Constraints on the spectrum of operators in two-dimensional Conformal Field Theories

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Declaration of authorship

The projects discussed in this thesis are ideas proposed to me by Alex Maloney and were done in collaboration with him. They are some of the many ideas that were suggested during general discussions and that lead to further investigation. The section on universal bounds has also been done with Ioannis Tsiares at McGill University. The section on $\mathcal{W}$-algebras was worked out with Kale Colville and Gim Seng Ng at McGill University. I worked out myself all the results that I present, even those that originated initially from my collaborators. Ideas coming from other people's work are referred to as much as possible in the text.

Abstract

The objective of this thesis is to derive constraints on the operator spectrum of Conformal Field Theories (CFTs) in two dimensions that have various properties. We start by describing the most basic notions about 2d CFTs along with an introduction to the representation theory of the conformal algebra and modular invariance of tori, which are important subjects for this work. The first type of constraints that are found is bounds on operator dimensions and the tool used is modular invariance of the torus partition function. Using the modular S transformation we find an upper bound on the conformal dimension of the lightest primary operator in any 2d CFT with $c > 1$ and then we improve this bound for CFTs with only even spin operators by using the ST transformation. We then show a way to obtain further constraints from any modular transformation that could lead to better bounds. The second part of the thesis is focused on extensions of the conformal algebra, known as $\mathcal{W}$-algebras, that possess higher spin conserved fields. We first derive the algebras $\mathcal{W}(2,3)$ and $\mathcal{W}(2,4)$ and then analyze the constraints coming from asking that their representations are unitary. We study the Gram matrix for the first few levels and find lower bounds on the conformal weight of every primary operator in a unitary theory that has one of these symmetries. We also find more constraints on the charges numerically in the case of $\mathcal{W}(2,4)$. Finally all these results about CFTs can be interpreted in gravity using the AdS/CFT correspondence so we introduce briefly the setup and use it to understand the meaning of the constraints. The modular bounds correspond to upper bounds on the mass of states at rest with respect to the spacetime and the extended algebra bounds can be interpreted in terms of higher spin gravity in a manner that is still not totally clear. We end with some ideas of future work that are inspired by the thesis.

Abrégé

L'objectif de cette thèse est de dériver des contraintes sur le spectre d'opérateurs de Théories Conformes des Champs (TCCs) en deux dimensions qui possèdent certaines propriétés. Nous commençons par décrire les notions de base sur les TCCs pour ensuite introduire la théorie des représentations de l'algèbre conforme et l'invariance modulaire des tores, qui sont des sujets importants pour ces travaux. Le premier type de contraintes que nous trouvons est des bornes sur les dimensions conformes en utilisant l'invariance modulaire de la fonction de partition sur un tore. Grâce à la transformation modulaire S nous trouvons une borne supérieure sur la dimension conforme du plus léger opérateur primaire de n'importe quelle TCC en 2d avec $c > 1$ et nous améliorons ensuite cette borne pour les TCCs qui possèdent uniquement des opérateurs avec un spin pair en utilisant la transformation ST. Nous montrons par la suite un moyen d'obtenir des contraintes additionnelles à partir de transformations conformes générales qui pourraient permettre d'améliorer nos bornes. La deuxième partie de cette thèse est centrée sur des extensions de l'algèbre conforme nommées algèbres $\mathcal{W}$ qui contiennent des champs conservés avec un spin élevé. Nous commencons par dériver les algèbres $\mathcal{W}(2,3)$ et $\mathcal{W}(2,4)$ et nous analysons ensuite les conséquences du fait de demander que leurs représentations soient unitaires. Nous étudions la matrice de Gram aux premiers niveaux et nous obtenons des bornes inférieures sur la dimension conforme de n'importe quel opérateur dans une TCC qui possède une de ces symétries. Nous trouvons aussi des contraintes sur les charges numériquement dans le cas de $\mathcal{W}(2,4)$. Finalement, toutes ces contraintes peuvent être interprétées dans un contexte gravitationnel avec l'aide de la correspondance AdS/TCC alors nous introduisons brièvement le formalisme afin de l'utiliser pour comprendre la signification de nos résultats. Les bornes modulaires correspondent à des bornes supérieures sur la masse des états au repos par rapport à l'espace-temps et les bornes pour les algèbres étendus peuvent être étudiées grâce à la gravité avec spin élevé d'une manière qui n'est toujours pas complètement claire. Nous terminons en discutant des idées de travaux futurs qui ont été inspirées par l'écriture de cette thèse.

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Chapter 1

Introduction

1.1 Motivation

Quantum field theory (QFT) is the most important tool used by physicists to study particles and small objects. It is the most accurately tested theory in all of physics and it can describe many different kinds of phenomena, from electrons and quarks to potentially gravity. Conformal field theories (CFT), the object of study in this thesis, are quantum field theories that are invariant under conformal transformations, which are defined by their property of preserving the angles between vectors. The simplest conformal transformation is a change of scale and very often asking that a theory is scale invariant leads directly to the invariance under all of the conformal transformations. As a contrast a generic QFT is only invariant under Poincaré transformations, a subset of conformal transformations. This means that CFTs are more restricted and more specialized objects than QFTs. Examples of theories that exhibit conformal invariance at the classical and quantum level are Maxwell theory in the vacuum, a free massless boson and a free massless Dirac fermion. An example that is conformally invariant classically but not when we quantize it is Yang-Mills theory in 4d. It fails to be conformal quantum mechanically because of the running of its coupling constant.

The examples of CFTs enumerated above don't seem to describe the world that we live in so why would be spend so much time studying them? First of all conformal field theories describe critical points in statistical physics. These are the points at the end of a phase equilibrium curve where a continuous phase transition occurs. Examples of critical points are where the liquid-gas transition happens in water or at the Curie temperature of a ferromagnet. This means that CFTs can actually describe real world phenomena. The next reason why CFTs are important is that a QFT at a fixed point of its renormalization group flow is in fact a CFT since the beta function vanishes there and the theory is scale

invariant. Studying the space of CFTs is then important for the study of the space of QFTs because we can obtain a QFT by deforming a CFT with a relevant operator, which makes it leave the fixed point.

Other reasons why CFT is important are coming from a more fundamental level. String theory is a theory that is supposed to unify all forces of nature in one framework and conformal field theory in 2d is a very important tool that is used to study it. In fact a CFT lives on the world-sheet of fundamental strings so it is essential to understand how these work. The last reason that we will state is the realization of the holographic principle in the AdS/CFT correspondence. This is a relation between a CFT and a theory of quantum gravity in Anti de Sitter space. Conformal field theories are then extremely important in that context because they help provide new insights about the quantization of gravity.

Because of all the reasons mentioned above it becomes very important to study conformal field theories in details from different angles. One interesting aspect that we need to understand is the properties of theories that satisfy a set of reasonable conditions, such as unitarity and modular invariance, to know what CFTs are allowed as physical models. This thesis summarizes work that has been done in that direction in the past years by different researchers and some new contributions from the author and its collaborators. We will actually not discuss CFTs in all dimensions but instead focus on $d = 2$. This seems unphysical because we live in $d = 4$ but there are in fact real systems that are described by 2d CFTs. However the major reason why we focus on low dimensions is that something special happens and it becomes easier to study CFTs in 2d. Hopefully many properties that we derive can be generalized to higher dimensions.

1.2 Outline

In this thesis we will discuss two different types of constraints that can be put on the spectrum of conformal field theories in two dimensions. First of all we will exploit a property known as modular invariance of the torus partition function to derive constraint equations that need to be satisfied by the operators in a CFT and we will use them to discover bounds on the allowed operator dimensions. The other topic will have to do with extensions of the conformal algebra to so-called $\mathcal{W}$-algebras. We will ask that the states in a theory with $\mathcal{W}$ symmetry satisfy unitarity and will find bounds on conformal weights along with more complicated constraints.

In Chapter 2 we will review carefully what conformal field theories are. In order to do that we will start with the basics, in particular the conformal group in general and then in $d = 2$. Following that we will discuss radial quantization and the operator product expansion, which will lead us to the discussion of the Hilbert space of a CFT and the Kac determinant. The last point that will be covered is modular invariance. In Chapter 3 we will present the constraints that are found by using modular invariance. First the well known Cardy's formula for the asymptotic density of states will be quickly discussed as a simple example and then we will show the derivation of an important bound discovered by Hellerman. An improved bound found by Qualls will then be presented and finally a more general constraint will be derived. In Chapter 4 it will be time to discuss extensions of the conformal algebra. We will introduce the concept of $\mathcal{W}$-algebras and determine the explicit expressions for $\mathcal{W}(2,3)$ and $\mathcal{W}(2,4)$. Unitarity of the representations of these algebras will be studied and we will present many interesting results. In Chapter 5 we will introduce simple aspects of the AdS/CFT correspondence in order to interpret our results in terms of a gravity theory. In particular the asymptotic symmetries of Anti de Sitter space and higher-spin gravity will be discussed and will allow us to rephrase the conclusions of the previous chapters in a different language. Finally we will conclude in Chapter 6 with a summary and some ideas for future work.

1.3 Notation

Here are some conventions for the notation that is used in this thesis. Everything is standard, but it is included for completeness.

- Like in any High Energy Physics text, we use the natural units in which $\hbar = c = k_B = G_N = 1$ throughout most of the thesis.
- Most of the work is done in Euclidean space language, but when we will work in Minkowski space the metric signature will have a mostly positive signature.
- A space in d dimensions refers to space+time.
- We use Einstein's summation convention where repeated indices are summed over. Greek indices run from 0 to $d-1$ with the index 0 referring to time. Latin indices are just spatial so they run from 1 to $d-1$.

Chapter 2

Conformal Field Theory in Two Dimensions

Conformal Field Theory (CFT) is a very rich and interesting subject that has found applications in many branches of physics and mathematics, like mentioned in Chapter 1. Motivation for studying it has already been given so we can jump right ahead and dedicate this chapter to the development of CFT. There are many books and lecture notes available on the subject and we will just cover the parts that are useful to know for the results presented later on in this thesis. We will follow mostly the books by Blumenhagen and Plauschinn [1] and by di Francesco, Mathieu and Sénéchal [2]. The first one discusses CFT with string theory in mind so we will only cover the basics and the second one is the bible for CFT so we will cover an infinitesimal part of it. There are also nice lecture notes by Qualls [3] on 2d CFT that have inspired a lot this chapter. Since all the work done in the context of this thesis is in two dimensions, little will be said here about $d \geq 3$. Indeed we will jump quickly to the operator formalism, which is very specific to $d = 2$. For a great review of the higher dimension case, lectures by Rychkov [4] are a must read. Other reviews by Simmons-Duffin [5], Schellekens [6] and Ginsparg [7] have also helped in writing this chapter.

We will start the chapter by introducing conformal transformations and deriving the conformal group, in $d = 2$ and in $d \geq 3$. We will then focus on 2d and discuss primary fields and the energy-momentum tensor. We will continue on to quantize the theory using radial quantization and we will introduce the important notion of operator product expansion. The Hilbert space of a 2d CFT will then be explained. After discussing the basic notions we will come to more specific material that will be useful to the thesis. Highest weight representations of the Virasoro algebra and modular invariance will be the last topics that we will cover in this chapter.

2.1 Basics

A Conformal Field Theory is a field theory that is invariant under conformal transformations so the first thing that we will do is study these. A conformal transformation is a change of coordinates that leaves the metric unchanged up to a scale factor. It is basically a local change of scale of the spacetime, that is obtained locally from a rotation and a dilation. We will only consider CFTs on flat spacetimes so that conformal transformations act in the following way

$$\eta_{\mu\nu} \to \eta'_{\mu\nu} = \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \eta_{\rho\sigma} = \Lambda(x)\eta_{\mu\nu} \,. \tag{2.1}$$

It's obvious that these transformations form a group that contains the Poincaré transformations as a subgroup since they correspond to $\Lambda(x) = 1$. We can also see from (2.1) that the distances are not preserved but the angles between vectors $\frac{v\cdot w}{\sqrt{(v\cdot v)(w\cdot w)}}$ are. The flat metric η corresponds to a space of d dimensions and we will take it to have Euclidean signature. The treatment of Lorentzian signature is almost identical and we will point out where something different will come up.

2.1.1 The conformal group in $d > 2$

We will now study the infinitesimal transformations $x^{\mu} \to x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x)$ in order to find the generators of the conformal transformations and the conformal algebra for a general $d > 2$. In the infinitesimal case the condition (2.1) becomes, to first order in $\epsilon \ll 1$,

$$\eta'_{\mu\nu} \approx \eta_{\rho\sigma} \left(\delta^{\rho}_{\mu} + \partial_{\mu}\epsilon^{\rho}\right) \left(\delta^{\sigma}_{\nu} + \partial_{\nu}\epsilon^{\sigma}\right) \approx \eta_{\mu\nu} + \partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = \Lambda(x)\eta_{\mu\nu} \approx \left(1 + K(x)\right)\eta_{\mu\nu}$$

$$\Rightarrow \;\; \partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = K(x)\eta_{\mu\nu} \,. \tag{2.2}$$

where $K(x)$ is for now an arbitrary function and we used the fact that $\frac{\partial x^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\nu}$. This is actually the conformal Killing equation, which is a generalization of the Killing equation for isometries ($K(x) = 0$) to conformal transformations.

We can take the trace of (2.2) by taking its contraction with $\eta^{\mu\nu}$ to obtain $2\partial_{\mu}\epsilon^{\mu} = dK(x)$. Putting this back into (2.2) itself leads to

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu} \tag{2.3}$$

where $\partial\cdot\epsilon = \partial_\mu\epsilon^\mu$. From there the scale factor can easily be read off to be $\Lambda(x) = 1+\frac{2}{d}(\partial\cdot\epsilon)$. Now we can contract (2.3) with $\partial^\mu\partial^\nu$ to get

$$(d-1)\Box(\partial\cdot\epsilon) = 0 \;\Rightarrow \Box(\partial\cdot\epsilon) = 0 \tag{2.4}$$

where $\Box = \partial_\nu\partial^\nu$. This is an important equation that will help us in finding the allowed transformations. There is actually something wrong when $d = 2$ and we will come back to it in the next section.

We will also need another useful relation in order to completely classify conformal transformations. To get it we act on (2.3) with ∂_ρ and consider the following three permutations of indices

$$\partial_\rho\partial_\mu\epsilon_\nu + \partial_\rho\partial_\nu\epsilon_\mu = \frac{2}{d}\eta_{\mu\nu}\partial_\rho(\partial\cdot\epsilon)$$
$$\partial_\mu\partial_\nu\epsilon_\rho + \partial_\mu\partial_\rho\epsilon_\nu = \frac{2}{d}\eta_{\nu\rho}\partial_\mu(\partial\cdot\epsilon)$$
$$\partial_\nu\partial_\rho\epsilon_\mu + \partial_\nu\partial_\mu\epsilon_\rho = \frac{2}{d}\eta_{\rho\mu}\partial_\nu(\partial\cdot\epsilon)\,.$$

Combining them in the right way gives

$$\partial_\mu\partial_\nu\epsilon_\rho = \frac{1}{d}(\eta_{\rho\mu}\partial_\nu + \eta_{\nu\rho}\partial_\mu - \eta_{\mu\nu}\partial_\rho)(\partial\cdot\epsilon)\,. \tag{2.5}$$

At this point we have everything we need with (2.4) and (2.5) to find the infinitesimal conformal transformations. First of all (2.4) tells us that $\epsilon(x)$ can be at most quadratic so we will write it as $\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho$ where $c_{\mu\nu\rho}$ must be symmetric in its last two indices. The constraints that we derived for a conformal transformation are independent of the position x^μ so we can analyze each term in ϵ^μ separately. Note that generators of transformations are discussed in Appendix B.

- The term $\epsilon_\mu = a_\mu$ corresponds to an infinitesimal translation, which at the finite level is of course still a translation $x'^\mu = x^\mu + a^\mu$. The generator of translations is well known to be momentum $P_\mu = -i\partial_\mu$.

- The term $\epsilon_\mu = b_{\mu\nu}x^\nu$ can be studied by inserting it in (2.3), which gives

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d}\eta_{\mu\nu}b^\lambda{}_\lambda \equiv \alpha\eta_{\mu\nu}\,.$$

  We can always split a matrix into a symmetric and an anti-symmetric part and this

equations tells us that the symmetric part of $b_{\mu\nu}$ must be proportional to the metric. This means that $b_{\mu\nu} = \alpha\eta_{\mu\nu} + m_{\mu\nu}$ with $m_{\mu\nu} = -m_{\nu\mu}$. The symmetric part corresponds to an infinitesimal transformation $x'^{\mu} = (1+\alpha)x^{\mu}$, which at the finite level is $x'^{\mu} = \lambda x^{\mu}$. This is a scaling and the operator that generates it is a dilation $D = -ix^{\mu}\partial_{\mu}$. The anti-symmetric part gives infinitesimally $x'^{\mu} = (\delta^{\mu}{}_{\nu} + m^{\mu}{}_{\nu})x^{\nu}$ and the corresponding finite transformations are rotations $x'^{\mu} = R^{\mu}{}_{\nu}x^{\nu}$ (or Lorentz transformations in Lorentzian signature). The generator is again well known to be the angular momentum operator $L_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$.

- To study the term $\epsilon_{\mu} = c_{\mu\nu\rho}x^{\nu}x^{\rho}$ we insert it into (2.5) to find

$$c_{\mu\nu\rho} = \eta_{\mu\rho}b_{\nu} + \eta_{\nu\mu}b_{\rho} + \eta_{\rho\nu}b_{\mu}$$

with $b_{\mu} = \frac{1}{d}c^{\lambda}{}_{\lambda\mu}$. To get this we had to compute $\partial\cdot\epsilon = 2c^{\lambda}{}_{\lambda\mu}x^{\mu} = 2db_{\mu}x^{\mu}$ and to use the symmetry of the last two indices. Plugging this back into ϵ_{μ} gives the actual infinitesimal transformation $x'^{\mu} = x^{\mu} + 2(x\cdot b)x^{\mu} - x^2 b^{\mu}$. This is called a special conformal transformation (SCT) and the generator associated with it is $K_{\mu} = -i(2x_{\mu}x^{\nu}\partial_{\nu} - x^2\partial_{\mu})$. The finite version is not easy to compute but turns out to be $x'^{\mu} = \frac{x^{\mu}-x^2 b^{\mu}}{1-2(b\cdot x)+x^2b^2}$. The physical meaning of a SCT is not so clear in that form but we can rewrite it as $\frac{x'^{\mu}}{x'^2} = \frac{x^{\mu}}{x^2} - b^{\mu}$ and then it can be interpreted as an inversion followed by a translation and another inversion.

