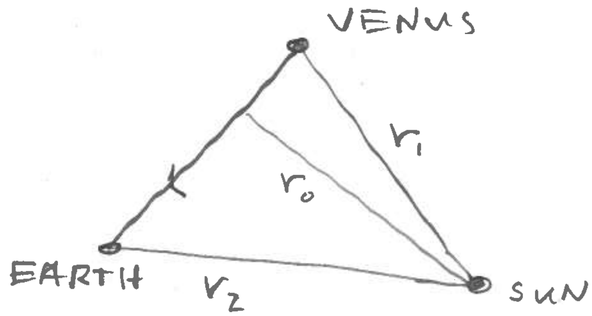
Assignment 9

Problem 1

In class there were two calculations.

The Newtonion one assumes that the light travels in a straight path

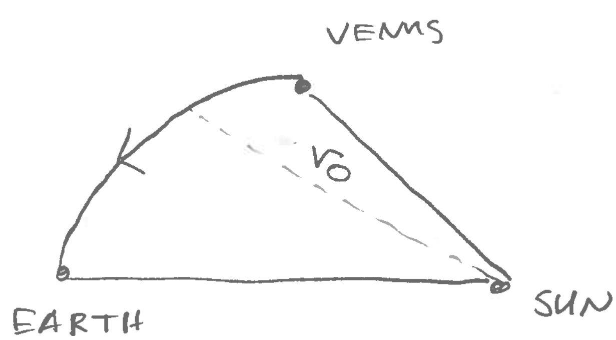


with r_0 being the point of closest contact.

This gives

$$t_{Newton} = \sqrt{r_1^2 - r_0^2} + \sqrt{r_2^2 - r_0^2}$$

The GR calculation follows a light ray along a geodesic



Here r_0 is not the point of closest distance but simply the point where

$$\frac{dr}{dt}\Big|_{r=r_0} = 0.$$

This analysis gives

$$t_{GR} = t(r_0, r_1) + t(r_0, r_2)$$

where

$$t(r_0, r_1) = \sqrt{r_1^2 - r_0^2} + 2m \ln\left(\frac{r_1 - \sqrt{r_1^2 - r_0^2}}{r_0}\right) + m\left(\frac{r_1 - r_0}{r_1 + r_0}\right)^{1/2}$$

with m the mass of the sun.

The problem with comparing these two answers is that r_0 has different meaning in the different expressions

We can assume that

$$(r_0)_{Newton} = (r_0)_{GR} + O(m).$$

Then

$$t_{Newton} = \sqrt{r_1^2 - (r_0)_{GR}^2} + \sqrt{r_2^2 - (r_0)_{GR}^2} + O(m)$$

where $O(m)$ is some higher order correction that would have to be calculated explicitly

Problem 2

The expansion of the universe is governed by the Friedmann equation

$$H^2 = \frac{8\pi G}{3}\rho$$

where $H = \frac{\dot{a}}{a}$ and in our case

$\rho = \rho_m + \rho_\Lambda$. Conservation of energy, along with the equation of state tells us that

$$\rho_m = \frac{\rho_{m0}}{a^3}, \quad \rho_\Lambda = \rho_{\Lambda 0}$$

where $\rho_{m0}, \rho_{\Lambda 0}$ are some constants

(see p. 334 in the textbook.)

The metric in a flat universe is

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2 d\Omega \right].$$

We can scale $a \to \lambda a$ as long as we do $r \to \frac{1}{\lambda} r$ as well.

Thus we're allowed to set $a=1$ today.

Letting H_0 be the Hubble constant today we see that

$$H_0^2 = \frac{8\pi G}{3}(\rho_{0m} + \rho_{0\Lambda})$$

Furthermore $\dfrac{\rho_{0m}}{\rho_{0m}+\rho_{0\Lambda}} = 0.3$, $\dfrac{\rho_{0\Lambda}}{\rho_{0m}+\rho_{0\Lambda}} = 0.7$

because of the distribution of energy today.

