{d\sigma^2}.$$

The geodesic equation

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\omega\chi} \frac{dx^\omega}{d\lambda} \frac{dx^\chi}{d\lambda} = 0.$$

is therefore

$$\frac{d^2\sigma}{d\lambda^2} \frac{dx^\mu}{d\sigma} + \left(\frac{d\sigma}{d\lambda}\right)^2 \frac{d^2 x^\mu}{d\sigma^2} + \Gamma^\mu_{\omega\chi} \left(\frac{d\sigma}{d\lambda}\right)^2 \frac{dx^\omega}{d\sigma} \frac{dx^\chi}{d\sigma} = 0$$

i.e. $$\frac{d^2 x^\mu}{d\sigma^2} + \Gamma^\mu_{\omega\chi} \frac{dx^\omega}{d\sigma} \frac{dx^\chi}{d\sigma} = - \frac{d^2\sigma/d\lambda^2}{(d\sigma/d\lambda)^2} \frac{dx^\mu}{d\sigma}.$$

It's nicer to write this without reference to some parameter λ we don't know.

We have that

$$0 = \frac{d}{d\sigma}(-1) = \frac{d}{d\sigma}\left(\frac{dx^\mu}{d\lambda} \frac{dx_\mu}{d\lambda}\right) = \frac{d}{d\sigma}\left(\left(\frac{d\sigma}{d\lambda}\right)^2 \frac{dx^\mu}{d\sigma} \frac{dx_\mu}{d\sigma}\right)$$

$$= \left(\frac{d\sigma}{d\lambda}\right)^2 \frac{d}{d\sigma}\left(\frac{dx^\mu}{d\sigma} \frac{dx_\mu}{d\sigma}\right) + \frac{dx^\mu}{d\sigma} \frac{dx_\mu}{d\sigma} \frac{d\lambda}{d\sigma} \frac{d}{d\lambda}\left(\frac{d\sigma}{d\lambda}\right)^2$$

$$= \left(\frac{d\sigma}{d\lambda}\right)^2 \frac{d}{d\sigma}\left(\frac{dx^\mu}{d\sigma} \frac{dx_\mu}{d\sigma}\right) + \frac{dx^\mu}{d\sigma} \frac{dx_\mu}{d\sigma} \, 2 \underbrace{\frac{d\lambda}{d\sigma} \frac{d\sigma}{d\lambda}}_{=1} \frac{d^2\sigma}{d\lambda^2}.$$

Thus $$- \frac{d^2\sigma/d\lambda^2}{(d\sigma/d\lambda)^2} = \frac{\frac{d}{d\sigma}\left(\frac{dx^\mu}{d\sigma} \frac{dx_\mu}{d\sigma}\right)}{2 \frac{dx^\mu}{d\sigma} \frac{dx_\mu}{d\sigma}}$$

and we get the same answer as above.

Q3: We have that

$$z = r\cos\theta, \qquad y = r\sin\theta\sin\varphi, \qquad x = r\sin\theta\cos\varphi$$

so

$$dx = \sin\theta\cos\varphi\,dr + r\cos\theta\cos\varphi\,d\theta - r\sin\theta\sin\varphi\,d\varphi,$$

$$dy = \sin\theta\sin\varphi\,dr + r\cos\theta\sin\varphi\,d\theta + r\sin\theta\cos\varphi\,d\varphi$$

$$dz = \cos\theta\,dr - r\sin\theta\,d\theta.$$

This means that

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$= (\sin\theta\cos\varphi\,dr + r\cos\theta\cos\varphi\,d\theta - r\sin\theta\sin\varphi\,d\varphi)^2$$

$$+ (\sin\theta\sin\varphi\,dr + r\cos\theta\sin\varphi\,d\theta + r\sin\theta\cos\varphi\,d\varphi)^2$$

$$+ (\cos\theta\,dr - r\sin\theta\,d\theta)^2$$

$$= dr^2(\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta)$$

$$+ d\theta^2 r^2(\cos^2\theta\cos^2\varphi + \cos^2\theta\sin^2\varphi + \sin^2\theta)$$

$$+ d\varphi^2 r^2\sin^2\theta(\sin^2\varphi + \cos^2\varphi)$$

$$+ dr\,d\theta\,2r(\sin\theta\cos\theta\cos^2\varphi + \sin\theta\cos\theta\sin^2\varphi - \sin\theta\cos\theta)$$

$$= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\varphi^2$$

where we've used that $\cos^2 A + \sin^2 A = 1$.