To summarize, the conformal group in d dimensions is made out of 1 dilation, d translations, $\frac{d(d-1)}{2}$ rotations and d SCTs for a total of $\frac{(d+2)(d+1)}{2}$ independent transformations.

After having found all the possible conformal transformations along with their generators, we can use the explicit form of the generators to compute the conformal algebra. The commutation relations are not very illuminating in their original form, apart from the fact that the Poincaré subgroup is obvious, so we will not enumerate them. Instead we will directly jump to a redefinition of the generators:

$$J_{\mu\nu} \equiv L_{\mu\nu}$$

$$J_{-2,-1} \equiv D$$

$$J_{-2\mu} \equiv \frac{1}{2}(P_{\mu} - K_{\mu})$$

$$J_{-1\mu} \equiv \frac{1}{2}(P_{\mu} + K_{\mu})\,.$$

The new indices differ from the literature because we took $\mu \in \{0, 1, ..., d-1\}$. With these new definitions the conformal algebra becomes

$$[J_{MN}, J_{PQ}] = i(\eta_{MQ}J_{NP} + \eta_{NP}J_{MP} - \eta_{MP}J_{NQ} - \eta_{NQ}J_{MP})\,. \tag{2.6}$$

The capital indices run from -2 to $d-1$ and the metric that appears is $\eta_{AB} = diag(-1, 1, ..., 1)$. These commutation relations characterize the groups $SO(...)$ so in the $\mathbb{R}^d$ case that we study the conformal group is $SO(d+1, 1)$. If we had done this in Minkowski space $\mathbb{R}^{d-1,1}$, the metric would be $\eta_{AB} = diag(-1, -1, 1, ..., 1)$ and the conformal group would be $SO(d, 2)$. In general for $\mathbb{R}^{p,q}$ the conformal group is $SO(p+1, q+1)$.

As a final comment, let's mention that the material of this section is the only thing we have to say about CFTs in $d \geq 3$ and everything will be focused on $d = 2$ in the rest of the text.

2.1.2 The conformal group in $d = 2$

When we focus on $d = 2$, most of what we did before is still valid. Actually everything is right up until (2.4), which turns out to be too restrictive. To see what the problem is with the equation we will derive it again from (2.3) but more slowly by taking only one derivative out of $\partial^\mu \partial^\nu$ first. This leads us to

$$\partial_\mu(\partial \cdot \epsilon) + \Box\epsilon_\mu = \frac{2}{d}\partial_\mu(\partial \cdot \epsilon)\,.$$

It's easy to see that for $d = 2$ this equation doesn't lead to the one that we used before to derive conformal transformations. Actually in 2d we don't need all of this machinery to study conformal transformations since the direct expansion of (2.3) leads to the simple equations

$$\begin{aligned} (\mu = \nu = 0) &\Longrightarrow \partial_0\epsilon_0 = \partial_1\epsilon_1 \\ (\mu = 0, \nu = 1) &\Longrightarrow \partial_0\epsilon_1 = -\partial_0\epsilon_1\,. \end{aligned} \tag{2.7}$$

These are the famous Cauchy-Riemann equations from complex analysis. A function whose real and imaginary parts satisfy (2.7) is holomorphic in some region. This suggests that it would be useful to work with the following complex variables:

$$z = x^0 + ix^1\,, \qquad \bar{z} = x^0 - ix^1$$

$$\epsilon = \epsilon^0 + i\epsilon^1\,, \qquad \bar{\epsilon} = \epsilon^0 - i\epsilon^1$$
$$\partial \equiv \partial_z = \frac{1}{2}(\partial_0 - i\partial_1)\,, \qquad \bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{2}(\partial_0 + i\partial_1)\,.$$

For future reference we will also write the metric with the new coordinates. Using the usual transformation rule for $\eta_{\mu\nu}$ gives us the expression in the $\{z, \bar{z}\}$ basis

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \qquad \eta^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \tag{2.8}$$

With these complex variables, (2.7) simply become $\partial\bar{\epsilon}(z,\bar{z}) = 0$ and $\bar{\partial}\epsilon(z,\bar{z}) = 0$, which mean that ϵ and $\bar{\epsilon}$ are respectively holomorphic and anti-holomorphic functions of z. They are arbitrary functions so the infinitesimal transformations $z \to z + \epsilon(z)$ and $\bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})$ correspond to the finite transformations $z \to f(z)$ and $\bar{z} \to \bar{f}(\bar{z})$. This was expected since the metric itself transforms as

$$ds^2 = \mathrm{d}z\mathrm{d}\bar{z} \to \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}}\mathrm{d}z\mathrm{d}\bar{z}\,,$$

which is exactly a conformal transformation with scale factor $\Lambda = \left|\frac{\partial f}{\partial z}\right|^2$. The result of all this is that any holomorphic function is a conformal transformation and the conformal group is immensely bigger in 2d than in higher dimensions.

The next step after having found the form of the infinitesimal conformal transformations is to find the generators and their algebra. We know that $\epsilon(z)$ is holomorphic in a certain region, but it's safe to assume that it's meromorphic on the whole space and expand it in Laurent modes about $z = 0$ such that a general conformal transformation looks like

$$z \to z + \sum_{n=-\infty}^{\infty} \epsilon_n(-z^{n+1})$$
$$\bar{z} \to \bar{z} + \sum_{n=-\infty}^{\infty} \bar{\epsilon}_n(-\bar{z}^{n+1})\,.$$

The generators are easy to find from these expressions and there is an infinite number of them

$$\begin{aligned} \ell_n &= -z^{n+1}\partial \\ \bar{\ell}_n &= -\bar{z}^{n+1}\bar{\partial}\,. \end{aligned} \tag{2.9}$$

Using these expressions, we can directly compute the commutators to find the conformal algebra in two dimensions. The result is two commuting copies of the so-called Witt

algebra

$$\begin{aligned}
[\ell_m, \ell_n] &= -z^{m+1}\partial(-z^{n+1}\partial) + z^{n+1}\partial(-z^{m+1}\partial) \\
&= (n+1)z^{m+n+1}\partial + z^{m+n+2}\partial^2 - (m+1)z^{m+n+1}\partial - z^{m+n+2}\partial^2 \\
&= (n-m)z^{m+n+1}\partial = (m-n)\ell_{m+n} \qquad (2.10) \\
\left[\bar{\ell}_m, \bar{\ell}_n\right] &= (m-n)\bar{\ell}_{m+n} \\
\left[\ell_m, \bar{\ell}_n\right] &= 0\,.
\end{aligned}$$

Since we have two independent copies of the same algebra, we will consider z and $\bar{z}$ as independent variables and only ask that they are complex conjugate to one another when we need physical results. The fact that there is an infinite number of generators confirms the statement made earlier that the conformal group is a lot bigger in $d = 2$ than in $d \geq 3$, where there is only a finite number of them.

Even if there are some major differences with the higher dimensional case, there is still a way to recover the usual conformal group with $d = 2$. Looking at the generators, it's easy to see that most of them are not well defined globally. Even if we insist on working on the Riemann sphere, which is the complex plane with a point at infinity included, the problems are not solved. For instance, the generators ℓ_n defined in (2.9) are not well behaved as $z \to 0$ for $n < -1$. There are also some problems near $z \to \infty$ that can be seen by doing a change of coordinates to $w = -\frac{1}{z}$ and looking at $w \to 0$. With the new coordinate the generators look like $\ell_n = -\left(-\frac{1}{w}\right)^{n+1} \frac{\partial w}{\partial z}\partial_w = -\left(-\frac{1}{w}\right)^{n+1} w^2\partial_w = -(-w)^{1-n}\partial_w$ and they are singular close to the origin if $n > 1$. The same story can be told for the anti-holomorphic generators so that the global conformal transformations on the Riemann sphere are only generated by $\{\ell_{-1}, \ell_0, \ell_1\} \cup \{\bar{\ell}_{-1}, \bar{\ell}_0, \bar{\ell}_1\}$. These global conformal transformations are the ones that are related to those discussed in higher dimensions. We can identity them in the following way:

- From their definitions it's clear that ℓ_{-1} and $\bar{\ell}_{-1}$ generate the translations $z \to z + b$ and $\bar{z} \to \bar{z} + \bar{b}$ since they are simply momentum operators.
- Working in polar coordinates $z = re^{i\phi}$ allows us to write

$$\ell_0 + \bar{\ell}_0 = -r\partial_r$$

$$i(\ell_0 - \bar{\ell}_0) = -\partial_\phi\,.$$

By comparing to the generators that we found in the $d \geq 3$ case we can see that $\ell_0 + \bar{\ell}_0$ generates dilations in 2d. Also $i(\ell_0 - \bar{\ell}_0)$ generates translations in ϕ, which are simply rotations.

- The only one left is ℓ_1, which should then correspond to special conformal transformations. This is indeed the case since it generates the finite transformation $z \to \frac{z}{cz+1}$, which is not clearly a SCT in that form but looks more familiar when written as $\frac{-1}{z} \to \frac{-1}{z} - c$. ($\bar{\ell}_1$ does the same thing to $\bar{z}$)

These transformations generate in 2d the group $SL(2,\mathbb{C})/\mathbb{Z}_2$, which is the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$ that act on a point as $z \to \frac{az+b}{cz+d}$. The $\mathbb{Z}_2$ is there simply because the elements (a, b, c, d) and $(-a, -b, -c, -d)$ give the same transformation. It is well known that the groups $SL(2,\mathbb{C})$ and $SO(3,1)$ are isomorphic so that we recover the usual conformal group with the global transformations. Note that there seems to be two copies of $SL(2,\mathbb{C})$ since we should also consider the transformations of the anti-holomorphic coordinate, but this is where the identification of $\bar{z}$ with the complex conjugate has to be used to make the second copy redundant and make the discussion fit with the physical expectations.

At this point everything that we said is classical. When we will actually quantize the theory we will find out that the symmetry algebra is not the Witt algebra (2.10) but its central extension, the Virasoro algebra. We will derive this algebra here and we will see later how it arises in conformal field theory. Since everything is exactly the same for both copies of the algebra we will focus on the holomorphic part and the anti-holomorphic part follows directly. The central extension of an algebra is obtained by adding a new term to the commutation relations that commutes with every element of the algebra. We will denote the elements of the extended algebra as L_n and will refer to them as Virasoro generators. The commutation relations then look like

$$[L_m, L_n] = (m-n)L_{m+n} + cp(m,n)\,.$$

We will be able to fix the arbitrary function $p(m,n)$ by using simple arguments and the so-called central charge c is an arbitrary real number that characterizes each individual CFT. There is also a central charge $\bar{c}$ associated with the anti-holomorphic algebra and it is independent of the holomorphic central charge.

First of all $p(m,n)$ must obviously be anti-symmetric since the commutator itself is. Second, we can redefine the generators in such a way that $p(n,0) = p(1,-1) = 0$. Indeed if we take $\hat{L}_n = L_n + \frac{cp(n,0)}{n}$ for $n \neq 0$ and $\hat{L}_0 = L_0 + \frac{cp(1,-1)}{2}$ we find the new commutators

$$\left[\hat{L}_n, \hat{L}_0\right] = nL_n + cp(n,0) = n\hat{L}_n$$

$$\left[\hat{L}_1, \hat{L}_{-1}\right] = 2L_0 + cp(1,-1) = 2\hat{L}_0\,.$$

We will thus consider this option and rename the $\hat{L}_n$ as L_n. The next thing to do is consider some specific cases of the Jacobi identity. The first one to use is

$$\begin{aligned}
0 =& [[L_m, L_n], L_0] + [[L_0, L_m], L_n] + [[L_n, L_0], L_m] \\
0 =& [(m-n)L_{m+n} + cp(m,n), L_0] + [-mL_m + cp(0,m), L_n] + [nL_n + cp(n,0), L_m] \\
0 =& (m-n)\left[(m+n)L_{m+n} + cp(m+n,0)\right] - m\left[(m-n)L_{m+n} + cp(m,n)\right] \\
&+ n\left[(n-m)L_{m+n} + cp(n,m)\right] \\
0 =& (m-n)p(m+n,0) - mp(m,n) + np(n,m) \\
0 =& (m+n)p(n,m)\,.
\end{aligned}$$

This means that $p(m,n)$ has to be zero if $m \neq -n$. Combining this with what we already know leads to the conclusion that the only non-zero functions are $p(n,-n)$ for $|n| \geq 2$. To get the explicit value we use another Jacobi identity

$$\begin{aligned}
0 =& [[L_{1-n}, L_n], L_{-1}] + [[L_n, L_{-1}], L_{1-n}] + [[L_{-1}, L_{1-n}], L_n] \\
0 =& (1-2n)p(1,-1) + (n+1)p(n-1,1-n) + (n-2)p(-n,n)
\end{aligned}$$

$$\Longrightarrow p(n,-n) = \left(\frac{n+1}{n-2}\right) p(n-1,1-n)\,.$$

This doesn't seem very useful but we can use this recursion relation up to the point where we get the last non-zero coefficient in the sequence, which is $p(2,-2)$, and we will obtain the desired result.

$$\begin{aligned}
p(n,-n) &= \left(\frac{n+1}{n-2}\right) p(n-1,1-n) = \left(\frac{n+1}{n-2}\right)\left(\frac{n}{n-3}\right) p(n-2,2-n) \\
&= \ldots = \frac{(n+1)n(n-1)\ldots 5\cdot 4}{(n-2)(n-3)\ldots 2\cdot 1}\, p(2,-2) = \frac{(n+1)n(n-1)}{6}\, p(2,-2)
\end{aligned}$$

We still need to normalize $p(2,-2)$ and we will take it to be $1/2$ by convention (to make the free boson have $c=1$). The resulting algebra is finally

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}\,. \tag{2.11}$$

It's good to note that the subalgebra $\{L_{-1}, L_0, L_1\}$ of the Virasoro algebra still generates the global conformal group $SL(2,\mathbb{C})/\mathbb{Z}_2$ since the central extension doesn't affect the generators ($p(m,n)=0$ for $n,m=-1,0,1$). Also everything that we said so far can be done in Lorentzian signature. The appropriate coordinates are however left-moving $v=x+t$ and right-moving $u=x-t$ light cone coordinates instead of complex ones. The symmetry algebra is then exactly the same.

2.1.3 Building blocks of CFTs

In this section we will introduce the building blocks of a conformal field theory: the fields. We will introduce primary and secondary fields and how conformal symmetry affects them and we will also talk about the energy-momentum tensor coming from conformal invariance. The fields will be taken to be functions of z and $\bar{z}$ and the identification $\bar{z}=z^*$ will be made when convenient.

The first notion that is relevant is the fact that we will call chiral a field that depends only on z and anti-chiral a field that depends only on $\bar{z}$. The next thing to do is to define a primary field as a field which has the following transformation property under conformal transformation $z \to f(z)$

$$\phi(z,\bar{z}) \to \phi'(z,\bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z),\bar{f}(\bar{z}))\,. \tag{2.12}$$

The real numbers h and $\bar{h}$ are called the holomorphic and anti-holomorphic conformal dimensions (or weights) of the fields and are often assembled into a doublet $(h,\bar{h})$. They are not complex conjugate to each other and they can be viewed as some kind of tensorial indices. We will see later that it's useful to define the scaling dimension $\Delta = h+\bar{h}$, which is just the dimension of the field, and the conformal spin $s=h-\bar{h}$. A field that transforms according to (2.12) only for global conformal transformations $f \in SL(2,\mathbb{C})/\mathbb{Z}_2$ is called a quasi-primary field. Finally a secondary field is a field that is not primary nor quasi-primary. An example is the derivative of a primary field.

We can already find consequences of the transformation properties of quasi-primary fields on their correlation functions. Actually, correlators of quasi-primary operators are extremely constrained by conformal symmetry since they are observable and must therefore be invariant under the symmetries. Let's start by studying the 2-point function of chiral quasi-primary fields for simplicity (so $\bar{h} = 0$). The correlator has to be a function of the two points, which we will denote by $g(z, w) = \langle \phi_1(z)\phi_2(w)\rangle$. We will use only global conformal transformations to constrain the function $g(z, w)$ and that's why it applies to quasi-primaries and not just primaries. The correlator should be invariant under translation $z \to z + a$ so we can only have $g(z, w) = g(z - w)$. The transformation of the fields under scaling $z \to \lambda z$ gives another constraint:

$$g(z-w) = \langle \phi_1(z)\phi_2(w)\rangle = \langle \phi_1'(z)\phi_2'(w)\rangle = \lambda^{h_1+h_2} \langle \phi_1(\lambda z)\phi_2(\lambda w)\rangle = \lambda^{h_1+h_2} g(\lambda(z-w)) \,.$$

This constraint on g says that it can only be of the form $g(z-w) = \frac{d_{12}}{(z-w)^{h_1+h_2}}$, where d_{12} is a structure constant. The last thing that we will need is the transformation property under $z \to -\frac{1}{z}$, which gives

$$g(z-w) = \frac{1}{z^{2h_1}w^{2h_2}} \left\langle \phi_1\left(\frac{-1}{z}\right)\phi_2\left(\frac{-1}{w}\right)\right\rangle = \frac{1}{z^{2h_1}w^{2h_2}} \frac{d_{12}}{\left(\frac{1}{w}-\frac{1}{z}\right)^{h_1+h_2}} = \frac{g(z-w)}{z^{h_1-h_2}w^{h_2-h_1}} \,.$$

This simply tells us that the conformal dimensions of the two fields must be the same to have a non-zero correlator. The final answer is then

$$\langle \phi_1(z)\phi_2(w)\rangle = \frac{d_{12}\delta_{h_1,h_2}}{(z-w)^{2h_1}} \,. \tag{2.13}$$

The exact same procedure can in fact be done for a quasi-primary field that is not chiral and the result is simple to guess

$$\langle \phi_1(z,\bar{z})\phi_2(w,\bar{w})\rangle = \frac{d_{12}\delta_{h_1,h_2}\delta_{\bar{h}_1,\bar{h}_2}}{(z-w)^{2h_1}(\bar{z}-\bar{w})^{2\bar{h}_1}} \,.$$

There is an interesting consequence of this form for the 2-point function. In addition to what we just did, we also need to ask that it is single valued on the complex plane, so that it is invariant under rotation $z \to e^{2\pi i}z$, which leads to the conclusion that the conformal weights must be integers of half-integers. This is basically the spin-statistics theorem. To continue, we can use the same techniques to also fix the 3-point function up to a constant.