Friedmann's equation gives that

$$\dot{a} = \frac{da}{dt} = \sqrt{\frac{8\pi G}{3}\rho}\, a$$

so the age of the universe is

$$T = \int_0^T dt = \int_0^1 da \frac{1}{a\sqrt{\frac{8\pi G}{3}\rho}}$$

since we know that $a \to 0$ at the beginning of the universe.

Thus

$$T = \int_0^1 \frac{da}{a\sqrt{\frac{8\pi G}{3}\rho}} = \int_0^1 \frac{da}{H_0 a \sqrt{\frac{\frac{8\pi G}{3}\rho}{\frac{8\pi G}{3}(\rho_{m0}+\rho_{\Lambda 0})}}}$$

$$= \frac{1}{H_0} \int_0^1 \frac{da}{a\sqrt{\frac{\rho_{m0}}{\rho_{m0}+\rho_{\Lambda 0}} \frac{1}{a^3} + \frac{\rho_{\Lambda 0}}{\rho_{m0}+\rho_{\Lambda 0}}}}$$

$$= \frac{1}{H_0} \int_0^1 \frac{da}{a\sqrt{0.3\frac{1}{a^3} + 0.7}}.$$

This integral can be evaluated numerically.

We get that

$$T = \frac{0.964099}{H_0} = \frac{0.964099}{0.7 \times 100 \text{ km s}^{-1} \text{ Mpc}^{-1}}$$

$$= \frac{0.964009}{0.7 \times 100 \text{ s}^{-1}} \frac{3.0857 \times 10^{22} \text{ m}}{1000 \text{m}}$$

$$= 4.25 \times 10^{17} \text{ s} \simeq 1.35 \times 10^{10} \text{ y}$$

$$\simeq \underline{\underline{13.5 \text{ billion years}}}$$

Problem 3

The equation of motion for a scalar field in an expanding universe is

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

and the Friedmann equation is

$$H^2 = \frac{1}{3m_p^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right).$$

(see eq. 8.191 and 8.192 in the textbook).

The slow rolling approximation means that the field changes gradually enough for inflation to carry on for some time

Thus $\dot{\phi}^2 \ll V(\phi)$ ⊛

$$|\ddot{\phi}| \ll |3H\dot{\phi}|, |V'|$$

(The expansion is rapid, i.e. H is big, which explains why we demand that $|\ddot{\phi}| \ll |3H\dot{\phi}|$.)

We need to show that ⊛ is equivalent to

$$\varepsilon := \frac{1}{2}\overline{m}_p^2\left(\frac{V'}{V}\right)^2 \qquad \text{and}$$

$$\eta := \overline{m}_p^2\left(\frac{V''}{V}\right)$$

being small quantities, $\varepsilon, \eta \ll 1$.

Since $\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$ and $\ddot{\phi}$ is small

we have that $H\dot{\phi} \sim V'(\phi)$.

Furthermore $H^2 \sim \dfrac{V}{\overline{m}_p^2}$.

Thus $(V')^2 \sim H^2\dot{\phi}^2 \sim \dfrac{V}{\overline{m}_p^2}\dot{\phi}^2 \ll \dfrac{V^2}{\overline{m}_p^2}$.

This shows that $\varepsilon = \frac{1}{2}\overline{m}_p^2\left(\frac{V'}{V}\right)^2 \ll 1$.

We have that

$$V' \sim H\dot{\phi}$$

so taking the time derivative gives

$$V''\dot{\phi} \sim \dot{H}\dot{\phi} + H\ddot{\phi}$$

Let's first assume that $V''\dot{\phi} \sim H\ddot{\phi}$.

Then

+

$$V''\dot{\phi} \sim H\ddot{\phi} \ll H^2\dot{\phi}$$

so $$V'' \ll H^2 \sim \frac{V}{\bar{m}_p^2}$$

i.e. $$\eta = \bar{m}_p^2 \frac{V''}{V} \ll 1.$$

If, however, $V''\dot{\phi} \sim \dot{H}\dot{\phi}$ in $\textcircled{+}$

then

$V'' \sim \dot{H}$. Using that $H^2 \sim \frac{V}{\bar{m}_p^2}$

and taking the time derivative we get that

$$H\dot{H} \sim \frac{\dot{V}'}{\bar{m}_p^2}\dot{\phi}$$

Thus $$V'' \sim \dot{H} \sim \frac{V'\dot{\phi}}{\bar{m}_p^2 H} \sim \frac{\dot{\phi}^2}{\bar{m}_p^2} \ll \frac{V}{\bar{m}_p^2}.$$