Clearly $g_{\mu\nu} = \begin{bmatrix} 1 & & 0 \\ & r^2 & \\ 0 & & r^2\sin^2\theta \end{bmatrix}$ and $g^{\mu\nu} = \begin{bmatrix} 1 & & 0 \\ & r^{-2} & \\ 0 & & r^{-2}\sin^{-2}\theta \end{bmatrix}$

Let's now calculate some Christoffel symbols.

E.g.

$$\Gamma^{\theta}_{r\theta} = \frac{1}{2} g^{\theta\mu} (\partial_r g_{\theta\mu} + \partial_\theta g_{r\mu} - \partial_\mu g_{r\theta})$$

$$= \frac{1}{2} g^{\theta\theta} (\partial_r g_{\theta\theta} + \partial_\theta g_{r\theta})$$

$$= \frac{1}{2} g^{\theta\theta} \partial_r g_{\theta\theta} = \frac{1}{2} \frac{1}{r^2} \partial_r r^2 = \frac{1}{r}$$

and

$$\Gamma^{\varphi}_{\theta\varphi} = \frac{1}{2} g^{\varphi\mu} (\partial_\theta g_{\varphi\mu} + \partial_\varphi g_{\theta\mu} - \partial_\mu g_{\varphi\theta})$$

$$= \frac{1}{2} g^{\varphi\varphi} \partial_\theta g_{\varphi\varphi} = \frac{1}{2r^2\sin^2\theta} \partial_\theta r^2 \sin^2\theta = \frac{\sin\theta}{\cos\theta} = \cot\theta.$$

The rest of the calculations are similar.

One gets that

$$\Gamma^{r}_{\theta\theta} = -r, \quad \Gamma^{r}_{\varphi\varphi} = -r\sin^2\theta$$

$$\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r}, \quad \Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta,$$

$$\Gamma^{\varphi}_{r\varphi} = \Gamma^{\varphi}_{\varphi r} = \frac{1}{r}, \quad \Gamma^{\varphi}_{\theta\varphi} = \Gamma^{\varphi}_{\varphi\theta} = \cot\theta$$

and all other components vanish.

Plugging this in the geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\sigma\chi}\frac{dx^\sigma}{d\lambda}\frac{dx^\chi}{d\lambda} = 0$$

gives

$$\frac{d^2r}{d\lambda^2} - r\left(\frac{d\theta}{d\lambda}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0 \qquad ①$$

$$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} - \sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0 \qquad ②$$

$$\frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\phi}{d\lambda} + 2\cot\theta\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} = 0. \qquad ③$$

We finally look at geodesics that go through the origin $r=0$. Muliplying ② and ③ by r and letting $r\to 0$ shows that

$$\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} = \frac{d\phi}{d\lambda}\frac{dr}{d\lambda} = 0.$$

We can assume that $\frac{dr}{d\lambda} \neq 0$ because otherwise we just have a solution that stays at the origin at all times.

Thus $\frac{d\theta}{d\lambda} = \frac{d\phi}{d\lambda} = 0$, i.e. $\theta = \text{const}$, $\phi = \text{const}$.

Plugging this into ① gives $\frac{d^2r}{d\lambda^2} = 0$ so $r = a\lambda + b$. This is exactly the equation for a straight line through the origin.

Q4:

(a) This argument is given in Appendix B in Carroll on pages 432 - 433.

(b), (c) Appendix B furthermore shows that

$$\mathcal{L}_V T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l}$$

$$= V^\sigma \partial_\sigma T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} - (\partial_\lambda V^{\mu_1}) T^{\lambda\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l}$$

$$- (\partial_\lambda V^{\mu_2}) T^{\mu_1\lambda\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} - \cdots$$

$$+ (\partial_{\nu_1} V^\lambda) T^{\mu_1\mu_2\cdots\mu_k}{}_{\lambda\nu_2\cdots\nu_l} + (\partial_{\nu_2} V^\lambda) T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\lambda\cdots\nu_l}$$

which is clearly linear in V.

Q5: We need to show that $\nabla_\mu g_{\omega x} = 0$

We have that

$$\nabla_\mu g_{\omega x} = \partial_\mu g_{\omega x} - \Gamma^\alpha_{\mu\omega} g_{\alpha x} - \Gamma^\alpha_{\mu x} g_{\omega\alpha}$$

$$= \partial_\mu g_{\omega x}$$

$$- \frac{1}{2} g^{\alpha\sigma}\left(g_{\sigma\mu,\omega} + g_{\sigma\omega,\mu} - g_{\omega\mu,\sigma}\right) g_{\alpha x}$$

$$+ \frac{1}{2} g^{\alpha\sigma}\left(g_{\sigma\mu,x} + g_{\sigma x,\mu} - g_{x\mu,\sigma}\right) g_{\omega\alpha}$$