The actual form is

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \frac{C_{123}}{(z_1-z_2)^{h_1+h_2-h_3}(z_2-z_3)^{h_2+h_3-h_1}(z_1-z_3)^{h_1+h_3-h_2}}\,. \tag{2.14}$$

This is everything that we can fix exactly because when we get to four points it's possible to build a conformally invariant combination of the points and we are always free to add a function of this ratio to the correlators. Of course, apart from that function there are major constraints that can be found, but we will not go into the details here.

Up until now we have studied how primary fields behave under finite conformal transformations, but it is even more useful to know how they behave under infinitesimal ones. To this end we will look at (2.12) with $f(z) = z + \epsilon(z)$. At first order in ϵ and $\bar{\epsilon}$ we get

$$\begin{aligned}\phi'(z,\bar{z}) &= \phi(z,\bar{z}) + \delta\phi(z,\bar{z}) = (1+\partial\epsilon)^h(1+\bar{\partial}\bar{\epsilon})^{\bar{h}}\phi(z+\epsilon,\bar{z}+\bar{\epsilon}) \\ &= (1+h\partial\epsilon+\bar{h}\bar{\partial}\bar{\epsilon})(\phi(z,\bar{z})+\epsilon\partial\phi(z,\bar{z})+\bar{\epsilon}\bar{\partial}\phi(z,\bar{z})) \\ &= \phi(z,\bar{z})+\epsilon\partial\phi(z,\bar{z})+\bar{\epsilon}\bar{\partial}\phi(z,\bar{z})+h\phi(z,\bar{z})\partial\epsilon+\bar{h}\phi(z,\bar{z})\bar{\partial}\bar{\epsilon} \\ \\ \implies \delta\phi(z,\bar{z}) &= (\epsilon\partial+\bar{\epsilon}\bar{\partial}+h\partial\epsilon+\bar{h}\bar{\partial}\bar{\epsilon})\phi(z,\bar{z})\,.\end{aligned} \tag{2.15}$$

This will be used in the next section to find an alternate definition for primary fields.

Another consequence of conformal invariance is familiar from regular QFT. Indeed whenever there is a continuous symmetry there has to be a conserved current. For the symmetry $x^\mu \to x^\mu + \epsilon^\mu(x)$ the current can be written as $j_\mu = T_{\mu\nu}\epsilon^\nu$ with the energy-momentum tensor $T_{\mu\nu}$ being symmetric. If we consider the consequences of translation invariance, that is $\epsilon^\mu(x)$ being a constant, and we ask that the current is conserved, we find

$$0 = \partial^\mu j_\mu = \epsilon^\nu \partial^\mu T_{\mu\nu}\,.$$

Since it should be true for an arbitrary transformation we conclude that the energy-momentum tensor is conserved $\partial^\mu T_{\mu\nu} = 0$. This is of course true in any QFT, not just CFTs, because translations are symmetries in general field theories. The special thing with CFTs happens when we consider $\epsilon^\mu(x)$ not a constant. Conservation of the current along with conservation of the energy-momentum tensor then say

$$0 = \partial^\mu j_\mu = \epsilon^\nu\partial^\mu T_{\mu\nu} + T_{\mu\nu}\partial^\mu\epsilon^\nu = \frac{1}{2}T_{\mu\nu}(\partial^\mu\epsilon^\nu+\partial^\nu\epsilon^\mu) = \frac{1}{2}T_{\mu\nu}K(x)\eta^{\mu\nu} = \frac{1}{2}T^\mu_\mu K(x)\,.$$

To get the final result we used equation (2.2) from Section 2.1, where $K(x)$ is the infinitesimal conformal factor. It's interesting to note here that for the symmetries of general QFTs the scale factor is zero and the current is automatically conserved. It's actually not the case for CFTs because of dilations and SCTs. This means that we need the energy-momentum to be traceless $T^{\mu}_{\mu} = 0$ in order for the current to be conserved. This is a fact that holds in any dimension since we only used the global conformal transformations. It is actually true even for theories that are scale invariant but not invariant under the full conformal group since dilations are sufficient to reach our conclusion.

Now that we have some information about the energy-momentum tensor in general dimensions, it's time to analyze what it means in 2d by changing to complex coordinates. We can use the usual transformation rule for a tensor to find the components in the $\{z, \bar{z}\}$ basis:

$$T_{zz} = \frac{\partial x^{\mu}}{\partial z}\frac{\partial x^{\nu}}{\partial z}T_{\mu\nu} = \frac{1}{4}(T_{00} - 2iT_{10} - T_{11})$$

$$T_{\bar{z}\bar{z}} = \frac{\partial x^{\mu}}{\partial \bar{z}}\frac{\partial x^{\nu}}{\partial \bar{z}}T_{\mu\nu} = \frac{1}{4}(T_{00} + 2iT_{10} - T_{11})$$

$$T_{z\bar{z}} = \frac{\partial x^{\mu}}{\partial z}\frac{\partial x^{\nu}}{\partial \bar{z}}T_{\mu\nu} = T_{\bar{z}z} = \frac{1}{4}(T_{00} + T_{11}) = \frac{1}{4}T^{\mu}_{\mu} = 0 \Rightarrow T_{00} = -T_{11}\,.$$

The last one is due to the tracelessness of the energy-momentum tensor and combining it with the other ones gives

$$T_{zz} = \frac{1}{2}(T_{00} - iT_{10})\,, \qquad T_{\bar{z}\bar{z}} = \frac{1}{2}(T_{00} + iT_{10})\,.$$

We can finally compute the following quantity

$$\begin{aligned}\bar{\partial}T_{zz} &= \frac{1}{4}(\partial_0 + i\partial_1)(T_{00} - iT_{10}) = \frac{1}{4}(\partial_0 T_{00} + \partial_1 T_{10} + i\partial_1 T_{00} - i\partial_0 T_{10}) \\ &= \frac{1}{4}(\partial^{\mu}T_{\mu 0} - i\partial_1 T_{11} - i\partial_0 T_{01}) = -\frac{i}{4}\partial^{\mu}T_{\mu 1} = 0\,.\end{aligned}$$

To get this result we have used the conservation of energy-momentum tensor $\partial^{\mu}T_{\mu\nu} = 0$ and the Euclidean metric to raise the indices. The equation that we derived means that T_{zz} is a chiral field. The same thing can be done for $T_{\bar{z}\bar{z}}$ to show that it is anti-chiral. The final form of the energy-momentum tensor in complex coordinates is then

$$\begin{pmatrix} T(z) & 0 \\ 0 & \bar{T}(\bar{z}) \end{pmatrix}.$$

2.2 Radial Quantization and the Operator Product Expansion

Again what we have done so far is all classical and we should think about quantizing the theory soon. This is what we will do in this section by using radial quantization. We will then discuss conserved charges and derive the operator product expansion (OPE) of two fields, first for special cases and then in a more general form.

2.2.1 OPE for primary fields

The first thing that we will do is consider flat Euclidean spacetime as we have already done and compactify the spatial dimension x^1 on a circle of radius 2π to avoid any infrared divergences. The size/radius of the circle is actually not important since a CFT is scale invariant so we chose a convenient value. After this compactification, the theory now lives on the surface of an infinite cylinder and we can define the complex variable $w = x^0 + ix^1$ on that space. To simplify our calculations we will map this cylinder to the complex plane by using the conformal transformation $z = e^w = e^{x^0}e^{ix^1}$. From that transformation we can see that fixed time slices on the cylinder are mapped to circles of fixed radius on the plane, increasing from the origin to infinity. A fixed spatial slice on the cylinder corresponds to a line that extends from the origin to infinity on the plane. See Figure 2.1 for a great visualization. From the mapping that we just described it's easy to see that spatial translations on the cylinder correspond to rotations on the plane such that the momentum operator of the theory is $P = i(L_0 - \bar{L}_0)$, the rotation generator of the plane. Since space is periodic we will also refer to this operator as angular momentum J. On the other hand, time translations on the cylinder are related to dilations on the plane such that the Hamiltonian, which moves time forward, is $H = L_0 + \bar{L}_0$. From now on every calculation will be done on the plane but we will often refer back to what it means on the cylinder since the theory is originally defined on that space.

In the context of the mapping from the cylinder to the plane, the quantization will be done by expanding the primary fields with conformal weight $(h, \bar{h})$ in Laurent modes:

$$\begin{aligned} \phi(z,\bar{z}) &= \sum_{n,\bar{n}\in\mathbb{Z}} z^{-n-h}\bar{z}^{-\bar{n}-\bar{h}}\phi_{n,\bar{n}} \\ \phi_{n,\bar{n}} &= \frac{1}{2\pi i}\oint \mathrm{d}z\, z^{n+h-1}\frac{1}{2\pi i}\oint \mathrm{d}\bar{z}\, \bar{z}^{\bar{n}+\bar{h}-1}\phi(z,\bar{z}) \end{aligned} \tag{2.16}$$

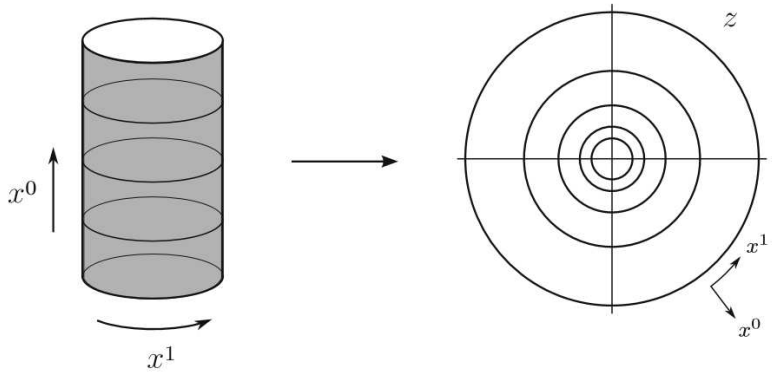


FIGURE 2.1: The map from the cylinder to the complex plane (taken from [1])

and promoting these modes to operators. This is actually pretty intuitive since it corresponds on the cylinder to expanding the field in Fourier modes and quantizing the modes. We will study the properties of the modes later in Section 2.3 and for now the important thing to know is that like as usual fields become operators upon quantization. This procedure is what we call radial quantization.

It is now time to investigate further the conserved current $j_\mu = T_{\mu\nu}\epsilon^\nu$ introduced in the previous section. Whenever there is a conserved current there is an associated conserved charge that is simply the integral over space of the first component of the current. On the cylinder, the charge coming from our current is $Q = \int_{fixed\ x^0} \mathrm{d}x^1 j_0$. We would like to rewrite this in terms of elements from the complex plane to make it simpler. We already know that constant x^0 on the cylinder corresponds to a circle on the complex plane so the integral will become a contour integral $\oint \mathrm{d}z$ on a circle that we will always take to be counter clockwise. The actual change from the integral on the cylinder to the one on the complex plane can be done in a straightforward manner and gives the natural answer

$$Q = \frac{1}{2\pi i}\oint_C \left(\mathrm{d}z\, T(z)\epsilon(z) + \mathrm{d}\bar{z}\bar{T}(\bar{z})\bar{\epsilon}(\bar{z})\right). \tag{2.17}$$

Of course we know that conserved charges generate symmetry transformations of an operator ϕ with the commutator $\delta\phi = [Q, \phi]$ so we can now use the expression (2.17) for the charge to compute the transformation of a generic field under conformal transformation:

$$\delta\phi(w,\bar{w}) = \frac{1}{2\pi i}\oint_C \mathrm{d}z\,[T(z)\epsilon(z), \phi(w,\bar{w})] + \frac{1}{2\pi i}\oint_C \mathrm{d}\bar{z}\,\left[\bar{T}(\bar{z})\bar{\epsilon}(\bar{z}), \phi(w,\bar{w})\right]. \tag{2.18}$$

Here and afterward w is another point on the complex plane and not a point on the complexified cylinder. This rule is valid for any field but one of the goals of this section is to rewrite it in a specific way to compare it with the actual transformation rule of primary fields (2.15). In order to do this we notice that the circle on which we integrate to compute the charge in (2.17) can have an arbitrary radius since it is conserved in time. This allows us to take circles that enclose w or not as we like in (2.18) to rewrite it as

$$\delta\phi(w,\bar{w}) = \frac{1}{2\pi i}\oint_{|z|>|w|} \mathrm{d}z\,\epsilon(z)T(z)\phi(w,\bar{w}) - \frac{1}{2\pi i}\oint_{|w|<|z|} \mathrm{d}z\,\epsilon(z)\phi(w,\bar{w})T(z) + anti\,. \tag{2.19}$$

At this point we will write only the holomorphic part since everything is similar for the anti-holomorphic one.

In a quantized theory, products of operators are not well defined just from the classical theory so we always need to introduce a prescription for the ordering of operators. In a regular QFT, the important quantity to compute correlation functions is the time ordered product of operators. This has to be used here for the theory on the cylinder, but as we have seen earlier different times on the cylinder correspond to circles of different radii on the plane. This means that on the plane time ordering becomes radial ordering, defined by

$$R[A(z)B(w)] = \begin{cases} A(z)B(w) & \text{for } |z|>|w| \\ B(w)A(z) & \text{for } |w|>|z| \end{cases}. \tag{2.20}$$

Using this definition for the products we can rewrite (2.19) as

$$\begin{aligned} \delta\phi(w,\bar{w}) &= \frac{1}{2\pi i}\left(\oint_{|z|>|w|} - \oint_{|w|<|z|}\right)\mathrm{d}z\,\epsilon(z)R[T(z)\phi(w,\bar{w})] + anti \\ &= \frac{1}{2\pi i}\oint_{C(w)} \mathrm{d}z\,\epsilon(z)R[T(z)\phi(w,\bar{w})] + anti\,. \end{aligned} \tag{2.21}$$

The change in integration contours is illustrated in Figure 2.2. This is the final form that we wanted to obtain for the general variation of a field under conformal transformations. Next we will rewrite (2.15) in order to compare both expressions. Since we will work quite a bit with complex integrals, we review quickly how they work in Appendix B. Using these tools, it's simple to rewrite the holomorphic terms as

$$h(\partial_w\epsilon(w))\phi(w,\bar{w}) = \frac{1}{2\pi i}\oint_{C(w)} \mathrm{d}z\,h\frac{\epsilon(z)}{(z-w)^2}\phi(w,\bar{w})$$

$$\epsilon(w)\partial_w\phi(w,\bar{w}) = \frac{1}{2\pi i}\oint_{C(w)} \mathrm{d}z\, \frac{\epsilon(z)}{(z-w)}\partial_w\phi(w,\bar{w})\,.$$

Using this in (2.15) and comparing to (2.21) gives

$$R[T(z)\phi(w,\bar{w})] = \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{(z-w)}\partial_w\phi(w,\bar{w}) + \dots\,. \tag{2.22}$$

The ellipsis are regular terms that are there because they are not fixed by our argument since they don't contribute to the integrals. The expression that we obtained is called an operator product expansion (OPE) and relates a product of two operators inserted at nearby points to a sum of single operators. It is an operator statement that should therefore be applied inside correlators. We will now omit the radial ordering symbol for simplicity and assume that every product of operators is radially ordered. The $T\phi$ OPE along with the anti-chiral version (exact same with bars) define what is meant by a primary field.

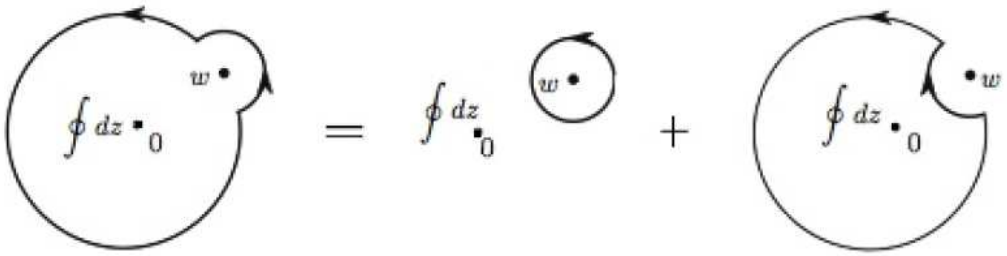


FIGURE 2.2: Change of integration contours in radial ordering (taken from [3])

2.2.2 OPE for energy-momentum tensor

Now that we have introduced the OPE of the energy-momentum tensor with a primary field it would be interesting to know it for two energy-momentum tensors. This is non-trivial since we never assumed that it is a primary and nothing points toward that direction. To find the TT OPE we will actually do exactly the opposite of what we did for $T\phi$. The essential ingredient to get $T\phi$ was the transformation rule of the primary field, but we don't know how the energy-momentum tensor transforms under the action of the conformal group so we will need to find another way to build the OPE. We will then find out how $T(z)$ transforms using the OPE.

