Casimirs of the conformal group

Yan Gobeil

September 2017

We discuss the quadratic, cubic and quartic Casimir operators of the conformal group, which commute with all the generators. We write their expression in terms of the usual conformal generators and their eigenvalue on an irreducible representation. We work in Euclidean signature, but generalizations are easy to find. The conventions are the same as Simmons-Duffins notes.

1 Conformal Algebra

The conformal group in d dimensions is composed of 1 dilation operator D, d translations P_i, d special conformal transformations K_i and $\frac{d(d-1)}{2}$ rotations J_{ij}, which is an antisymmetric matrix. The Roman indices run from 1 to d and there are a total of $\frac{(d+2)(d+1)}{2}$ conformal generators. The commutation relations of these generators are

$$\begin{aligned}
&[D,P_i]=P_i\\
&[D,K_i]=-K_i\\
&[M_{ij},P_k]=\delta_{jk}P_i-\delta_{ik}P_j\\
&[M_{ij},K_k]=\delta_{jk}K_i-\delta_{ik}K_j\\
&[K_i,P_j]=2\delta_{ij}D-2J_{ij}\\
&[J_{ij},J_{k\ell}]=\delta_{jk}J_{i\ell}-\delta_{ik}J_{j\ell}+\delta_{k\ell}J_{ki}-\delta_{i\ell}J_{kj}\,.
\end{aligned} \tag{1}$$

The rotations obviously satisfy the $SO(d)$ algebra.

It is useful to repackage these generators together into an antisymmetric matrix M_{AB} with indices running from -1 to d and metric $\eta_{-1-1}=-1$, $\eta_{00}=\eta_{ii}=1$. The relations used to define this are the following

$$\begin{aligned}
M_{-10}&=D\\
M_{ij}&=J_{ij}\\
M_{0i}&=\frac{P_i+K_i}{2}\\
M_{-1i}&=\frac{P_i-K_i}{2}\,.
\end{aligned} \tag{2}$$

The reason for doing this is that the M_{AB} satisfy the algebra

$$[M_{AB},M_{CD}]=\eta_{BC}M_{AD}-\eta_{AC}M_{BD}+\eta_{BD}M_{CA}-\eta_{AD}M_{CB}\,, \tag{3}$$

which is simply the $SO(d+1,1)$ algebra.

Rrepresentations of the conformal group are built as highest weight representation from a primary state. The states are labelled by the eigenvalue of D, which are the conformal dimensions Δ, and the raising/lowering operators are P_i/K_i. A primary state $|\Delta\rangle$ has the smallest dimension and satisfies $K_i\,|\Delta\rangle=0$. The rest of the reprentation is built from that primary by applying P_i repeatedly. As for J_{ij}, the primaries can be organized into any irreps of $SO(d)$ and the generators act as they usually do.

2 Quadratic Casimir

A Casimir operator for an algebra is a combination of the generators that commutes with all the generators of the algebra. The one that is used most often is the quadratic Casimir, which is build from quadratic combinations of the generators. For the conformal group, this quadratic Casimir is

$$c_2 = \frac{1}{2} M^{AB} M_{BA} \,. \tag{4}$$

We can easily check that it commutes with every generator

$$\begin{aligned}
[c_2, M_{CD}] \propto \left[M^{AB} M_{BA}, M_{CD}\right] &= M^{AB}[M_{BA}, M_{CD}] + \left[M^{AB}, M_{CD}\right] M_{BA} \\
&= -M^{AB}(\eta_{BC} M_{AD} - \eta_{AC} M_{BD} + \eta_{BD} M_{CA} - \eta_{AD} M_{CB}) \\
&- (\eta^B{}_C M^A{}_D - \eta^A{}_C M^B{}_D + \eta^B{}_D M_C{}^A - \eta^A{}_D M_C{}^B) M_{AB} \\
&= -M^A{}_C M_{AD} + M_C{}^B M_{BD} - M^A{}_D M_{CA} + M_D{}^B M_{CB} \\
&- M^A{}_D M_{AC} + M^B{}_D M_{CB} - M_C{}^A M_{AD} + M_C{}^B M_{DB} = 0
\end{aligned} \tag{5}$$