$$= \partial_\mu g_{\omega x} - \frac{1}{2}\delta^\sigma_x\left(g_{\sigma\mu,\omega} + g_{\sigma\omega,\mu} - g_{\omega\mu,\sigma}\right)$$

$$+ \frac{1}{2}\delta^\sigma_\omega\left(g_{\sigma\mu,x} + g_{\sigma x,\mu} - g_{x\mu,\sigma}\right)$$

$$= \partial_\mu g_{\omega x} - \frac{1}{2}\left(g_{x\mu,\omega} + g_{x\omega,\mu} - g_{\omega\mu,x}\right)$$

$$+ \frac{1}{2}\left(g_{\omega\mu,x} + g_{\omega x,\mu} - g_{x\mu,\omega}\right)$$

$$= \partial_\mu g_{\omega x} - \frac{1}{2}\partial_\mu g_{\omega x} - \frac{1}{2}\partial_\mu g_{\omega x} = 0.$$

Q5:

(a) With the metric $ds^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$

the only non-zero Christoffel symbols are

$$\Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta, \qquad \Gamma^{\varphi}_{\theta\varphi} = \Gamma^{\varphi}_{\varphi\theta} = \cot\theta$$

Thus the geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^{\mu}_{\omega\chi}\frac{dx^\omega}{d\lambda}\frac{dx^\chi}{d\lambda} = 0$$

becomes

$$\begin{cases} \dfrac{d^2\theta}{d\lambda^2} - \sin\theta\cos\theta\left(\dfrac{d\varphi}{d\lambda}\right)^2 = 0 \\ \dfrac{d^2\varphi}{d\lambda^2} + 2\cot\theta \dfrac{d\theta}{d\lambda}\dfrac{d\varphi}{d\lambda} = 0 \end{cases}$$

If $\varphi = \text{const}$ (lines of constant longitude) the equations reduce to $\dfrac{d^2\theta}{d\lambda^2} = 0$ which is solved by

$$\theta = a\lambda + b.$$

If, on the other hand, $\theta = \theta_0$ (constant) we get

$$\begin{cases} \sin\theta\cos\theta\left(\dfrac{d\varphi}{d\lambda}\right)^2 = 0 \\ \dfrac{d^2\varphi}{d\lambda^2} = 0. \end{cases}$$

One solution is $\varphi = \text{const}$ but that's uninteresting (that's just a fixed point).

Otherwise $\frac{d^2\varphi}{d\lambda^2} = 0$ and $\sin\theta_0 \cos\theta_0 = 0$

which is only solved by $\theta_0 = \frac{\pi}{2}$.

(The solutions $\theta_0 = 0$, $\theta_0 = \pi$ correspond to a fixed point, i.e the north pole or the south pole).

Thus the only geodesic of constant latitude is the equator.

(b) The initial vector is $V^\mu = (1, 0)$ which we parallel transport around a circle of constant θ.

If we travel the circle with speed 1 the parallel transport equation is

$$\nabla_\varphi V^\mu = 0, \quad \text{i.e.}$$

$$\partial_\varphi V^\mu + \Gamma^\mu_{\varphi\nu} V^\nu = 0.$$

This gives

$$\begin{cases} \partial_\varphi V^\theta - \sin\theta\cos\theta\, V^\varphi = 0 \\ \partial_\varphi V^\varphi + \cot\theta\, V^\theta = 0 . \end{cases}$$

Thus $$\partial_\varphi^2 V^\theta = \sin\theta\cos\theta\, \partial_\varphi V^\varphi$$

$$= -\sin\theta\cos\theta\cot\theta\, V^\theta$$

$$= -\cos^2\theta\, V^\theta$$

This equation has the solution

$$V^\theta = A\cos((\cos\theta)\varphi) + B\sin((\cos\theta)\varphi)$$

Since $V^\theta(\varphi=0) = 1$ we get that $A=1$.

Furthermore

$$V^\varphi = \frac{1}{\sin\theta\cos\theta}\partial_\varphi V^\theta$$

$$= -\frac{1}{\sin\theta}\sin(\cos\theta\,\varphi) + \frac{B}{\sin\theta}\sin(\cos\theta\,\varphi)$$

and $V^\varphi(\varphi=0)=0$ shows that $B=0$.

After one round around the circle we thus get

$$\underline{V^\theta = \cos(2\pi\cos\theta)}, \qquad \underline{V^\varphi = -\frac{1}{\sin\theta}\sin(2\pi\cos\theta)}$$

As a check we see that we get the original vector if $\theta = \frac{\pi}{2}$ (then the circle is a geodesic).