The strategy that we will take is finding the most general OPE allowed with some assumptions. This is inspired by Tong's lectures [8]. We know that the energy-momentum tensor must have scaling dimension $\Delta = 2$ since we get energy by integrating it over space. This means that every term in the TT OPE must scale like $\Delta = 4$ and the singular terms must look like $\mathcal{O}_n(z-w)^{-n}$ with the scaling dimension of the operator being $\Delta_n = 4 - n$. We will show later that there can't be operators with negative scaling dimension in a unitary CFT so the highest possible n in the OPE is 4. Remember that we actually compute the radial ordering of the fields so in fact $T(z)T(w) = T(w)T(z)$ so that the OPE must be invariant under $z \leftrightarrow w$ and odd n are not allowed. With all of these assumptions we find the most general OPE to be

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \dots \tag{2.23}$$

At this point c is just a number but it will turn out to really be the central charge of the theory. The term with $n = 1$ is allowed even if we ask for the OPE to be even under exchange because of the derivative of T in the numerator. This OPE shows that the energy-momentum tensor is not a primary because of the term with a constant. We can however show that in fact $T(z)$ is a quasi-primary field. To do so we take the expectation value of the OPE (2.23) to obtain the two-point function. We use the fact that one-point functions must vanish to get the simple result $\langle T(z)T(w)\rangle = \frac{c/2}{(z-w)^4}$. One-point functions must vanish in a CFT because as in QFT they have to be constants but furthermore scale invariance sets this constant to zero. Comparing the result with (2.13) shows that $T(z)$ is a quasi-primary field of weight $(2,0)$. Of course we already knew the weight from the dimension but we didn't know the nature of the field. The same thing can be said for the $\bar{T}\bar{T}$ OPE and we find that it has weight $(0,2)$. The $T\bar{T}$ OPE contains only regular terms and is thus not very interesting.

After using the OPE we will show that (2.23) corresponds to some statements that we made earlier so that it is the right answer. We know that OPEs are related to commutators and the energy-momentum tensor is related to symmetry transformations so it should be possible to use the TT OPE to find the symmetry algebra. In order to do that we will consider the mode expansion motivated by (2.16)

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \Rightarrow L_n = \frac{1}{2\pi i}\oint \mathrm{d}z\, z^{n+1} T(z)\,. \tag{2.24}$$

The fact that we named the modes of the energy-momentum tensor L_n is not random.

They really are the generators of conformal transformations. This can be seen by considering a transformation $\epsilon(z) = -\epsilon_n z^{n+1}$ in the expression (2.17)

$$Q_n = \oint \frac{\mathrm{d}z}{2\pi i} T(z)(-\epsilon_n z^{n+1}) = -\epsilon_n \sum_{m\in\mathbb{Z}} \oint \frac{\mathrm{d}z}{2\pi i} L_m z^{n-m-1} = -\epsilon \sum_{m\in\mathbb{Z}} L_m \delta_{mn} = -\epsilon_n L_n \,.$$

Since the charge generates conformal transformations then from this so do L_n and they are really the Virasoro generators. The obvious way to confirm this is to compute the commutator of the modes by using the OPE and find the Virasoro algebra:

$$\begin{aligned}
[L_m, L_n] &= \frac{1}{2\pi i}\oint \mathrm{d}z\, z^{m+1} \frac{1}{2\pi i}\oint \mathrm{d}w\, w^{n+1}[T(z), T(w)] \\
&= \oint_{C(0)} \frac{\mathrm{d}w}{2\pi i}\, w^{n+1} \oint_{C(w)} \frac{\mathrm{d}z}{2\pi i}\, z^{m+1} R[T(z)T(w)] \\
&= \oint_{C(0)} \frac{\mathrm{d}w}{2\pi i}\, w^{n+1} \oint_{C(w)} \frac{\mathrm{d}z}{2\pi i}\, z^{m+1} \left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \ldots \right) \\
&= \oint_{C(0)} \frac{\mathrm{d}w}{2\pi i}\, w^{n+1} \left((m+1)m(m-1)w^{m-2}\frac{c}{2\cdot 3!} + 2(m+1)w^m T(w) + w^{m+1}\partial_w T(w) \right) \\
&= \oint_{C(0)} \frac{\mathrm{d}w}{2\pi i} \left((m^3-m)w^{n+m-1}\frac{c}{12} + 2(m+1)w^{n+m+1}T(w) + w^{n+m+2}\partial_w T(w) \right) \\
&= (m^3-m)\delta_{m+n,0}\frac{c}{12} + 2(m+1)L_{m+n} + \oint_{C(0)} \frac{\mathrm{d}w}{2\pi i} \left[\partial_w(w^{n+m+2}T(w)) - \partial_w(w^{n+m+2})T(w) \right] \\
&= (m^3-m)\delta_{m+n,0}\frac{c}{12} + 2(m+1)L_{m+n} + 0 - (n+m+2)\oint_{C(0)} \frac{\mathrm{d}w}{2\pi i}\, w^{n+m+1}T(w) \\
&= (m^3-m)\delta_{m+n,0}\frac{c}{12} + 2(m+1)L_{m+n} - (m+n+2)L_{m+n} \\
&= (m-n)L_{m+n} + (m^3-m)\delta_{m+n,0}\frac{c}{12}\,.
\end{aligned}$$

We performed an integration by parts to compute the last integral of $\partial_w T(w)$. Again the formulas to do the integrals are discussed in Appendix B.

We can do basically the same kind of computation that lead to the Virasoro algebra from the TT OPE but using the $T\phi$ OPE to find

$$[L_m, \phi_n] = ((h-1)m - n)\phi_{m+n}\,. \tag{2.25}$$

This is true for any primary and if it's valid only with $\{L_{-1}, L_0, L_1\}$ then the field is a quasi-primary. Taking $\phi_n = L_n$ leads directly to the conclusion that the energy-momentum is a quasi-primary field with weight $h = 2$, which we have already found out by looking at

the 2-point function.

The last thing that we will do in this part is use the TT OPE to determine how the energy-momentum tensor behaves under conformal transformations. Of course we already know what happens for global transformations since it's a quasi-primary field, but we have no idea for other conformal transformations. We already know how the transformation should look like for a chiral infinitesimal conformal transformation from the OPE and (2.21)

$$\begin{aligned}\delta T(w) &= \oint_{C(w)} \frac{\mathrm{d}z}{2\pi i}\, \epsilon(z) R[T(z)T(w)] \\ &= \oint_{C(w)} \frac{\mathrm{d}z}{2\pi i}\, \epsilon(z) \left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \ldots \right) \\ &= \frac{c}{12}\partial^3 \epsilon(z) + 2T(z)\partial\epsilon(z) + \epsilon(z)\partial T(z)\,.\end{aligned}$$

The finite transformation can be found to be

$$T(z) \to T'(z) = \left(\frac{\partial f}{\partial z}\right)^2 T(f(z)) + \frac{c}{12} S(f(z), z) \tag{2.26}$$

where $S(w, z)$ is the Schwarzian derivative defined by

$$S(w,z) = \frac{1}{(\partial_z w)^2}\left((\partial_z w)(\partial_z^3 w) - \frac{3}{2}(\partial_z^2 w)^2 \right)\,.$$

It's straightforward to check that this transformation rule coincides with the infinitesimal one and also that it satisfies the group product rule that applying two successive transformations $z \to w \to u$ gives the same result as the single transformation $z \to u$. One can also show that the Schwarzian derivative vanishes for elements of the global conformal group $SL(2,\mathbb{C})/\mathbb{Z}_2$ in order for $T(z)$ to be a quasi-primary field.

2.2.3 General form for the OPE

Previously in this section we have found the operator product expansion of a primary field with the energy-momentum tensor and of the energy-momentum tensor with itself. We will now find a general expression for the OPE of two chiral quasi-primary fields together as a sum of other chiral fields and their derivatives. We will start from the following

ansatz:

$$\phi_i(z)\phi_j(w) = \sum_{k,n\geq 0} C_{ij}^k \frac{a_{ijk}^n}{n!} \frac{1}{(z-w)^{h_i+h_j-h_k-n}} \partial^n \phi_k(w)\,. \tag{2.27}$$

The scaling with $(z-w)$ comes from scale invariance and the ansatz is arranged such that a_{ijk}^n depends only on the conformal weights and C_{ij}^k contains more information about the fields. We will take $w=1$ and insert this into the following 3-point function

$$\langle(\phi_i(z)\phi_j(1))\phi_k(0)\rangle = \sum_{l,n\geq 0} C_{ij}^l \frac{a_{ijl}^n}{n!} \frac{1}{(z-1)^{h_i+h_j-h_l-n}} \langle \partial^n \phi_l(1)\phi_k(0)\rangle\ .$$

We can evaluate the left-hand side of this by using (2.14)

$$\langle \phi_i(z)\phi_j(1)\phi_k(0)\rangle = \frac{C_{ijk}}{(z-1)^{h_i+h_j-h_k} z^{h_i+h_k-h_j}}$$

and the right-hand side by using (2.13)

$$\langle \partial^n \phi_l(1)\phi_k(0)\rangle = \partial_z^n \langle \phi_l(z)\phi_k(0)\rangle \Big|_{z=1} = \partial_z^n \left(\frac{d_{lk}\delta_{h_l,h_k}}{z^{2h_k}}\right)\Big|_{z=1} = (-1)^n n! \binom{2h_k+n-1}{n} d_{lk}\delta_{h_l,h_k}\,.$$

Combining the last two results into (2.27) gives the constraint

$$\sum_{l,n\geq 0} d_{lk} C_{ij}^l a_{ijk}^n \frac{(-1)^n}{(z-1)^{h_i+h_j-h_k-n}} \binom{2h_k+n-1}{n} = \frac{C_{ijk}}{(z-1)^{h_i+h_j-h_k} z^{h_i+h_k-h_j}}$$

$$\implies \sum_{l,n\geq 0} d_{lk} C_{ij}^l a_{ijk}^n (-1)^n (z-1)^n \binom{2h_k+n-1}{n} = \frac{C_{ijk}}{(1+(z-1))^{h_i+h_k-h_j}}\,.$$

The last step needed to get the result is to use the identity

$$\frac{1}{(1+x)^H} = \sum_{n=0}^{\infty} (-1)^n x^n \binom{H+n-1}{n}$$

with $x = z-1$ and $H = h_i+h_k-h_j$ to rewrite the right-hand side in order to find

$$\begin{aligned} C_{ijk} &= \sum_l C_{ij}^l d_{lk} \\ a_{ijk}^n &= \binom{2h_k+n-1}{n}^{-1} \binom{h_i+h_k-h_j+n-1}{n}\,. \end{aligned} \tag{2.28}$$

These coefficients together with the ansatz (2.27) give the OPE of any two chiral quasi-primary fields. This is nice but the Laurent modes of the fields are the quantities that are actually useful. We can do the same computation that lead to the Virasoro algebra and to (2.25), that is writing the commutator of the modes as contour integrals over the fields and then use the general OPE (2.27) and compute the integrals, to find the commutation relations of the modes of any chiral quasi-primary fields as a sum of the other chiral primaries in the theory. The derivation is not at all illuminating so we will just state the result:

$$\left[\phi_{(i)m}, \phi_{(j)n}\right] = \sum_k C_{ij}^k p_{ijk}(m,n)\phi_{(k)m+n} + d_{ij}\delta_{m,-n}\binom{m+h_i-1}{2h_i-1} \tag{2.29}$$

with

$$p_{ijk}(m,n) = \sum_{\substack{r,s\in\mathbb{Z}_0^+ \\ r+s=h_i+h_j-h_k-1}} C_{r,s}^{ijk}\binom{h_i-m-1}{r}\binom{h_j-n-1}{s}$$

where

$$C_{r,s}^{ijk} = \frac{(-1)^r(2h_k-1)!}{(h_i+h_j+h_k-2)!}\prod_{t=0}^{s-1}(2h_i-2-r-t)\prod_{u=0}^{r-1}(2h_j-2-s-u)\,.$$

This general form for the commutators will be extremely useful in Chapter 4 when we will derive $\mathcal{W}$-algebras. Note that by looking carefully at the coefficients $p_{ijk}(m,n)$ we can see that only fields with dimension $h_k < h_i + h_j$ can appear in (2.29). Also since $p_{ijk}(m,n)$ only depends on the weights of the fields we can label it by h_i, h_j, h_k as well as i, j, k.

Before moving on it's important to mention that we can derive Ward identities from conformal invariance and that it is possible to base the derivation of almost everything in 2d CFT on these identities. However we chose a different path and will therefore not need to introduce the identities since they are not relevant to the rest of the thesis.

2.3 Hilbert Space

At this point we know a lot about the operators in the theory but we don't know anything about the states so the goal of this section will be to study the Hilbert space of a 2d CFT. We will start by presenting the operator/state correspondence and then we will derive some necessary properties that the modes of the fields obey. Next we will introduce descendants and highest weight representations. The last thing that we will discuss here is constraints on the representations coming from unitarity.

We will start by defining in-states from conformal fields. Normally we obtain these states by applying an operator on the vacuum at past infinity $|\phi\rangle = \lim_{t\to-\infty}\phi(x,t)\,|0\rangle$ but when we quantize a CFT on the plane, past infinity becomes the origin so now we define an in-state by

$$|\phi\rangle = \lim_{z,\bar{z}\to 0}\phi(z,\bar{z})\,|0\rangle\ .$$

However we need this state to be well defined when we take the limit and by looking at the mode expansion for a field (2.16) leads us to require that

$$\phi_{n,\bar{n}}\,|0\rangle = 0 \text{ for } n > -h,\ \bar{n} > -\bar{h}\,. \tag{2.30}$$

If we actually take the limit while using the mode expansion almost every term will vanish at the origin and we are left with the actual definition of an in-state

$$|\phi\rangle = \phi_{-h,-\bar{h}}\,|0\rangle\ . \tag{2.31}$$

We will also label the states by the (not necessarily conformal) weights of the operators that created them so we can write $|\phi\rangle = \left|h,\bar{h}\right\rangle$. The correspondence between the fields and the states is known as operator/state correspondence and has the particularity in a CFT to be one-to-one, which it is not in a general QFT. The obvious next step is to define the out-states as the hermitian conjugate of the in-states $\langle\phi| = |\phi\rangle^\dagger$, but for that we need to define what we mean by hermitian conjugate. Hermitian conjugate is normally defined on Minkowski space and the coordinates are not affected but when we switch to Euclidean space the component $x^0 = it$ has to change to $-x^0$. In terms of the complex coordinates this means that $z^\dagger = 1/\bar{z}$ when we identify $\bar{z} = z^*$. Keeping this in mind we define the hermitian conjugate of a field as

$$\phi(z,\bar{z})^\dagger = \bar{z}^{-2h}z^{-2\bar{h}}\phi\left(\frac{1}{\bar{z}},\frac{1}{z}\right)\ .$$

We can expand the right-hand side in modes to get

$$\phi(z,\bar{z})^\dagger = \bar{z}^{-2h}z^{-2\bar{h}}\sum_{n,\bar{n}\in\mathbb{Z}}\left(\frac{1}{\bar{z}}\right)^{-n-h}\left(\frac{1}{z}\right)^{-\bar{n}-\bar{h}}\phi_{n,\bar{n}} = \sum_{n,\bar{n}\in\mathbb{Z}}\bar{z}^{n-h}z^{m-\bar{h}}\phi_{n,\bar{n}}\,.$$

We can also expand directly the hermitian conjugate in modes to get

$$\phi(z,\bar{z})^\dagger = \sum_{n,\bar{n}\in\mathbb{Z}} z^{-n-h}\bar{z}^{-\bar{n}-\bar{h}}\phi^\dagger_{n,\bar{n}}\,.$$

The two expansions are compatible only if $\phi^\dagger_{n,\bar{n}} = \phi_{-n,-\bar{n}}$. The actual example of this relation that will be useful is for the modes of the energy-momentum tensor $L_n^\dagger = L_{-n}$. Taking the hermitian conjugate of everything that we did in this paragraph gives the definition of an out-state

$$\langle\phi| = \lim_{z,\bar{z}\to 0}\langle 0|\,\phi(z,\bar{z})^\dagger = \lim_{w,\bar{w}\to\infty} w^{2h}\bar{w}^{2\bar{h}}\,\langle 0|\,\phi(w,\bar{w}) = \langle 0|\,\phi_{h,\bar{h}} \tag{2.32}$$

with $w = z^{-1}$ and

$$\langle 0|\,\phi_{n,\bar{n}} = 0 \text{ for } n < h,\ \bar{n} < \bar{h}\,.$$

At this point we have already started to introduce the vacuum state $|0\rangle$ in our discussion but we didn't explain what it is. The natural way of defining the vacuum is to ask that it is the most symmetric state, which means that it is annihilated by as many symmetry generators as possible. Normally we ask that it is annihilated by all of them, but here the symmetry generators are the Virasoro modes and the presence of the central charge in the algebra prevents us from having the full symmetry. Indeed if for example we required $L_2\,|0\rangle = L_{-2}\,|0\rangle = 0$ and $L_0\,|0\rangle = 0$ then there would be a contradiction with the actual calculation of the norm of the state $L_{-2}\,|0\rangle$ by using the algebra since $\|L_{-2}\,|0\rangle\,\|^2 = \langle L_2 L_{-2}\rangle = \langle L_{-2}L_2\rangle + 4\,\langle L_0\rangle + \frac{c}{2} = \frac{c}{2} \neq 0$. We are then lead to require that $L_n\,|0\rangle = 0$ for $n > -2$. The out vacuum is defined simply by the conjugate of this: $\langle 0|\,L_n = 0$ for $n < 2$. Of course this was obvious from the fact that the Virasoro generators are modes of the energy-momentum tensor, which has $h = 2$, but it's now justified using symmetry. Note that the vacuum is invariant under the global conformal group $\{L_{-1}, L_0, L_1\}$ like in higher dimensional CFTs. Of course all this discussion is applicable to anti-holomorphic modes as well.