It is important to be able to rewrite this operator in terms of the usual conformal generators. For this we just need to expand the sum and use the relation introduced above along with the antisymmetry of M_{AB}.

$$\begin{aligned}
c_2 &= \frac{1}{2} M^{AB} M_{BA} = \frac{1}{2}(M^{ij} M_{ji} + M^{-10} M_{0-1} + M^{-1j} M_{j-1} + M^{0-1} M_{-10} + M^{0j} M_{j0}) \\
&= \frac{1}{2} M^{ij} M_{ji} + M_{-10} M_{-10} + M^{-1i} M_{-1i} - M^{0i} M_{0i} \\
&= \frac{1}{2} J_{ij} J_{ji} + D^2 + \frac{(P_i - K_i)(P_i - K_i)}{4} - \frac{(P_i + K_i)(P_i + K_i)}{4} \\
&= \frac{1}{2} J_{ij} J_{ji} + D^2 - \frac{1}{2} P_i K_i - \frac{1}{2} K_i P_i = \boxed{\frac{1}{2} J_{ij} J_{ji} + D(D - d) - P_i K_i}
\end{aligned} \tag{6}$$

In the last step we used the conformal algebra to rewrite the last terms.

Schur's lemma tells us that an operator that commutes with all the generators of an algebra has to be proportional to the identity when evaluated on an irreducible representation of the algebra. This must be the case for the quadratic Casimir, and since the value doesn't depend on the state it is simpler to use the primary directly. The value of D on such state is Δ and K_i kill it. For a state belonging to a symmetric traceless spin ℓ representation of $SO(d)$, the value of $\frac{1}{2} J_{ij} J_{ji}$ is known to be $\ell(\ell + d - 2)$. The value of the quadratic Casimir is then

$$\boxed{c_2 = \Delta(\Delta - d) + \ell(\ell + d - 2)} \,. \tag{7}$$

3 Cubic Casimir

There is always the possibility of having an operator that commutes with every generator and is made of cubic combinations of the elements of the algebra. For the conformal algebra the guess would be

$$c_3 = \frac{1}{2} M^{AB} M_{BC} M^C{}_A \,. \tag{8}$$

However after expanding this in terms of the conformal generators it turns out that it is simply proportional to the quadratic Casimir. The explicit relation is $c_3 = \frac{d}{2} c_2$. This can be simply explained by the fact that the guess for the cubix Casimir is the trace of a product of an odd number of anti-symmetric matrices, which vanishes. The reason why we find a nonzero result is because of the substlety that the trace vanishes up to commutators, which however reduce the order of the terms and elads to the quadratic Casimir. This can be said for every odd Casimir.

4 Quartic Casimir

Continuing on the idea, there is a quartic Casimir for the conformal group, which is

$$c_4 = \frac{1}{2} M^{AB} M_{BC} M^{CD} M_{DA} \,. \tag{9}$$

The actual calculations needed to write it in terms of the conformal generators are long so we did it using mathematica and the result is

$$\begin{aligned} c_4 =& D^2(D-d)^2 + \frac{d(d-1)}{2} D(D-d) - \frac{1}{2} J_{ij} J_{ji} + \frac{1}{2} J_{ij} J_{jk} J_{k\ell} J_{\ell i} + \frac{(d-1)(4-3d)}{2} P_i K_i \\ &+ \frac{1}{2} P_i P_i K_j K_j + \frac{1}{2} P_i P_j K_j K_i + 3(d-1) P_i D K_i + 3(d-1) P_i J_{ij} K_j \\ &- 2 P_i D^2 K_i - 2 P_i J_{ij} D K_j - 2 P_i J_{ij} J_{jk} K_k \,. \end{aligned} \tag{10}$$

Given that the value of the $\frac{1}{2} J^4$ term on a symmetric traceless tensor is $\ell^2(\ell+d-2)^2 + \frac{(d-2)(d-3)}{2} \ell(\ell+d-2)$, the total eigenvalue of the quartic Casimir is

$$c_4 = \Delta^2(\Delta-d)^2 + \frac{d(d-1)}{2} \Delta(\Delta-d) + \ell^2(\ell+d-2)^2 + \frac{(d-1)(d-4)}{2} \ell(\ell+d-2) \,. \tag{11}$$