In other words, we see again that

$$\eta = \bar{m}_p^2 \frac{V''}{V} \ll 1.$$

Looking at the case of $V(\phi) = \frac{1}{4}\lambda\phi^4$ we see

that $$\varepsilon = \bar{m}_p^2 \frac{(\lambda\phi^3)^2}{(\frac{1}{4}\lambda\phi^4)^2} = \frac{16\,\bar{m}_p^2}{\phi^2}$$

and

$$\eta = \overline{m}_P^2 \frac{V''}{V} = \overline{m}_P^2 \frac{3\lambda\phi^2}{\frac{1}{4}\lambda\phi^4} = \frac{12\,\overline{m}_P^2}{\phi^2}.$$

Thus $\varepsilon, \eta \ll 1$ only requires $\overline{m}_P \ll \phi$.

The slow-rolling approximation ceases to be true when $\phi \sim \overline{m}_P$.

Problem 4

In a flat FRW spacetime the metric is

$$ds^2 = -dt^2 + a^2(t) R_0^2 \left[d\chi^2 + \chi^2 d\Omega^2\right].$$

Thus a circle at fixed χ at time t has radius

$$2\pi a R_0 \chi$$

and the angle (in radians) of an object of proper size L on this circle is

$$\theta = \frac{L}{2\pi a R_0 \chi} 2\pi = (1+z) \frac{L}{R_0 \chi(z)}$$

where we've used that $a = \frac{1}{1+z}$. The only thing left is to find $R_0 \chi(z)$ which corresponds to the coordinate distance a light ray has travelled if it was emitted at redshift z and reaches us now.

For light we have that

$$0 = -dt^2 + a^2 R_0^2 d\chi^2$$

so $$R_0 \chi = \int \frac{dt}{a} \ldots$$

We denote the scale factor at the time of emission by a_* and choose the normalization so that $a = 1$ today.

Since $da = \dot{a}\,dt$ we see that

$$R_0 \chi = \int \frac{dt}{a} = \int_{a_*}^{1} \frac{da}{\dot{a}a} = \int_{a_*}^{1} \frac{da}{a^2 H}$$

where $H = \dot{a}/a$ is the Hubble factor.

Friedmann's equation tells us that

$$H^2 = \frac{8\pi G}{3} \rho_m$$

where $\rho_m = \rho_{m0}/a^3$ for matter with ρ_{m0} some constant. Denoting today's Hubble constant by H_0 we see that

$$H_0^2 = \frac{8\pi G}{3} \rho_{m0} \qquad \text{so}$$

$$H^2 = H_0^2/a^3.$$

Thus finally

$$R_0 \chi = \int_{a_*}^{1} \frac{da}{a^2 H} = \frac{1}{H_0} \int_{a_*}^{1} \frac{da}{a^{1/2}} = \frac{2}{H_0} (1 - a_*^{1/2}).$$

This means that

$$\theta(z) = (1+z)\,\frac{L}{R_0 \chi}$$

$$= \frac{L H_0}{2}\,\frac{1+z}{1-\frac{1}{\sqrt{1+z}}}$$

where z is the redshift when the light was emitted.

We now want to find which z minimizes θ.

It's easy to see that

$$\theta'(z) = \frac{H_0 L}{2}\,\frac{1}{1-\frac{1}{\sqrt{1+z}}}\left[1 - \frac{1}{2}\,\frac{1/\sqrt{1+z}}{\left(1-\frac{1}{\sqrt{1+z}}\right)}\right]$$

and that $\theta'(z)=0$ when $z = 5/4 = 1.25$.

(One can also check that this is indeed a minimum.)

Substituting $z = 1.25$ we get that

$$\theta_{min} = \frac{27}{8}\,\frac{H_0 L}{c}$$

(I've added a factor of c to make the units consistent.)