Now that we know the behavior of the vacuum under the symmetry algebra, let's look at primary states, that is in-states created by applying a primary field to the vacuum. For simplicity we will look at chiral primaries, but the generalization to non-chiral is simple. We already have the necessary tool in (2.25) to study the conformal transformation of

primary states. The first interesting result is

$$L_0 |\phi\rangle = L_0\phi_{-h} |0\rangle = \phi_{-h}L_0 |0\rangle + h\phi_{-h} |0\rangle = h |\phi\rangle \ .$$

In other words the holomorphic conformal weight is the eigenvalue of the operator L_0. Of course we can do the same thing to show that $\bar{h}$ is the eigenvalue of $\bar{L}_0$. This means that primary states are eigenstates of the Hamiltonian $L_0 + \bar{L}_0$ and this is why the sum of the weights (the scaling dimension) can be interpreted as the energy. For chiral fields the weight h can be directly interpreted as energy. The fact that the angular momentum is $i(L_0 - \bar{L}_0)$ justifies also the fact that we called $h - \bar{h}$ the spin. Another interesting result is

$$L_n |\phi\rangle = L_n\phi_{-h} |0\rangle = \phi_{-h}L_n |0\rangle + (h(n+1) - n)\phi_{-h+n} |0\rangle \ .$$

This is equal to zero for $n > 0$ and it means that positive Virasoro modes kill primary states. This is the best that can be done because any states other than the vacuum has to have less symmetry. We will investigate primary states further later when we will discuss the representations of the Virasoro algebra.

The last ingredients that we need to introduce are ladder operators and normal ordering. In the Fourier expansion of a field in a general QFT the modes are creation and annihilation operators so we expect that for a CFT on the plane the Laurent modes would be the same. Indeed we have already seen in (2.30) that field modes with $n > -h$ annihilate the vacuum. We can better analyze the role of the Laurent modes by looking at how they affect the L_0 eigenvalue of a state.

$$L_0\phi_n |0\rangle = (L_0\phi_n - \phi_n L_0) |0\rangle = [L_0, \phi_n] |0\rangle = -n\phi_n |0\rangle$$

This shows that positive n decrease the energy and negative ones increase it. However we already know that the result is zero for $n > -h$ so it means that the energy is bounded from below and we can conclude that the modes with $n \leq -h$ are creation operators since they increase the energy. Normal ordering is then simply the usual statement that we put all the creation operators to the left and the annihilation operators to the right. An example of such an ordering is

$$N(\chi\phi)_n = \sum_{k>-h^\phi} \chi_{n-k}\phi_k + \sum_{k\leq -h^\phi} \phi_k\chi_{n-k} \, . \tag{2.33}$$

It can be shown that normal ordered products appear in the regular part of OPEs in the following way

$$\phi(z)\chi(w) = singular + \sum_{n=0}^{\infty} \frac{(z-w)^n}{n!} N(\chi\partial^n\phi)(w)\,.$$

Comparing this with the general form for the OPE (2.27) we see that it's possible to write the regular part in two ways. We can first simply write the term at each order as the normal ordering of the fields, for example at level $(z-w)^0$ it looks like $N(\phi\chi)$. We can also write it as a sum of derivatives of possibly every quasi-primary in the theory, like $\sum_k \partial\phi_{(k)}$. It is simpler to divide the sum over quasi-primaries into terms with the fields of which we compute the OPE and all the others as $\mathcal{N}(\phi\chi) + \partial\phi + \partial\chi$. This basically means that $N(\phi\chi) = \mathcal{N}(\phi\chi) + \partial\phi + \partial\chi$ where $\mathcal{N}(\phi\chi)$ includes the contribution from all the other quasi-primaries. The important point in this expansion is that the normal ordered product and the derivatives are not quasi-primary fields but the new term introduced $\mathcal{N}(\phi\chi)$ is. This is very schematic but it shows the idea that the whole field space present in the OPE can be found by using derivatives of fields and quasi-primary normal ordered products. This will be useful when we will actually use (2.27) to build $\mathcal{W}$-algebras.

2.3.1 Highest weight representations and Kac determinant

After having discussed concretely how to build the states in a CFT, we will now show more abstractly how to build the whole Hilbert space in a cleaner way. In fact the Hilbert space is going to be a sum of representations of the Virasoro algebra. We will proceed very similarly to the way that we build representations of the $SU(2)$ algebra for angular momentum. We need to first pick the biggest set of commuting operators in the algebra in order to label the states by their eigenvalues. However in the Virasoro algebra it's impossible to find such a set with more than one operator so we naturally pick L_0 since it's hermitian and label the states by their weight. L_0 will play the role of J_z from $SU(2)$ and h will be equivalent to the component of spin along the z direction. Using the Virasoro algebra we can show that the L_n with $n > 0$ decrease the eigenvalue of L_0 by n and with $n < 0$ increase the eigenvalue by n. This just comes from $L_0 L_n |h\rangle = (L_n L_0 - nL_n)|h\rangle = (h-n)L_n|h\rangle$. Therefore L_n with $n > 0$ correspond to $(J_-)^n$ and L_{-n} correspond to $(J_+)^n$. With these operators we build what is called a highest weight representation of the algebra. In fact here we will build a lowest weight representation because the states are labeled by energy and energy is bounded from below. To construct the representation we need to start with a lowest weight state that is annihilated by the lowering operators, that are here L_n. The

rest of the states in the Hilbert space are then built by applying the raising operators L_{-n} on the lowest weight state. As we have seen in the previous section, states that satisfy the properties that we have just enumerated for the lowest weight states are actually primary states in a CFT. This means that we build different representations of the Virasoro algebra by starting with different primary states and by applying the negative generators. The states generated are not primaries and we will call them descendants. They have higher L_0 eigenvalues and are therefore excited states. The basis of the representation is then the following set of states

$$\{L_{k_1}...L_{k_n}\left|h\right\rangle : k_i \leq -1\}\,. \tag{2.34}$$

The condition $k_i \leq -1$ is replaced by $k_i \leq -2$ in the case where we start with the vacuum since $L_{-1}\left|0\right\rangle = 0$. By convention we order the states in increasing index from left to right. It turns out that the states with fixed $\sum_i k_i = -N$ form subspaces of energy $h + N$ that we call levels and that are orthogonal to the each other.

We will build a representation of one copy of the Virasoro algebra that way and we will call each representation a Verma module. A Verma module only depends on the eigenvalue h of the primary that it is built on and on the central charge c so we call it $V(h, c)$. The actual Hilbert space consists of representations of two copies of Virasoro so we do the same thing for the anti-holomorphic part and at the end the full Hilbert space is $\sum_{h,\bar{h}} V(h, c) \otimes \bar{V}(\bar{h}, \bar{c})$.

It is possible to relate the states that compose a Verma module to the operators that create them. We will give some examples for the simple Verma module of the identity and then state the generalization. First of all it's obvious that the state $L_{-2}\left|0\right\rangle$ comes from the energy-momentum tensor. We can use the derivative of the energy-momentum tensor $\partial T(z) = \sum_n (-n-2) z^{-n-3} L_n$ to see that it corresponds to the state $L_{-3}\left|0\right\rangle$. The state associated with the normal ordered product $N(TT)$, which has weight $h = 4$, is found by using (2.31) and (2.33)

$$N(TT)_{-4}\left|0\right\rangle = \left(\sum_{k>-2} L_{-4-k}L_k + \sum_{k\leq -2} L_k L_{-4-k}\right)\left|0\right\rangle = L_{-2}L_{-2}\left|0\right\rangle\,.$$

I think at this point the pattern is easy to see. In general the energy-momentum tensor will always be part of the descendants because every state is built with Virasoro modes. For a primary field ϕ of weight h the Verma module and the associated descendants look

like:

Level	Weight	Operator	State
0	h	ϕ	$\phi_{-h}\|0\rangle = \|h\rangle$
1	$h+1$	$\partial\phi$	$L_{-1}\|h\rangle$
2	$h+2$	$\partial^2\phi$	$L_{-1}L_{-1}\|h\rangle$
2	$h+2$	$N(T\phi)$	$L_{-2}\|h\rangle$
3	$h+3$	$\partial^3\phi$	$L_{-1}L_{-1}L_{-1}\|h\rangle$
3	$h+3$	$N(T\partial\phi)$	$L_{-2}L_{-1}\|h\rangle$
3	$h+3$	$N(\partial T\phi)$	$L_{-3}\|h\rangle$

We call the set of operators that generate a Verma module a conformal family. In general a conformal family is composed of normal ordered products involving the primary field and the energy-momentum tensor along with their derivatives.

We now have a systematic way of computing the Hilbert space of a 2d CFT, but nothing tells us that every space built that way satisfies unitarity. We will now use the tools that we developed up to this point to constrain the values of h and c in a unitary theory. In order to satisfy unitarity we will ask that there are no negative norm states in the Hilbert space. The first constraint that this gives is very simply obtained from $\|L_{-2}|0\rangle\|^2 = \frac{c}{2}$, which was computed earlier. In order for the state to have a positive norm we must ask that $c > 0$. We can also compute $\|L_{-1}|h\rangle\|^2 = \langle h|L_1L_{-1}|h\rangle = \langle h|L_{-1}L_1 + 2L_0|h\rangle = 2h\langle h|h\rangle$. We ask that norms are positive so $\langle h|h\rangle$ is positive and we must ask that $h > 0$ to preserve unitarity.

On top of negative norm states that ruin unitarity, there are also null states that have to be taken out of the Hilbert space in order to not have problems with probability conservation. We will thus develop a procedure to find the null states and it will in fact allow us to find negative norm states as well. Consider a general vector in a Verma module that we expand in the basis that we already introduced $|v\rangle = \sum_a \lambda_a |a\rangle$. The norm of this state is $\|v\|^2 = \sum_{a,b} \lambda_a^* \langle a|b\rangle \lambda_b = \vec{\lambda}^\dagger M \vec{\lambda}$ where we defined the so-called Gram matrix $M_{ab} = \langle a|b\rangle$, which is simply the matrix of all the possible inner products of basis states. Such a state can have a null norm if the vector $\vec{\lambda}$ is an eigenvector of the matrix with eigenvalue zero. This means that null states arise when the Gram matrix has a null eigenvalue, in which case its determinant, called the Kac determinant, will be zero. We know what the basis states look like in a Verma module so the actual matrix that we will consider is

$$M = \langle h| \prod_i L_{m_i} \prod_j L_{-n_j} |h\rangle \ . \tag{2.35}$$

In fact this matrix will be block diagonal because of the property stated before that states in different levels are orthogonal. This will allow us to consider the determinant of the Gram matrix level by level. At level 1 there is only one state and we find

$$\det M_1(h,c) = M_1(h,c) = \langle h| L_1 L_{-1} |h\rangle = 2h$$

where we have taken the primary state to be unit normalized. The zero of this determinant is at $h = 0$, which means that for this Verma module there is a null state at level 1. We can next compute the Gram matrix at level 2, where there are two states:

$$M_2(h,c) = \begin{pmatrix} \langle h| L_2 L_{-2} |h\rangle & \langle h| L_1 L_1 L_{-2} |h\rangle \\ \langle h| L_2 L_{-1} L_{-1} |h\rangle & \langle h| L_1 L_1 L_{-1} L_{-1} |h\rangle \end{pmatrix} = \begin{pmatrix} 4h + \frac{c}{2} & 6h \\ 6h & 4h(2h+1) \end{pmatrix}$$

$$\implies \det M_2(h,c) = 32h\left(h^2 - \frac{5h}{8} + \frac{hc}{8} + \frac{c}{16}\right) = 32(h - h_{1,1}(c))(h - h_{2,1}(c))(h - h_{1,2}(c))$$

where the roots of the determinant are

$$h_{1,1} = 0$$

$$h_{2,1} = \frac{5-c}{16} + \frac{1}{16}\sqrt{(1-c)(25-c)}$$

$$h_{2,1} = \frac{5-c}{16} - \frac{1}{16}\sqrt{(1-c)(25-c)}\,.$$

This means that there are 3 null states at level 2, with the one at $h = 0$ coming from the null state at level 1. The fact that null states carry out to the following levels is a generic thing. At this point it starts to be tedious work to compute the determinant, but Kac found a general formula to compute it at every level. The amazing result is

$$\det M_N(h,c) = \alpha_N \prod_{0<p,q\leq N} (h - h_{p,q}(c))^{P(N-pq)} \tag{2.36}$$

where

$$h_{p,q}(m) = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}$$

and

$$m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25-c}{1-c}}\,.$$

The function $P(n)$ counts the number of partitions of the number n and is there because null states carry on to every following levels. This is a general formula that lets us know

when there are null states in a Verma module. We will not explain the details of the Kac determinant but we will expose the conclusions that we can get by studying it carefully. If we look at the $h - c$ plane, the zeros of the Kac determinant are curves and we can check that any point that lie between these curves leads to a negative determinant, which means that there is a negative norm state in the Hilbert space. In fact this is only the case for $c < 1$ and one can show that for $c > 1$ there are no constraints coming from the Kac determinant. For $c < 1$ the allowed theories must lie on curves with a zero of the determinant and by studying further the actual expression it's possible to show that there is only a discrete set of points that are consistent with unitarity: the intersection points of vanishing curves. The allowed central charges are rational numbers so we call the theories rational CFTs and for each of them there is only a finite number of weights that can appear so unitarity is very constraining. Finally, the conditions derived from the Kac determinant are necessary for a theory to be unitary, but nothing says they are sufficient. In fact they are since there exist statistical models that have exactly the properties predicted in rational CFTs. These theories are called minimal models.

2.4 Modular Invariance

In the last section we put constraints on CFTs by asking that they are unitary. It seems like we have found as much as we could with this property and it is now time to study another property that constraints conformal field theories. As we saw everywhere in the previous sections, holomorphic and anti-holomorphic parts of conformal fields separate when we study a CFT on the whole complex plane. It simplifies a lot our calculations but it would also be interesting to study what happens when there is a coupling between the two sectors. This arises when we leave the conformal fixed point of a theory, but to study it while staying on the fixed point we will look at a theory living on a different geometry. The complex plane is equivalent to the Riemann sphere, that has genus zero, so the next geometry to consider would be the torus, which has genus one. In the last section of this chapter we will first introduce a property of tori known as modular invariance and we will then apply it to the partition function of a 2d CFT.

2.4.1 Modular transformations of the torus

Here we will discover an important property of a torus. To construct a torus, all we have to do is take a cylinder and glue its ends together. However since the cylinder itself is a parallelogram with a pair of opposite edges identified we can define a torus by taking a parallelogram on the complex plane and identifying its pairs of opposite sides together. The shape of the torus is then entirely defined by the shape of what is called the fundamental domain (see Figure 2.3). Every point on the complex plane is identified by following the rule $w \sim w + \alpha_1 m + \alpha_2 n$ with $m, n \in \mathbb{Z}$ so that the fundamental domain generates a lattice. We can characterize the lattice by one complex parameter defined by $\tau = \alpha_2/\alpha_1$ and this defines at the same time the torus. However many different complex parameters can define the same lattice so that there is a redundancy in the description. Indeed if (α_1, α_2) describes a torus, any basis satisfying

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

with $a, b, c, d \in \mathbb{Z}$ will describe the same lattice, so the same torus. Some examples, that will be discussed later, are shown in Figure 2.4. Note that the determinant of the transformation matrix must be ± 1 in order for the inverse transformation to exist and have integer coefficients. This means that such transformation preserves the area of the fundamental domain. Also since (α_1, α_2) and $(-\alpha_1, -\alpha_2)$ describe the same lattice we can identify the transformations under a $\mathbb{Z}_2$ group. These reparametrizations of the lattice are called modular transformations and form the modular group $SL(2, \mathbb{Z})/\mathbb{Z}_2$. We can take advantage of scale invariance to rewrite the lattice (α_1, α_2) as $(1, \tau)$ to find that modular transformations are written as

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{2.37}$$

with $ad - bc = 1$.

The modular group is generated by the so-called S and T transformations, defined as follows

$$T : \tau \to \tau + 1 \Longrightarrow T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$S : \tau \to -\frac{1}{\tau} \Longrightarrow S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Any element of the group can be written as a succession of applications of these two. An

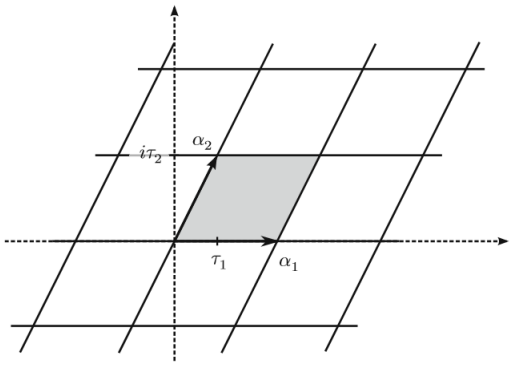


FIGURE 2.3: Lattice generating a torus (taken from [1])

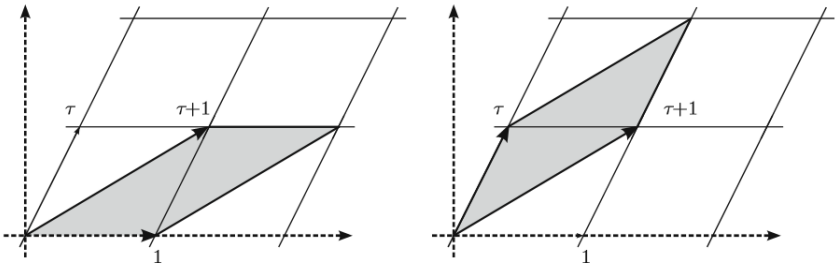


FIGURE 2.4: Some different basis that generate the same lattice (taken from [1])

important property of these transformations is that $S^2 = (ST)^3 = 1$. The modular group takes a point on the complex plane to another point, but it's possible to find a region where no point is transformed into another one. This region is (also) called the fundamental domain and is illustrated in Figure 2.5 along with its transformation under some simple elements of the modular group.