5 Casimirs in 2d CFT

In a 2d CFT, we often work with a different basis for the generators that is related to the Virasoro algebra. We introduce a complex coordinate $z = x + iy$ for the plane and use the generators $L_n = -z^{n+1} \partial_z$ along with their complex conjugates $\bar{L}_n$. Since we study only global conformal symmetry in general dimensions, we need only $n = -1, 0, 1$. Their algebra closes to two commuting copies of the Witt algebra

$$[L_m, L_n] = (m-n) L_{m+n} \,. \tag{12}$$

It is easy to convert the usual conformal generators to the new basis by using the chain rule on the differential operators realization of the conformal symmetries.

$$\begin{aligned} D &= -x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} = L_0 - \bar{L}_0 \\ P_1 &= -\frac{\partial}{\partial x} = L_{-1} + \bar{L}_{-1} \\ P_2 &= -\frac{\partial}{\partial y} = i(L_{-1} - \bar{L}_{-1}) \\ K_1 &= (x^2+y^2) \frac{\partial}{\partial x} - 2x \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) = L_1 + \bar{L}_1 \\ K_2 &= (x^2+y^2) \frac{\partial}{\partial y} - 2y \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) = -i(L_1 - \bar{L}_1) \\ J_{12} &= x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} = -i(L_0 - \bar{L}_0) \,. \end{aligned} \tag{13}$$

We can now use this in the general expression for the quadratic Casimir to obtain $c_2 = 2(\mathcal{C} + \bar{\mathcal{C}})$ with

$$\mathcal{C} = L_0(L_0-1) - L_{-1} L_1 \,. \tag{14}$$

When we use the new basis for the quartic Casimir, we get

$$\begin{aligned} \frac{1}{2} c_4 =& L_0(L_0-1)(L_0^2 - L_0 + 1) - L_{-1} L_1 + L_{-1}^2 L_1^2 - 2 L_{-1} L_0^2 L_1 \\ &+ \bar{L}_0(\bar{L}_0-1)(\bar{L}_0^2 - \bar{L}_0 + 1) - \bar{L}_{-1} \bar{L}_1 + \bar{L}_{-1}^2 \bar{L}_1^2 - 2 \bar{L}_{-1} \bar{L}_0^2 \bar{L}_1 + 6 \mathcal{C} \bar{\mathcal{C}} \,. \end{aligned} \tag{15}$$

However the last term commutes with every generator by itself since it is built from the quadratic Casimir and we can substract $\mathcal{C}$ and $\bar{\mathcal{C}}$ from our c_4 such that we can rewrite the quartic Casimir as $c_4 = 2(\mathcal{C}' + \bar{\mathcal{C}}')$ with

$$\mathcal{C}' = L_0^2(L_0 - 1)^2 + L_{-1}^2 L_1^2 - 2L_{-1}L_0^2 L_1\,. \tag{16}$$

Actually, looking carefully at this expression, we notice quickly that it is simply equal to $\mathcal{C}^2$ so overall we can rewrite the original quartic Casimir as

$$\frac{1}{2}c_4 = \mathcal{C}^2 + \bar{\mathcal{C}}^2 + \mathcal{C} + \bar{\mathcal{C}} + 6\mathcal{C}\bar{\mathcal{C}}\,. \tag{17}$$

The fact that the quartic Casimir can be expressed in terms of the quadratic ones in 2d comes from the factorization of the 2d conformal group $SO(3,1)$ into $SU(2) \times SU(2)$. As a rank two algebra, $SO(3,1)$ has two independent Casimirs and they can simply be chosen to be the quadratic Casimirs of the two $SU(2)$, which is exactly what $\mathcal{C}$ and $\bar{\mathcal{C}}$ are. We don't expect such a simplification to happen in higher dimensions.

6 Eigenvalues of rotation Casimirs