Substituting $L = 10$ kpc and $H_0 = 70$ km/s/Mpc

we get that $\theta_{min} \approx 7.88 \times 10^{-6}$ //.

Problem 5/6

(a) The idea here is that before $t=0$ the universe is contracting until it collapses in a 'Big Crunch' at $t=0$. After that (for $t>0$) the universe starts expanding. Thus, in this model Big Bang / Big Crunch is preceded by a contracting phase. We want to see whether this model can reproduce some of the sucesses of inflationary cosmology.

The Friedmann equation is $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho$ and in a matter dominated universe $\rho \sim \frac{1}{a^3}$. We will stick with parametric estimates in this part. We see that

$$\dot{a} \sim \pm a^{-1/2}$$

and we choose the minus sign.

Thus $a^{1/2}\, da \sim -dt$

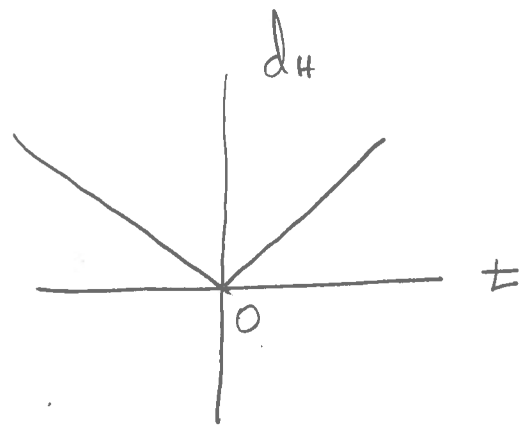
so $a^{3/2} \sim -t$, i.e. $a \sim (-t)^{2/3}$ for $t<0$.

This means that that the Hubble parameter

is $H = \frac{\dot{a}}{a} \sim -\frac{1}{t}$

and the Hubble radius is

$d_H = H^{-1}c \sim -ct$

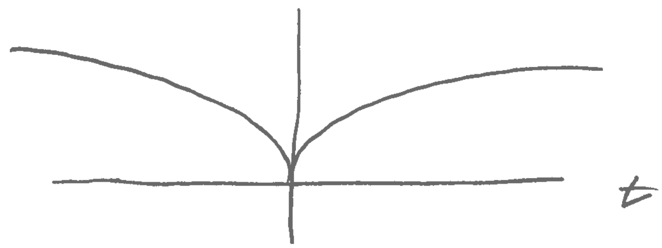


We next consider the particle horizon which is the distance a photon can have travelled within the age of the universe.

Since the universe started at $t = -\infty$ a photon can have travelled arbitrarily far since the beginning of time. This means that the particle horizon is infinite at all times. In particular, there is no horizon problem

We finally consider the wavelength of a fixed comoving scale. If some wavelength is λ in the coordinate distance then the physical distance is $a\lambda \sim (-t)^{2/3}$

It's behaviour with time is



(b) We have that the cosmological fluctuations evolve according to

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0.$$

where

$$v = a\left(\delta\varphi + \frac{\varphi_0'}{H}\phi\right)$$

with $\delta\varphi$ the fluctuations in the matter and ϕ the metric perturbations.

z is <u>not</u> the redshift but rather proportional to the scale factor a.