2.4.2 The partition function

The next thing to consider is the partition function of a CFT. We consider a 2d conformal field theory living on a torus generated by a lattice $(1, \tau)$ with complex parameter $\tau = \tau_1 + i\tau_2$. For simplicity we take time along the imaginary axis and space along the real axis but it doesn't really matter since the only important thing is the relation of the complex parameter compared to the coordinates. The operators that generate translations along the real and imaginary axis are then obviously the momentum P and the Hamiltonian H. Since the complex parameter of the torus has a non-trivial real part, moving in the

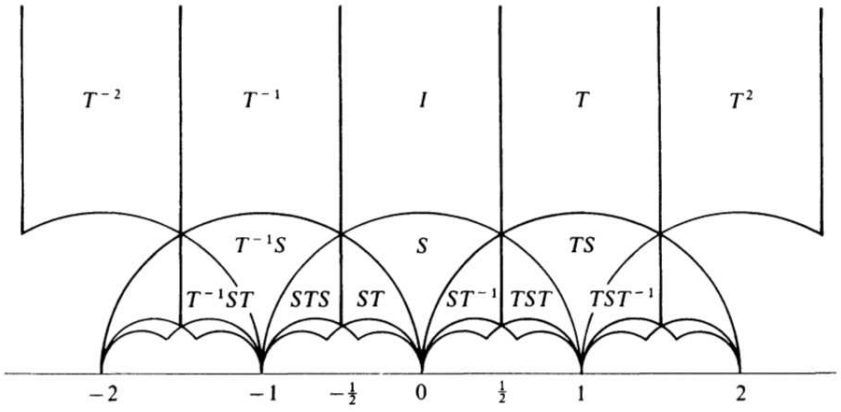


FIGURE 2.5: Fundamental domain of the complex plane and its modular transformations (taken from [9])

time direction by τ_2 doesn't bring us back to the starting point. This means that we cannot use the usual Boltzmann factor because time is not periodic by itself. In fact to come back to the same point we need to do a spatial translation as well so that the operator that generates periodic translations involves momentum and the Hamiltonian and the right partition function to consider is

$$Z(\tau_1, \tau_2) = \mathrm{Tr}\left(e^{-2\pi\tau_2 H + 2\pi i \tau_1 P}\right) \tag{2.38}$$

where the trace is taken over the whole Hilbert space of the theory. The momentum and Hamiltonian operators used there are defined on the torus, which can in fact be seen as a cylinder with periodic time. We would like to relate them to the operators on the plane to compare with everything we did so far. To do so we have to remember the fact that $H = (L_0)_{cyl} + (\bar{L}_0)_{cyl}$ and $P = ((L_0)_{cyl} - (\bar{L}_0)_{cyl})$ (this can be seen from T_{00} and T_{01}). We then need to express the energy-momentum tensor on the cylinder in terms of the one on the plane, which can be done by using the transformation rule (2.26). Remember that the transformation from the plane z to the cylinder w is given by $z(w) = e^w$ so that the transformation gives

$$T_{cyl}(w) = \left(\frac{\partial z(w)}{\partial w}\right)^2 T_{plane}(z(w)) + \frac{c}{12} S(z(w), w) = z^2 T(z) - \frac{c}{24}\,.$$

Expanding this in modes leads to $(L_0)_{cyl} = L_0 - \frac{c}{24}$ with the same for the anti-holomorphic one and the modes without labels are on the plane. Taking these changes into (2.38) gives

$$\begin{aligned} Z(\tau,\bar{\tau}) &= \mathrm{Tr}\left(e^{-2\pi\tau_2\left(L_0+\bar{L}_0-\frac{c}{24}-\frac{\bar{c}}{24}\right)+2\pi i\tau_1\left(L_0-\bar{L}_0-\frac{c}{24}+\frac{\bar{c}}{24}\right)}\right) \\ &= \mathrm{Tr}\left(q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right) \end{aligned} \tag{2.39}$$

with $q = e^{2\pi i\tau}$ and $\bar{\tau} = \tau_1 - i\tau_2$. This is the final expression for the partition function on a torus and it is with this that we can now exploit modular invariance to find interesting results. We will explore the actual consequences of modular invariance later in Chapter 3.

As a final remark, this partition function corresponds physically to taking a CFT on a spatial circle and looking at it at finite temperature. The partition function is then computed by a path integral on a torus since time must be periodic at finite temperature. The component τ_2 is the inverse temperature $\beta/2\pi$, which is always there in a partition function, and τ_2 is an angular potential $K/2\pi$, which is there since momentum is conserved. The partition function in these terms is

$$Z(\beta, K) = \mathrm{Tr}\left(e^{-\beta H + iKP}\right) . \tag{2.40}$$

Chapter 3

Universal Constraints from Modular Invariance

In this chapter we will explore the consequences of modular invariance of the partition function of a CFT on its spectrum. Indeed, by looking at a 2d CFT on a torus and asking that the partition function be invariant under modular transformations of the torus' complex structure we can obtain non-trivial constraints that must be satisfied by the primary operators in the theory. We will first discuss a simple constraint on the asymptotic growth of states that is coming from invariance under the S transformation. The presentation of this first constraint is included mostly to show the power of modular invariance and the kind of idea that are studied. Then we will present a universal bound on operator dimensions that comes from a more elaborate use of the S transformation. Next we will show an improvement on this bound for CFTs with only even spin primaries, which is obtained using S and ST invariance. Finally we will show a tentative method to improve the original bound that uses the whole modular group. The results presented here are particularly interesting because they are very general since they are coming from general consistency conditions that every CFT must satisfy.

3.1 Cardy's Formula

In 1986, John Cardy started studying the spectrum of 2d CFTs using modular invariance [10]. He constrained the asymptotic growth of the density of states in a CFT simply by using the invariance of the partition function under the S transformation. This tool is now very popular because it's as simple as it is powerful, as we will soon see. In this section we will derive a special case of Cardy's formula by following the presentation in Chapter 25 of a set of lecture notes on quantum gravity and black holes by Hartman [11]

and the paper [12]. The goal of this section is just to show how the idea works so we will not focus on the details.

Let's consider a CFT in 2d with space compactified on a circle of radius 2π. The partition function of this CFT at finite temperature $T = 1/\beta$ and with potential K has been introduced in Section 2.4.2. For simplicity we will consider the case where the potential is zero. The partition function is then

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta H} = \sum_{states} e^{-\beta E} = \sum_{E} \rho(E) e^{-\beta E}$$

with $\rho(E)$ being the number of states with energy E. Since the partition function can be computed by a path integral on a torus, it has to be invariant under a modular transformation of the complex parameter $\tau = \frac{K+i\beta}{2\pi}$ that characterizes the torus. We will focus on the S transformation $\tau \to -\frac{1}{\tau}$, which for $K = 0$ means $\beta \to \frac{4\pi^2}{\beta}$. S invariance then becomes

$$Z(\beta) = \sum_{E} \rho(E) e^{-\beta E} = Z\left(\frac{4\pi^2}{\beta}\right) = \sum_{E} \rho(E) e^{-\frac{4\pi^2 E}{\beta}}\,.$$

We would like to study this equation at high temperature, so at $\beta \to 0$. In that case the right-hand side is dominated by the vacuum state, for which $\rho(E) = 1$ and we already found that on a torus $E_{vac} = -\frac{c}{12}$, and the sum in the left-hand side can be approximated by an integral. Invariance under the S transformation then becomes at high temperature

$$\int \mathrm{d}E\, \rho(E) e^{-\beta E} = e^{-\frac{4\pi^2 E_{vac}}{\beta}} = e^{\frac{\pi^2 c}{3\beta}}\,.$$

This equation is basically the Laplace transform of the density of states, which can be inverted by using a well-known formula called Mellin inversion integral [13]:

$$\rho(E) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{zE} e^{\frac{\pi^2 c}{3z}} dz = \int_{-\infty}^{\infty} e^{-i\left(2\pi y E - \frac{\pi c}{6y}\right)} dy\,.$$

Here we chose to integrate on the imaginary axis but the location of the contour on the real axis is arbitrary since the integrand doesn't have a singularity. We also defined $z = -2\pi i y$ to rewrite the integral, which can now be evaluated for large E by a saddle point approximation. The maximum of the integrand is at $y = i\sqrt{\frac{c}{12E}}$ and the result is then the asymptotic density of states $\rho(E) = \exp\left(2\pi\sqrt{\frac{c}{3}E}\right)$, which comes purely from modular invariance.

Cardy's formula is actually not the kind of result that is of major interest to this thesis, even though it is interesting in a bigger context, but its simple derivation is what sparked the interest in the study of modular invariance in 2d CFT.

3.2 Hellerman's Bound

The main result that motivated the work of this chapter was derived by Hellerman [14], who found a constraint on the spectrum of 2d CFTs by using the invariance of the partition function under the S transformation. Stated quickly, the result is that any unitary Conformal Field Theory in 2 dimensions must contain a primary operator whose scaling dimension satisfies $0 < \Delta < \frac{c+\bar{c}}{12} + \mathcal{O}(1)$. It is rather different from Cardy's formula in the sense that it bounds the dimension of a single operator instead of the asymptotic growth of the spectrum. This is a fairly general result and it can be interpreted in theories of gravity using the AdS/CFT correspondence, as will be done in Chapter 5. This section is devoted to describing the details of the derivation of the bound and follows closely Hellerman's paper [14].

3.2.1 Setup

Techniques related to the ones that are used in this section were originally used in [15] to bound operator dimensions in 4d CFTs. The idea that we apply is very similar to it, but the actual tools are different. Indeed here we make use of modular invariance instead of crossing symmetry to obtain a constraint on the spectrum.

Let's again consider a 2d Conformal Field Theory, with positive norm and discrete spectrum, that has a compact spatial direction of length 2π. Different expressions for the partition function of this theory at temperature $1/\beta$ with a potential K were given in Section 2.4.2. As noted in Sections 2.4 and 3.1, the partition function $Z(\tau, \bar{\tau})$ must be invariant under any modular transformation since τ corresponds to the complex structure of a torus. In this section we will focus more specifically on invariance under the S transformation $\tau \to -\frac{1}{\tau}$. We should note that invariance under the T transformation $\tau \to \tau + 1$ is already taken into account since it simply corresponds to the condition that $h - \bar{h}$ be an integer. This can be easily seen from the expression (2.39) for the partition function.

The constraint coming from S invariance is $Z(\tau, \bar{\tau}) = Z(S\tau, S\bar{\tau}) = Z(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}})$. It is particularly useful to study this equation close to the point $\tau = i$, which is the fixed point

of the S transformation, satisfying $Si = i$. This point corresponds to a "square" torus that has the time and space periods equal. In other words the temperature is equal to $1/2\pi$ and is fixed under S transformation. Higher temperature partition functions are in general related to lower temperature ones by S transformation, so in that sense we can think of $\tau = i$ as being in the middle of the temperature spectrum. We will parametrize the neighborhood of this fixed point by writing $\tau = ie^s$ and in terms of this new variable we require that $Z(ie^s, -ie^{\bar{s}}) = Z(ie^{-s}, -ie^{-\bar{s}})$. Differentiating this new constraint with respect to s and $\bar{s}$ gives us

$$\left(\frac{\partial}{\partial s}\right)^{N_R}\left(\frac{\partial}{\partial \bar{s}}\right)^{N_L} Z(ie^s, -ie^{\bar{s}})\Big|_{s=0} = 0 \text{ for } N_L + N_R \text{ odd}$$

since every s or $\bar{s}$ derivative brings out a minus sign on the right-hand side. This means that if we apply an odd number of derivatives we will get an equation of the form $X = -X$, which of course leads to $X = 0$. This constraint can now be rewritten in terms of τ as

$$\left(\tau\frac{\partial}{\partial \tau}\right)^{N_R}\left(\bar{\tau}\frac{\partial}{\partial \bar{\tau}}\right)^{N_L} Z(\tau, \bar{\tau})\Big|_{\tau=i} = 0 \text{ for } N_L + N_R \text{ odd}\,.$$

Up until now we have a general constraint, but here we will focus on purely imaginary complex structures $\tau = \frac{i\beta}{2\pi}$ (no potential) so we can consider the simpler constraint

$$\left(\beta\frac{\partial}{\partial \beta}\right)^{N} Z(\beta)\Big|_{\beta=2\pi} = 0 \text{ for } N \text{ odd}\,. \tag{3.1}$$

All those constraints are independent and are potentially useful, but we will only use the ones for $N = 1, 3$ to obtain a satisfying constraint. An interesting fact about equation (3.1) is that it gives complementary information to what we can obtain from studies of high and low temperature partition functions. Indeed at low temperature the partition function is a sum of exponentials with positive coefficients so unitarity is manifest but modular invariance is not. The high temperature limit is redundant with low temperature specifically because of modular invariance so we don't gain anything new. This is in contrast with the medium temperature expansion, which makes modular invariance manifest but unitarity is hidden so that the information gained comes from a different condition.

To use invariance under the S transformation in order to get a bound on dimensions

of some primary operators in a CFT we will first need to write the partition function only in terms of h and $\bar{h}$. We will restrict our attention to theories with $c, \bar{c} > 1$ since the representations are simple in that case. Furthermore we will assume that there is no chiral algebra other than Virasoro, meaning that there is no primary operator with dimensions $(h, 0)$ or $(0, \bar{h})$. The objective will now be to express the full partition function in terms of the partition function of individual primary fields.

Let's start by separating the trace over the whole Hilbert space into sums over conformal families, keeping in mind the method to build highest weight representations of the Virasoro algebra discussed in Section 2.3. Since we assumed no chiral extension, the primaries either have $h = \bar{h} = 0$ (the identity) or $h, \bar{h} > 0$. The partition function can then be written as

$$Z(\tau) = Z_{id}(\tau) + \sum_A Z_A(\tau)\,.$$

The indices A are labeling the primary fields of the CFT, which have weights h_A and $\bar{h}_A$, and each term contains the contributions from an individual primary and all its descendants. These individual terms are known as characters. To compute them, remember that the descendants are build out of a primary state by applying the negative Virasoro generators:

$$|n_1, n_2, ..., \bar{n}_1, \bar{n}_2, ...\rangle = ...(\bar{L}_{-2})^{\bar{n}_2}(\bar{L}_{-1})^{\bar{n}_1}...(L_{-2})^{n_2}(L_{-1})^{n_1}\left|h, \bar{h}\right\rangle\,.$$

The most important thing to understand is that each factor of L_{-n} contributes to the energy L_0 of the state by n. This can be seen by using the Virasoro algebra to calculate the value of $L_0\,|n_1, n_2, ..., \bar{n}_1, \bar{n}_2, ...\rangle$. The commutation relations that are needed reduce to $[L_0, L_{-n}] = nL_{-n}$, which show clearly how passing L_0 through one L_{-n} will after all increase the energy by n. This means that, when we consider the actual energy of the primary state itself, the value of L_0 on a state $|n_1, n_2, ..., \bar{n}_1, \bar{n}_2, ...\rangle$ is $(h + n_1 + 2n_2 + 3n_3 + ...)$. The same goes for the bar variables, since they behave exactly like the non-bar and they

commute with them. The total Verma module is generated by the states with all the possible values of n and $\bar{n}$ so the individual parts of the partition function are

$$\begin{aligned} Z_A &= \mathrm{Tr}_A\left(q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right) = q^{-\frac{c}{24}}\bar{q}^{-\frac{\bar{c}}{24}}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\ldots\sum_{\bar{n}_1=0}^{\infty}\sum_{\bar{n}_2=0}^{\infty}\ldots q^{h_A+n_1+2n_2+\ldots}\bar{q}^{\bar{h}_A+\bar{n}_1+2\bar{n}_2+\ldots} \\ &= q^{h_A-\frac{c}{24}}\bar{q}^{\bar{h}_A-\frac{\bar{c}}{24}}\left(\sum_{n_1=0}^{\infty}q^{n_1}\right)\left(\sum_{n_2=0}^{\infty}q^{2n_2}\right)\ldots\left(\sum_{\bar{n}_1=0}^{\infty}\bar{q}^{\bar{n}_1}\right)\left(\sum_{\bar{n}_2=0}^{\infty}\bar{q}^{2\bar{n}_2}\right) \\ &= q^{h_A-\frac{c}{24}}\bar{q}^{\bar{h}_A-\frac{\bar{c}}{24}}\left(\frac{1}{1-q}\right)\left(\frac{1}{1-q^2}\right)\ldots\left(\frac{1}{1-\bar{q}}\right)\left(\frac{1}{1-\bar{q}^2}\right)\ldots \\ &= q^{h_A-\frac{c}{24}}\bar{q}^{\bar{h}_A-\frac{\bar{c}}{24}}\prod_{n=1}^{\infty}(1-q^n)^{-1}\prod_{\bar{n}=1}^{\infty}(1-\bar{q}^{\bar{n}})^{-1}\ . \end{aligned} \tag{3.2}$$

The sums converge if $|q|<1$, which happens for τ in the upper-half plane. If we expand the result in the form $\sum_n p(n)q^n$, the coefficients are just the number of partitions of the number n. This is to be expected since $p(n)$ is actually the number of states in the nth level of the Verma module.

The expression is a little bit different for the descendants of the identity because we have to leave out the states $L_{-1}|0\rangle$ and $\bar{L}_{-1}|0\rangle$ since they are null. All descendants of the vacuum with any powers of L_{-1} or $\bar{L}_{-1}$ will also have a zero norm, as can be seen in the Kac determinant. This simply takes out the sums over n_1 and $\bar{n}_1$ in (3.2) so that we get

$$Z_{id} = q^{-\frac{c}{24}}\bar{q}^{-\frac{\bar{c}}{24}}\prod_{n=2}^{\infty}(1-q^n)^{-1}\prod_{\bar{n}=2}^{\infty}(1-\bar{q}^{\bar{n}})^{-1}\ . \tag{3.3}$$

We can rewrite the characters in a simpler way by introducing the Dedekind eta function

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$$

so that the expression for the total partition function is obtained by combining equations (3.2) and (3.3)

$$Z(\tau) = q^{-\frac{c-1}{24}}\bar{q}^{-\frac{\bar{c}-1}{24}}|\eta(\tau)|^{-2}\left((1-q)(1-\bar{q})+\sum_A q^{h_A}\bar{q}^{\bar{h}_A}\right)\ . \tag{3.4}$$

In the next section we will focus on purely imaginary complex structure $\tau = \frac{i\beta}{2\pi}$ so we can consider a simpler version of the partition function by using $q=\bar{q}=e^{-\beta}$ in equation

(3.4)

$$Z(\beta) = M(\beta)Y(\beta) + B(\beta)\,. \tag{3.5}$$

The functions used in this equation are the following

$$M(\beta) = q^{-\frac{c-1}{24}}\bar{q}^{-\frac{\bar{c}-1}{24}}|\eta(\tau)|^{-2} = e^{-\beta\left(E_0+\frac{1}{12}\right)}\left|\eta\left(\frac{i\beta}{2\pi}\right)\right|^{-2}$$

$$B(\beta) = M(\beta)(1-q)(1-\bar{q}) = M(\beta)\left(1-e^{-\beta}\right)^2$$

$$Y(\beta) = \sum_A q^{h_A}\bar{q}^{\bar{h}_A} = \sum_A e^{-\beta\Delta_A}$$

with $E_0 = -\frac{c_{total}}{24}$ being the energy of the vacuum and $\Delta_A = h_A + \bar{h}_A$ the scaling dimension of the primary operator $\mathcal{O}_A$.

