Characters of conformal field theories

Yan Gobeil

March 2018

We compute the contribution to the partition function of conformal field theories in general dimensions coming from a single conformal family. We also consider turning on a chemical potential for angular momentum in 3d and look at the special case of 2d, where the Virasoro algebra is important.

1 General dimensions

In a CFT, the states are organized in terms of conformal families, which include a primary state $|\mathcal{O}\rangle$ and all of its descendants, obtained from the primary by acting with the momentum operator P_μ many times. If the primary state has scaling dimension Δ (eigenvalue of the dilation operator D), each application of the momentum operator increases that dimension by 1. All the states with n applications of momentum are said to be at level n and the states in different levels are orthogonal. To know how many such states there are at a given level n in d dimensions, we simply count the number of independant components of a totally symmetric tensor (since the P's commute) of rank n in d dimensions. This is done in another note and the result is

$$N_d(n) = \binom{d+n-1}{n}.$$

Given this, the object that we are interested in is the partition function of the theory at temperature $T = \frac{1}{\beta}$, defined as

$$Z(\beta) = \mathrm{Tr}\big[e^{-\beta H}\big] = \sum_i \langle i|e^{-\beta H}|i\rangle = \sum_i e^{-\beta E_i}\,.$$

This basically counts the number of states at a given energy. In a CFT, the Hamiltonian is the dilation operator so we define $q = e^{-\beta}$ and the energies are just the scaling dimensions of the states. We separate the sum into conformal famillies to write

$$Z(\beta) = \sum_{\mathcal{O}} q^\Delta \sum_{n=0}^{\infty} N_d(n) q^n\,.$$

The contribution to the partition function from a single primary, called the character, is then

$$\chi_{\mathcal{O}}(q) = q^\Delta \sum_{n=0}^{\infty} N_d(n) q^n = q^\Delta \sum_{n=0}^{\infty} \binom{d+n-1}{n} q^n = \boxed{\frac{q^\Delta}{(1-q)^d}}\,.$$

The sum was done using Mathematica.

2 Including angular momentum

It's clear that labeling the states in a representation of the conformal algebra (a conformal family) by their scaling dimensions is not enough to determine them uniquely since all the states in a given level have the same label. It's then natural to include extra labels coming from the eigenvalue of operators that commute with D so they can all be diagonalized simultaneously. This needs to be done separately for different dimensions so we do it for 3d Euclidean space here.

The rotation group in the 3d case is just $SO(3) = SU(2)$ so we can organize the states into the spin representations and label them by the eigenvalues of J^2 and J_z. It turns out that at level $n = 2k$, the representations that are present have any even spin starting at 0 and up to n. For level $n = 2k+1$, the representations that are involved have any odd spin from 1 to n. Of course in a representation of spin ℓ there are states with eigenvalue j of J_z going from $-\ell$ to ℓ. The counting of the number of states in a level is right this way and we can even construct the basis explicitly, but we don't need to here.

The object that we are now interested in includes all the Cartan generators of the conformal algebra

$$Z(q,y) = \mathrm{Tr}\big[q^D y^{J_z}\big] = \sum_i q^{\Delta_i} y^{j_i} .$$

The contribution to this from a single primary is given by

$$\chi_{\mathcal{O}}(q,y) = q^{\Delta}\left(\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{j=-2\ell}^{2\ell} q^{2k}y^j + \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{j=-2\ell-1}^{2\ell+1} q^{2k+1}y^j\right) .$$

To get that sum we separated the even and odd levels and summed over all the spin representations present to account for the states in a given level. Using Mathematica, we can compute these sums explicitly to get

$$\chi_{\mathcal{O}}(q,y) = q^{\Delta}\frac{y}{(1-q)(q-y)(qy-1)} .$$

Because each power y^j need to appear in the sum with its inverse y^{-j}, it is convenient to express the result in terms of the variable $u = y + y^{-1} - 2$, which is normalized so that the limit $y \to 1$ corresponds to $u \to 0$. The final answer is then

$$\chi_{\mathcal{O}}(q,u) = \boxed{\frac{q^{\Delta}}{(1-q)^3 - (1-q)qu}}$$

and it is clear that the case $u = 0$ reproduces the original character without angular momentum.

3 Virasoro algebra

In a 2d CFT, the conformal algebra is bigger because of Virasoro symmetry so we can consider bigger conformal families, known as Verma modules. The algebra can be factorized and the descendants are obtained from the primaries by applying any combination of L_{-n} and $\bar{L}_{-n}$. The states are now labeled by the conformal weights h and $\bar{h}$, the eigenvalues of L_0 and $\bar{L}_0$. A generic Virasoro descendant looks like

$$\prod_{k=1}^{\infty}(L_{-k})^{n_k}\,|h\rangle$$

where $n_k = 0, 1, ...$ and we neglected the antiholomorphic part. Each L_{-n} increases the value of h by n and the same is true for the bar part so the weight of some generic descendant is $h + \sum_{k=1}^{\infty} kn_k$.

We again consider the partition function with momentum turned on:

$$Z(\beta, K) = \mathrm{Tr}\big[e^{-\beta H + iKP}\big] .$$

The Hamiltonian is $H = L_0 + \bar{L}_0 - \frac{c+\bar{c}}{24}$ and the momentum generator is $P = L_0 - \bar{L}_0 - \frac{c-\bar{c}}{24}$ such that the partition function can be rewritten as

$$Z(\tau,\bar{\tau}) = \mathrm{Tr}\Big[q^{L_0 - \frac{c}{24}}\bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}\Big]$$

with $\tau = \frac{K+i\beta}{2\pi}$ and $q = e^{2\pi i}$. Since the algebra factors out, the contribution to this from a single Verma module will factorize into characters

$$\chi_h(q) = \mathrm{Tr}_h[q^{L_0 - \frac{c}{24}}]\,,$$

where we sum only over the descendants built from the primary by applying the holomorphic generators. Given the form of the descendants and their weight, the character can be computed by the following sum

$$\begin{aligned}\chi_h(q) &= q^{h-\frac{c}{24}} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \dots q^{n_1+2n_2+\dots} = q^{h-\frac{c}{24}} \left(\sum_{n_1=0}^{\infty} q^{n_1} \right) \left(\sum_{n_2=0}^{\infty} q^{2n_2} \right) \dots \\ &= q^{h-\frac{c}{24}} \left(\frac{1}{1-q} \right) \left(\frac{1}{1-q^2} \right) \dots = \boxed{q^{h-\frac{c}{24}} \prod_{n=1}^{\infty} (1-q^n)^{-1}}\,.\end{aligned}$$

The partition function can then be expressed as

$$Z(\tau.\bar{\tau}) = \sum_{h,\bar{h}} M_{h,\bar{h}} \chi_h(q) \chi_{\bar{h}}(\bar{q})\,.$$

All of this is for $c > 1$ because we need to assume that there are no null states to take out of the spcetrum. It is also important that this doesn't work for the Verma module of the identity since the state $L_{-1}|0\rangle$ is null (as can be seen from the Virasoro algebra) so all the descendants with $n_1 \neq 0$ must be excluded from the sum. In this case, it is easy to see that the result it

$$\chi_0(q) = q^{-\frac{c}{24}} \prod_{n=2}^{\infty} (1-q^n)^{-1}\,.$$

Since the identity should be unique in a nice theory, we need to have $M_{0,0} = 1$

Finally, this section is different from what we did before because of Virasoro symmetry but we can recover the $d = 2$ result from earlier by taking only the global primaries $L_{-1}^n|h\rangle$ in the sum for the characters.