Thus $\dfrac{z''}{z} = \dfrac{a''}{a}.$

Finally, $f' = \frac{df}{d\tau}$ where

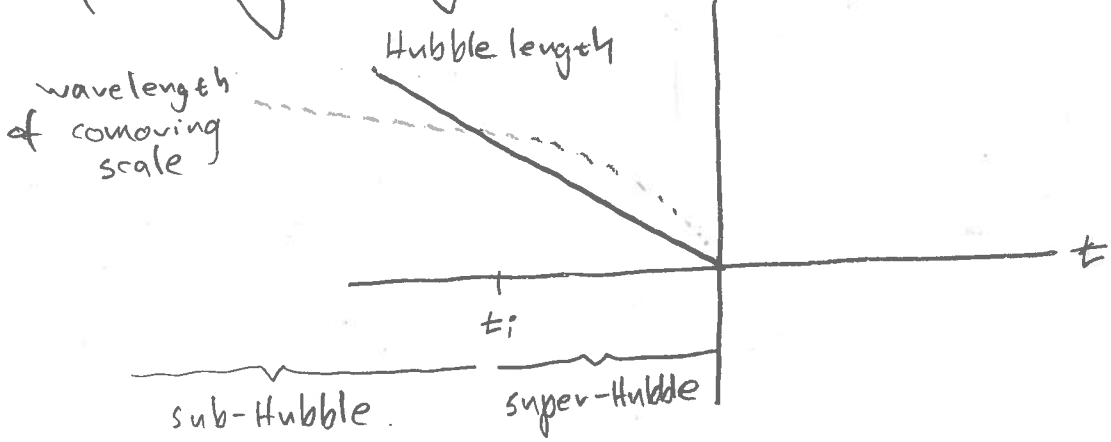
$dt = a\, d\tau$. (τ is called conformal time)

We work in Fourier space where k is the wavenumber.

There are two different regimes:

- If $k^2 > \frac{a''}{a}$ then v_k oscillates. We call this the sub-Hubble regime.
- If $k^2 < \frac{a''}{a}$ then v_k grows. We call this the super-Hubble regime and this is what we're interested in.

Using the analysis of (a) we can draw the following figure:



Let's find the conformal time τ_i when we go to the super-Hubble regime for given k_i i.e. let's find when

$$k^2 = \frac{a''}{a}.$$

We have that $a = b(-t)^{2/3}$

where b is some constant.

Then (setting $\tau = 0$ at $t = 0$)

$$\tau = \int d\tau = \int \frac{dt}{a} = \frac{1}{b} \int \frac{dt}{(-t)^{2/3}} = \frac{3}{b}(-t)^{1/3}$$

This means that

$$a = b \frac{b^2}{9} \tau^2 = \frac{b^3}{9} \tau^2.$$

and

$$a'' = 2 \frac{b^3}{9}.$$

Thus

$$\frac{a''}{a} = \frac{2}{\tau^2}.$$

The equation $\frac{2}{\tau_i^2} = k^2$

has the solution $\tau_i = -\frac{\sqrt{2}}{k}.$

Since k is small we can approximate the differential equation by

$$v_k'' - \frac{a''}{a} v_k = 0,$$

i.e.

$$v_k'' - \frac{2}{\tau^2} v_k = 0$$

This has solutions of the form

$$v_k = A\,(-\tau)^2 + B\,\frac{1}{-\tau}.$$

We apply the vacuum initial conditions

$$v_k(\tau_i) = \frac{1}{\sqrt{2k}}$$

$$v_k'(\tau_i) = \sqrt{\frac{k}{2}}$$

i.e.

$$\begin{cases} A\,\dfrac{2}{k^2} + B\,\dfrac{k}{\sqrt{2}} = \dfrac{1}{\sqrt{2k}} \\[2ex] -2A\,\dfrac{\sqrt{2}}{k} + B\,\dfrac{k^2}{2} = \sqrt{\dfrac{k}{2}} \end{cases}$$

This can be solved to give

$$A = \frac{-1}{6}\left(1 - \frac{1}{\sqrt{2}}\right) k^{3/2}, \qquad B = \frac{1}{3}\left(2 + \sqrt{2}\right)\frac{1}{k^{3/2}}$$

Thus

$$V_k = -\frac{1}{6}\left(1-\frac{1}{\sqrt{2}}\right) k^{3/2}(-\tau)^2$$

$$+ \frac{1}{3}\left(2+\sqrt{2}\right)\frac{1}{k^{3/2}}\frac{1}{-\tau}.$$

We're interested in the solution for low k, close to the bouncing point $\tau = 0$. Thus we can drop the first term giving

$$V_k = \frac{1}{3}\left(2+\sqrt{2}\right)\frac{1}{k^{3/2}}\frac{1}{-\tau}.$$

The power spectrum is then

$$\Delta^2 = k^3 |V_k|^2$$

$$= \frac{1}{9}\left(2+\sqrt{2}\right)^2 \frac{1}{(-\tau)^2}.$$

(c) The power spectrum we found in (b) is independent of the scale k. This matches the predictions of the inflationary scenario.