3.2.2 Derivation of the bound

Now that we have found the expression (3.5) for the full partition function in the previous section, let us discuss quickly the strategy that we will follow to derive a bound in this section. We will use this expression along with the constraints (3.1) to obtain identities involving the primary partition functions evaluated at $\tau = i$ that are consequences of modular invariance. We will show that the equalities for $N = 1, 3$ are inconsistent if the dimension of the lightest primary operator other than the vacuum is too big and this will give an upper bound on that dimension. We will actually derive an implicit expression for the bound and just cite the explicit solution.

The first step that we will need to do is compute explicitly the required derivatives for the terms $M(\beta)Y(\beta)$ and $B(\beta)$ and evaluate them at $\beta = 2\pi$ to apply the constraints (3.1) with $N = 1$ and $N = 3$. The results are the following

$$\left(\beta\frac{\partial}{\partial\beta}\right)^p M(\beta)Y(\beta)\Big|_{\beta=2\pi} = (-1)^p M(2\pi)\sum_A f_p\left(\Delta_A + E_0 + \frac{1}{12}\right)e^{-2\pi\Delta_A}$$

$$\left(\beta\frac{\partial}{\partial\beta}\right)^p B(\beta)\Big|_{\beta=2\pi} = (-1)^p M(2\pi)\left[f_p\left(E_0+\frac{1}{12}\right) - 2e^{-2\pi}f_p\left(E_0+\frac{13}{12}\right) + e^{-4\pi}f_p\left(E_0+\frac{25}{12}\right)\right]$$

with the functions f_p defined by

$$f_1(z) = 2\pi z - \frac{1}{2}$$

$$f_3(z) = (2\pi z)^3 - \frac{9}{2}(2\pi z)^2 + \left(\frac{41}{8} + 6r_{20}\right)(2\pi z) - \left(\frac{17}{16} + 3r_{20}\right)$$

and the number r_{20} defined by

$$r_{20} \equiv \frac{\eta''(i)}{\eta(i)} = \left.\frac{\mathrm{d}\log(\eta(\tau))}{\mathrm{d}\tau}\right|_{\tau=i} - \left(\frac{\eta'(i)}{\eta(i)}\right)^2 \approx 0.0120528\,.$$

In deriving these, we used the properties (A.7) and (A.8) of the η function, that are derived in Appendix A

$$\eta'(i) = \frac{i}{4}\eta(i)$$
$$\eta'''(i) = \frac{15i}{32}\left(\eta(i) + 8\eta''(i)\right)\,.$$

We define $\hat{E}_0 = E_0 + \frac{1}{12} = \frac{2-c_{total}}{24}$ for simplicity and we can now combine the results together to get the actual constraints from $p = 1$ and $p = 3$:

$$\sum_A f_1(\Delta_A + \hat{E}_0)e^{-2\pi\Delta_A} = -b_1(\hat{E}_0) \tag{3.6}$$

$$\sum_A f_3(\Delta_A + \hat{E}_0)e^{-2\pi\Delta_A} = -b_3(\hat{E}_0) \tag{3.7}$$

with

$$b_p(x) = f_p(x) - 2e^{-2\pi}f_p(x+1) + e^{-4\pi}f_p(x+2)\,.$$

Now we will combine these two constraints to obtain our result. To that end we define the following ratios:

$$I_{31}(x) \equiv \frac{f_3(x)}{f_1(x)}$$
$$K_{31}(x) \equiv \frac{b_3(x)}{b_1(x)}$$

such that we can divide the constraint (3.7) by (3.6) and put everything on the left-hand side to obtain

$$\frac{\sum_A [I_{31}(\Delta_A + \hat{E}_0) - K_{31}(\hat{E}_0)]f_1(\Delta_A + \hat{E}_0)e^{-2\pi\Delta_A}}{\sum_B f_1(\Delta_B + \hat{E}_0)e^{-2\pi\Delta_B}} = 0\,. \tag{3.8}$$

At this point we assume that the scaling dimensions of the primary operators in the CFT are organized in order to have $0 = \Delta_0 < \Delta_1 \leq \Delta_2 \leq \ldots$ and we will show an upper bound

of the form $\Delta_1 \leq \Delta_+$. The specific value for the upper bound Δ_+ is defined to be the biggest real zero of the cubic polynomial $I_{31}(\Delta + \hat{E}_0) - K_{31}(\hat{E}_0)$. Equivalently it's defined as the biggest solution of the equation

$$f_3(\Delta + \hat{E}_0) - K_{31}(\hat{E}_0) f_1(\Delta + \hat{E}_0) = 0\,.$$

If Δ_1 was larger than Δ_+, by definitions of Δ_+ as the biggest root and Δ_1 as the smallest dimension it would mean that every factor of $I_{31}(\Delta_A + \hat{E}_0) - K_{31}(\hat{E}_0)$ in each term in the numerator of (3.8) would have the same sign. We will also soon see that $f_1(\Delta_+ + \hat{E}_0) > 0$ and since $f_1(z)$ is monotonically increasing we can see that every term in the numerator would be the same sign and the whole sum could not possibly be zero. The denominator is a positive number for the same reason and combining the results for the numerator and the denominator leads us to conclude that if Δ_1 was larger than Δ_+ the left-hand side of (3.8) would never be zero and there would be an inconsistency with modular invariance. This proves that there must be a primary operator with the scaling dimension $\Delta \leq \Delta_+$ for the CFT to be consistent with invariance under the S transformation.

In the appendices of his paper, Hellerman shows that the upper bound $\Delta_+(\hat{E}_0)$ is a smooth function of c_{total} by studying its defining equation. Next, he derives a value for the bound in the large central charge limit and then shows that this bound has a maximum value. He also shows that $\Delta_+ > \frac{1}{4\pi} - \hat{E}_0$ so that we can now see easily the fact that $f_1(\Delta_+ + \hat{E}_0) > 0$. The details of all those calculations are not very illuminating, but the overall result is pretty important. The conclusion is that for any c_{total} in the range $[2, \infty)$ the bounding value for Δ_1 is

$$\Delta_+ \leq \frac{c_{total}}{12} + 0.473695\,. \tag{3.9}$$

This is the final result that we wanted to obtain in this section.

Other people actually continued to use the idea of this section to improve the numerical value of the bound that we obtained. Indeed the authors of [16] used the constraints with higher order in derivatives that Hellerman found and somewhat improved his bound using numerical tools. Remember that here we used only the first and third derivatives and using higher odd derivatives was possibly a promising way of getting a better result.

In [17], the authors used the same constraints as Hellerman but separated the terms differently in (3.8) to obtain bounds on the dimension of the next operators. They found

results like $\Delta_2 \leq \frac{c_{total}}{12} + 0.5338$ and $\Delta_3 \leq \frac{c_{total}}{12} + 0.8795$. In the asymptotic limit of large c_{total} and for $n \lesssim e^{\frac{\pi c_{total}}{12}}$, they found that in general $\Delta_n \leq \frac{c_{total}}{12} + \mathcal{O}(1)$.

The work in [18] has shown that Hellerman's bound is still valid even when there are chiral extensions. This comes from a simple generalization of the argument that we presented here where they included the appropriate characters in the partition function (3.4) and used exactly the same calculations to show that the bound is the same. Of course the existence of a chiral operator, for example a conserved current, adds more symmetries to the theory, but the bound is still in terms of representations of the Virasoro algebra and not of the full symmetry algebra.

3.3 Qualls' Bound

In the previous section, we derived an upper bound on the scaling dimension of the lightest primary field in any unitary CFT with $c, \bar{c} > 1$ by using invariance of the partition function under the S transformation. In this section, we will use the ST transformation along with the S transformation to improve significantly that bound, but uniquely for CFTs that contain only even spin primaries. This is a result by Qualls and we will follow closely the derivation of his paper [19].

Let's consider once more exactly the same setup as in Section 3.2, so a unitary 2d CFT on a circle. The partition function is again given by (2.39) and must be invariant under modular transformations. This time we will study the consequences of invariance under the ST transformation $\tau \to \frac{-1}{\tau+1}$, which as a fixed point at $\tau = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \equiv w$. Since we only want one constraint from this and not all the possible ones we will study modular invariance for complex parameters close to the fixed point by writing $\tau = we^s \approx w(1+s)$. For a small parameter s, which means close to the fixed point, the ST transformation becomes $s \to w^2 s$. We can see this by actually doing the transformation.

$$we^s \to \frac{-1}{we^s + 1} \approx \frac{-1}{w + ws + 1} = \frac{-1}{(w+1)(1 + \frac{ws}{w+1})} = \frac{w}{1 - sw^2} \approx w(1 + sw^2) \approx we^{sw^2}$$

All of this is to first order in s, but it is sufficient since we will get constraints evaluated at $s = 0$. We used the defining property of the fixed point $w = \frac{-1}{w+1}$ to do this calculation. In that language, invariance of the partition function under the ST transformation becomes

$$Z(we^s, \bar{w}e^{\bar{s}}) = Z(we^{w^2 s}, \bar{w}e^{\bar{w}^2 \bar{s}}) \,.$$

Taking derivatives of this equation leads us to the constraints

$$\left(\frac{\partial}{\partial s}\right)^{N_R}\left(\frac{\partial}{\partial \bar{s}}\right)^{N_L} Z(we^{s},\bar{w}e^{\bar{s}})\Big|_{s=0} = 0 \text{ for } N_L \neq N_R \,(\text{mod } 3)\,.$$

The weird looking restriction on the number of derivatives comes from the fact that $w\bar{w} = 1$ and $w^6 = \bar{w}^6 = 1$. Since each derivative with respect to s / $\bar{s}$ brings down a factor of w^2 / $\bar{w}^2$, the equation that we obtain when $N_L \neq N_R$ (mod 3) are of the form $X = aX$ so that we need $X = 0$. However when $N_L = N_R$ (mod 3), the properties stated above make the equation become simply $X = X$, which doesn't give anything special. We can next rewrite the derivatives in terms of the complex parameter τ as

$$\left(\tau\frac{\partial}{\partial \tau}\right)^{N_R}\left(\bar{\tau}\frac{\partial}{\partial \bar{\tau}}\right)^{N_L} Z(\tau,\bar{\tau})\Big|_{\tau=w} = 0 \text{ for } N_L \neq N_R \,(\text{mod } 3)\,.$$

The actual constraints on the partition function that would come from applying these equations would involve sums of complex numbers and it would be hard to extract any information on the spectrum from them. That's why we will do like in Section 3.2 and find an equation involving derivatives with respect to β instead of τ. However we need to keep the potential K in the expression $\tau = \frac{K+i\beta}{2\pi}$ since the fixed point has an imaginary part this time. We can very easily combine the constraints with $N_L = 0$, $N_R = 1$ and $N_L = 1$, $N_R = 0$ together to find

$$\left(\beta\frac{\partial}{\partial \beta}\right) Z(\beta,K)\Big|_{\tau=w} = 0\,. \tag{3.10}$$

The result will come from applying the procedure of Section 3.2 with this constraint and Hellerman's one for $p = 1$

$$\left(\beta\frac{\partial}{\partial \beta}\right) Z(\beta,K)\Big|_{\tau=i} = 0\,.$$

The next step is to obtain an expression for the partition function in order to compute (3.10). To do this we start with the general expression (3.4) and write it in terms of β and K by using $\tau = \frac{K+i\beta}{2\pi}$. In terms of these variables we have $q = e^{iK}e^{-\beta}$ and $\bar{q} = e^{-iK}e^{-\beta}$. The partition function then looks like

$$Z(\beta,K) = \tilde{M}(\beta,K)\tilde{Y}(\beta,K) + \tilde{B}(\beta,K) \tag{3.11}$$

with

$$\tilde{M}(\beta,K) = q^{-\frac{c-1}{24}}\bar{q}^{-\frac{\bar{c}-1}{24}}|\eta(\tau)|^{-2} = e^{-\beta\hat{E}_0}e^{iK\Delta c}\left|\eta\left(\frac{K+i\beta}{2\pi}\right)\right|^{-2}$$

$$\tilde{B}(\beta,K) = \tilde{M}(\beta,K)(1-q)(1-\bar{q}) = \tilde{M}(\beta,K)\left(1-e^{-\beta}e^{iK}\right)\left(1-e^{-\beta}e^{-iK}\right)$$

$$\tilde{Y}(\beta,K) = \sum_A q^{h_A}\bar{q}^{\bar{h}_A} = \sum_A e^{-\beta\Delta_A}e^{iKJ_A}\,.$$

The quantities $\hat{E}_0$ and Δ_A are defined as before while $J_A = h_A - \bar{h}_A$ is the conformal spin of the primary operator $\mathcal{O}_A$ and $\Delta c = -\frac{c-\bar{c}}{24}$. We can now compute the appropriate derivative of each term in (3.11) to get

$$\left(\beta\frac{\partial}{\partial\beta}\right)\tilde{M}(\beta,K)\tilde{Y}(\beta,K)\Big|_{\beta=\pi\sqrt{3},\,K=-\pi} = -\tilde{M}(w)\sum_A e^{-\pi\sqrt{3}\Delta_A}e^{-i\pi J_A}g(\Delta_A+\hat{E}_0) \tag{3.12}$$

$$\left(\beta\frac{\partial}{\partial\beta}\right)\tilde{B}(\beta,K)\Big|_{\beta=\pi\sqrt{3},\,K=-\pi} = -\tilde{M}(w)d(\hat{E}_0)\,. \tag{3.13}$$

The functions $g(z)$ and $d(z)$ are polynomials that are defined by

$$g(z) = \pi\sqrt{3}z - \frac{1}{2}$$

$$d(z) = g(z)\left(1+e^{-\pi\sqrt{3}}\right)^2 + 2\pi\sqrt{3}e^{-\pi\sqrt{3}}\left(1+e^{-\pi\sqrt{3}}\right)\,.$$

In deriving these results we used the property of the eta function that is derived in Appendix A to relate its derivative to the function itself

$$\eta'(w) = \frac{i\sqrt{3}}{6}\eta(w)\,.$$

After obtaining the expressions for the derivative of the partition function we can assemble everything together to get a differential constraint that is a consequence of invariance under the ST transformation. Combining (3.12) and (3.13) into (3.10) leads to

$$\sum_A e^{-\pi\sqrt{3}\Delta_A}e^{-i\pi J_A}g(\Delta_A+\hat{E}_0) = -d(\hat{E}_0)\,. \tag{3.14}$$

This is the first order constraint due to ST invariance and we will consider its consistency with the first order constraint due to S invariance, in the form of (3.6). However this is the point where we will need to make a major assumption about the properties of the

CFT that we are studying. If we want to reproduce the same kind of argument that lead to Hellerman's bound, we will rely entirely on the positivity of each term in a constraint like (3.14). In this case the factors $e^{-i\pi J_A}$ will cause problems since the terms of the sum will have different signs depending on the spin J_A, which makes it impossible to use the argument. This is why we assume that the CFT has only even spin primary operators such that the factors of $e^{-i\pi J_A}$ are all equal to one. To support this reasoning remember that T invariance lead us to show in Section 3.2 that the spin of operators is an integer.

To obtain a bound on the dimension of the lightest primary operator in the CFT we will proceed as we did before to get (3.9), simply with different equations. Let's define $G_0(\hat{E}_0) \equiv \frac{b_1(\hat{E}_0)}{d(\hat{E}_0)}$ so that we divide (3.6) by (3.14) to find

$$\frac{\sum_A f_1(\Delta_A + \hat{E}_0)e^{-2\pi\Delta_A}}{\sum_B g(\Delta_B + \hat{E}_0)e^{-\pi\sqrt{3}\Delta_B}} = G_0(\hat{E}_0)\,.$$

If we gather everything on the left-hand side of the equality, this looks like

$$\frac{\sum_A \left[e^{-(2-\sqrt{3})\pi\Delta_A} f_1(\Delta_A + \hat{E}_0) - g(\Delta_A + \hat{E}_0)G_0(\hat{E}_0)\right] e^{-\pi\sqrt{3}\Delta_A}}{\sum_B g(\Delta_B + \hat{E}_0)e^{-\pi\sqrt{3}\Delta_B}} = 0\,.$$

Now we multiply everything by $e^{-(2-\sqrt{3})\pi\hat{E}_0}$ and define $G \equiv G_0 e^{-(2-\sqrt{3})\pi\hat{E}_0}$ to get the final expression for the constraint involving S and ST invariance

$$\frac{\sum_A P(\Delta_A)e^{-\pi\sqrt{3}\Delta_A}}{\sum_B g(\Delta_B + \hat{E}_0)e^{-\pi\sqrt{3}\Delta_B}} = 0 \tag{3.15}$$

where the function $P(\Delta_A)$ is defined as

$$P(\Delta_A) = e^{-\pi(2-\sqrt{3})(\hat{E}_0+\Delta_A)} f_1(\Delta_A + \hat{E}_0) - g(\Delta_A + \hat{E}_0)G(\hat{E}_0)\,.$$

We will proceed in proving the desired bound by contradiction, as we did in Section 3.2. Let's define the largest root of the function $P(\Delta_A)$ to be Δ^+ and the zero of $g(\Delta_A + \hat{E}_0)$ to be g^+. We assume that the scaling dimension of the lightest primary operator in the CFT satisfies $\Delta_1 > max(g^+, \Delta^+)$. All the scaling dimensions are real and it's easy to check with the expression for g that it's positive for arbitrarily large argument so that if $\Delta_1 > g^+$ (thus $\Delta_n > g^+$ for all n) the denominator of (3.15) is positive. It's also possible using the explicit expression to check that for $\Delta_1 > \Delta^+$ the function P is negative such that the whole sum in the numerator of (3.15) is negative. This means that under our assumption

it's impossible for the left-hand side of the constraint to be zero and the assumption must then be false for the theory to respect modular invariance. The conclusion is then that $\Delta_1 \leq max(g^+, \Delta^+)$.

To compute the explicit value for the bound we can first look at the expression for $g(\Delta + \hat{E}_0)$ and we find that

$$g^+ = \frac{c_{total}}{24} + \frac{\sqrt{3}}{6\pi} - \frac{1}{12}\,.$$

We can also study the expression for $P(\Delta)$ and find that for $c_{total} > 2.33544$

$$\Delta^+ < \frac{c_{total}}{24} + \frac{\sqrt{3}}{6\pi} - \frac{1}{12} + \frac{2-\sqrt{3}}{\pi(6G - 4\sqrt{3})}\,.$$

It's then pretty clear that the bound is $\Delta_1 < \Delta^+$. Qualls studied the value of Δ^+ numerically and found a better bound that is valid for all values of the central charge in the range $[2, \infty)$

$$\Delta_1 < \frac{c_{total}}{24} + 0.0927\,. \tag{3.16}$$

This is a bound on the lightest primary operator other than the identity in any unitary 2d CFT with $c, \bar{c} > 1$, with only even spin operators and with no extra chiral operator. It is nice to note that the scaling with the central charge is half the one in Hellerman's bound (3.9). This will be important in the next section.

3.4 Generalized Constraint

After deriving two different bounds by using modular invariance in the previous sections of this chapter we will now discuss a tentative way of getting a new one. Indeed there are some reasons to believe that it could be possible to improve Hellerman's bound by a factor of two such that basically something similar to Qualls' bound would be valid for every CFT, not just the ones that contain exclusively even spin primaries. In the first part of this section we will explain why we think there could exist a better universal bound and in the second part we will show a possible way of deriving it. We actually don't have explicit results yet but we will present the idea.

3.4.1 Motivation

The first reason to believe in the existence of an improved bound is actually Qualls' bound itself $\Delta_1 \lesssim \frac{c+\bar{c}}{24}$. The way we derived it, it's valid only for even spin CFTs, but this assumption was the simplest thing that could have been used to get a result. It's possible or not that dropping this assumption would still give the bound, but using a somehow different method. Of course this is very speculative, but there are more contexts in which that kind of bound has been derived. For example in [20] it was shown that CFTs that exhibit holomorphic factorization have their lightest primary field being either holomorphic or anti-holomorphic with dimension $\Delta_1 \leq 1+min\left(\frac{c}{24}, \frac{\bar{c}}{24}\right)$. Another example is in [21], where some (2,2) supersymmetric extensions are considered and it is shown that the lightest primary has again $\Delta \leq \frac{c}{24}$ for large c. Looking at these cases leads us to think that if they are typical examples of 2d CFTs it would be possible to show the same bound in general. It's of course important to note that many aspects of these examples are very atypical for a 2d CFT so that it may as well be possible that they are exceptions and that $\Delta \leq \frac{c_{total}}{12}$ is the best we can do universally. Finding an example of a theory that saturates Hellerman's bound would be a good way of confirming the non-existence of a better bound.

From the optimistic point of view, perhaps the best reason to expect a better bound is because of the AdS/CFT correspondence. The details will be explained in Chapter 5 but we just need to know that operators with $\Delta = \frac{c_{total}}{12}$ correspond in the gravity dual to black holes with mass $M = \frac{1}{4G_N}$. Knowing that black holes in AdS_3 exist for masses as low as $M = \frac{1}{8G_N}$, which correspond to $\Delta = \frac{c_{total}}{24}$, we expect that an improved bound would at least be valid in the semi-classical limit $c_{total} \to \infty$ and could possibly be extendable to any values of the central charge.

3.4.2 Using the full modular group

Starting with the assumption that Hellerman didn't derive the most optimal bound but that he used the right tools, we are lead to believe that one possible problem with his derivation is that he only used a small set of the full modular group to build constraints. To account for that problem we will derive here a set of constraints coming from modular invariance, one for each element in the modular group. We will then explain how they could be used to find a new bound on operator dimensions.

Let's study the consequences of invariance of the partition function under a general element γ of the modular group acting as $\tau \to \frac{a\tau+b}{c\tau+d}$ with $ad-bc=1$. The partition function must satisfy $Z(\tau,\bar{\tau}) = Z(\gamma\tau,\bar{\gamma}\bar{\tau})$. We can study this constraint close to the fixed point w of the transformation γ by writing $\tau = w(1+s)$ and taking s small. After a sequence of first order approximations we can find how this point transforms under γ

$$\tau = w + ws \to \gamma\tau = \gamma w(1+s) = \frac{aw+aws+b}{cw+cws+d} \approx w + \frac{ws}{(cw+d)^2}\,.$$

To derive this we used the fact that $w = \frac{aw+b}{cw+d}$. If we use this in the invariance of the partition function and we Taylor expand for small s we find

$$Z(w+ws,\bar{w}+\bar{w}\bar{s}) \approx Z(w,\bar{w}) + wsZ_\tau(w,\bar{w}) + \bar{w}\bar{s}Z_{\bar{\tau}}(w,\bar{w}) =$$
$$Z\left(w+\frac{ws}{(cw+d)^2},\bar{w}+\frac{\bar{w}\bar{s}}{(c\bar{w}+d)^2}\right) \approx Z(w,\bar{w}) + \frac{ws}{(cw+d)^2}Z_\tau(w,\bar{w}) + \frac{\bar{w}\bar{s}}{(c\bar{w}+d)^2}Z_{\bar{\tau}}(w,\bar{w})\,.$$

For the first order terms to be equal we need either $cw+d = c\bar{w}+d = 1$ or

$$\left.\frac{\partial Z(\tau,\bar{\tau})}{\partial\tau}\right|_{\tau=w} = \left.\frac{\partial Z(\tau,\bar{\tau})}{\partial\bar{\tau}}\right|_{\tau=w} = 0\,.$$

The condition $cw+d = c\bar{w}+d = 1$ is only satisfied for w on the real axis so we will have to forget about these transformations. For the others we can combine the two independent equations involving complex derivatives into one that contains a single real derivative like in Section 3.3

$$\left.\frac{\partial Z(\beta,K)}{\partial\beta}\right|_{\tau=w} = 0\,. \tag{3.17}$$

We can also use the τ and $\bar{\tau}$ derivatives to get

$$\left.\frac{\partial Z(\beta,K)}{\partial K}\right|_{\tau=w} = 0\,. \tag{3.18}$$

To do anything useful with these equations we will again need to use the expression (3.11) for the partition function. We take the derivative with respect to β and evaluate it at

the fixed point $\tau = w = \frac{K_0 + i\beta_0}{2\pi}$ (with $q_0 = e^{2\pi i w}$) such that (3.17) gives

$$\sum_A e^{-\beta_0 \Delta_A + i K_0 J_A} \left(\hat{E}_0 + \Delta_A + \frac{i}{2\pi} \left[\frac{\eta'(w)}{\eta(w)} + \frac{\eta'(\bar{w})}{\eta(\bar{w})} \right] \right) = \\ q_0 + \bar{q}_0 - 2q_0\bar{q}_0 - \left(\hat{E}_0 + \frac{i}{2\pi} \left[\frac{\eta'(w)}{\eta(w)} + \frac{\eta'(\bar{w})}{\eta(\bar{w})} \right] \right) (1 - q_0)(1 - \bar{q}_0) \,. \tag{3.19}$$

We do the same thing with the derivative with respect to K and (3.18) gives

$$\sum_A e^{-\beta_0 \Delta_A + i K_0 J_A} \left(\Delta c + J_A + \frac{i}{2\pi} \left[\frac{\eta'(w)}{\eta(w)} + \frac{\eta'(\bar{w})}{\eta(\bar{w})} \right] \right) = \\ q_0 - \bar{q}_0 - \left(\Delta c + \frac{i}{2\pi} \left[\frac{\eta'(w)}{\eta(w)} + \frac{\eta'(\bar{w})}{\eta(\bar{w})} \right] \right) (1 - q_0)(1 - \bar{q}_0) \,. \tag{3.20}$$

To get these written that way we used the simple fact that $\bar{\eta}(\bar{\tau}) = \eta(-\bar{\tau})$.

Now if we look carefully at the constraints (3.19) and (3.20) we see that we have the same problem as we originally had with (3.14) since they contain complex exponentials that alternate and spoil the positivity of the whole sum on the left-hand side. We will suggest a way of canceling these oscillations other than by assuming that every primary has an even spin. Start by considering a specific modular transformation X with fixed point τ_0. For example pick the S transformation with $\tau_0 = i$ since it's the simplest one. For any transformation γ the point γi will be a fixed point of the transformation $\gamma S \gamma^{-1}$ such that we can apply our generalized constraints to this new fixed point. No modular transformation can take the point i to the real axis so the constraints are always valid. Now consider the infinite number of constraints coming from applying (3.19) and (3.20) to the points $w = \gamma i = \frac{ai+b}{ci+d} = \frac{ac+bd}{c^2+d^2} + \frac{i}{c^2+d^2}$ for any possible $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$. Most of them will still have the problem of the complex exponential, but there might be a way of averaging all the constraints coming from (3.19) together and all the ones from (3.20) such that the phases get averaged away. This kind of sum over the whole modular group, known as Poincaré sums, has already been studied in [22] and we would like in the future to use the tools developed there to construct the appropriate average of our generalized constraints. These combined equations could then be used in exactly the same way as how we derived Hellerman's and Qualls' bounds to find a constraint on the spectrum of generic 2d CFTs.

The results that we presented in this chapter are all very interesting by themselves but they are even more so in the context of the AdS/CFT correspondence. We will discuss this duality in Chapter 5 and will interpret the results of this chapter in gravity.

Chapter 4

Constraints on the Spectrum of $\mathcal{W}$-algebras

In the previous chapter we discussed a way of putting constraints on the spectrum of general conformal field theories in 2d by using modular invariance. There however exist CFTs that possess more symmetries than just the Virasoro algebra and it would be interesting to understand how these extra symmetries put constraints on the allowed theories. Algebras that contain conserved currents or supersymmetric extensions have been studied a lot but we will focus here on theories that contain conserved operators with spin higher than two. Such objects are called $\mathcal{W}$-algebras and we will study conformal field theories that have different types of $\mathcal{W}$-algebras as symmetries. In this chapter we will first discuss what $\mathcal{W}$-algebras are and why we care about them. Then we will show quickly some examples where strong constraints have already been found. Finally we will derive the $\mathcal{W}(2,3)$ and $\mathcal{W}(2,4)$ algebras and describe a way of constraining their spectra. $\mathcal{W}$-algebras form actually a rich and complicated subject in mathematics but we will discuss it in the context of physics, more specifically conformal field theory, and will work with the simplest formalism. A great review on the subject in the context of physics is [23]. It contains much more than what we will cover here but has most of what we need.

As a warning, a lot of the computations in this section were made using Mathematica. In other words, when results seem to come out of nowhere and we skip many steps, it's because Mathematica helped us get an answer. For example all the coefficients $p_{ijk}(m,n)$ were computed using Mathematica and the calculations involving simplifications of operator expressions by using the algebra were done with a modified version of a code developed by Matthew Headrick. On this line of thought we would like to thank Headrick for making available this Mathematica notebook on his personal webpage [24].

4.1 Rational CFTs with $\mathcal{W}$ Symmetry

Conformal symmetry in two dimensions is extremely constraining, as can be seen from the infinite number of symmetry generators in the Virasoro algebra. This even allows one to solve exactly conformal field theories when the central charge is less than one. Indeed the minimal models discussed in Section 2.3 are examples of how powerful conformal symmetry is. Someone could wonder if it's possible to have a theory that exhibits even more symmetry and Zamolodchikov was the first one to find such a theory [25]. His symmetry algebra is the simplest example of what we now call $\mathcal{W}$-algebras. He found an extension to the Virasoro algebra where there is also a conserved chiral operator of spin 3 and derived the OPEs involving that field, which can easily give the algebra from what we discussed in Chapter 2. To set the notation straight for the rest of the discussion, Zamolodchikov's theory will be called $\mathcal{W}_3$ since it contains conserved operators with spins up to 3 (starting at 2). With the same reasoning $\mathcal{W}_N$ contains spins $2, 3, ..., N$. We will however discuss some theories which don't have all of the spins up to N so we will have to specify directly every spin present by writing for example $\mathcal{W}(2, 4)$. Many $\mathcal{W}$-algebras have been built throughout the years and have found applications. The major reasons why people care about extended symmetry algebras are because they are useful in applications of CFT (like string theory and statistical mechanics) and are helpful in the classification of rational CFTs (RCFTs). Indeed the standard extension of the Virasoro algebra that is used in string theory is supersymmetry and there are some models that try to incorporate also $\mathcal{W}$-symmetry. Some lattice models in statistical mechanics also exhibit some kind of $\mathcal{W}$-symmetry in their scaling limit at criticality, for example the $\mathbb{Z}_N$ symmetric lattice model. There are some applications in perturbative CFTs as well and in the coupling of a CFT to 2d gravity. On the other hand the presence of extra symmetries allows for more constraints on the spectrum of operators in a CFT and can lead to the discovery of new rational theories, which can be associated with actual models. Without the addition of these symmetries, RCFTs can only be found for $c < 1$, as discussed in Section 2.3. Another indirectly related reason to study these theories is the AdS/CFT correspondence. We will not discuss it in this chapter but will briefly discuss it in the next one.

At this point we know why people study theories with $\mathcal{W}$-symmetry so we should now describe what they actually are. We will show how to build the actual algebras later when we will discuss specific examples but we can already assume that the algebras exist and explain how to build the Hilbert space of a CFT that has $\mathcal{W}$-symmetry. The discussion

of the representations and the RCFTs is based on [26]. We already know that the (holomorphic part of the) energy-momentum tensor is the conserved operator that leads to the Virasoro algebra when we expand it in Laurent modes. It's then natural to expect that an extended symmetry algebra would be built out of the Laurent modes of a higher spin conserved operator, that we will call $W(z)$. The generators that will be part of the algebra are then the modes W_n defined by the usual expansion

$$W(z) = \sum_{n\in\mathbb{Z}} z^{-n-h} W_n \,.$$

The Hilbert space of the theory is then a representation of the symmetry algebra that is built in a similar way to what was explained in Section 2.3 for the Virasoro algebra. Let's discuss the case where there is only one additional conserved operator and the generalization will be clear. We start with a primary state that is defined by being annihilated by all the L_n with positive n and also by all the positive W_n. This is a more constraining definition than in the Virasoro case so there are less primaries. Since $W(z)$ is conserved there is a conserved charge associated to it that we will call w. A primary state is then labeled by its energy h and its charge w. It means that a primary is an eigenstate of L_0 with eigenvalue h and of W_0 with eigenvalue w. This implies that the Verma modules of a $\mathcal{W}$-algebra are specified by 3 numbers, $V(h, w, c)$. The weight h must be positive for unitarity but at this point there is no restriction on w other than the fact that it is real since W_0 is hermitian. After having defined the primary states, the descendants associated with them are built by applying the negative modes of $T(z)$ and $W(z)$. Like before any mode L_{-n} and W_{-n} increase the L_0 eigenvalue of a state by n and descendants are grouped in levels with the same weight and that are orthogonal to each other. However descendants are not necessarily eigenstates of W_0 anymore so we can't define clearly their charge. It would in principle be possible to consider linear combinations of descendants that have a definite weight and charge since L_0 and W_0 commute but we will not bother doing this. In summary highest weight representations of a $\mathcal{W}$-algebra are built by applying L_{-n} and W_{-n} on a primary state $|h, w\rangle$ that is defined by

$$\begin{aligned} &L_n\,|h,w\rangle = W_n\,|h,w\rangle = 0 \text{ for } n>0 \\ &L_0\,|h,w\rangle = h\,|h,w\rangle \text{ and } W_0\,|h,w\rangle = w\,|h,w\rangle \ . \end{aligned} \tag{4.1}$$

Of course if one doesn't notice at first that there is this extra symmetry in a theory and builds the Hilbert space just from the Virasoro algebra, the resulting states will be the

same since it's the same theory. However there will turn out to be many similar properties between states in different Verma modules that seemingly can't be explained. On the other hand if we consider the classification of the Hilbert space into representations of the extended algebra then the accidents are simply explained by the fact that some Virasoro primary states are actually not primary with respect to the full symmetry algebra and are thus supposed to be descendants of another true primary. In other words some Virasoro primary state is annihilated by all the positive L_n but not necessarily by the positive W_n so it and all of its Virasoro descendants are actually descendants of a primary that is annihilated by all the right operators. This fact was not noticed at first because they can be built from the primary by acting with negative W_n, which are not used in Virasoro representations.

Now that we know how to build representations, we will discuss a little bit some work that has already been done in constraining the allowed ones. Since one of the main interests of studying $\mathcal{W}$-algebras is to find rational CFTs, most of the results available are example of that type of theory. I will present here some of the possible cases. To find some of these results, people have simply found by constructing explicitly the algebras that only some values of the central charge are allowed to satisfy the consistency conditions that we will introduce later and have studied these specific models directly. Of course some other RCFTs are found by studying more carefully the representations, but actually in simple ways. Often the central charges found correspond to already known minimal models so we will not mention them. A new example is $\mathcal{W}(2,6)$ with $c=-47$, which has only 7 allowed primaries, with only 2 of them that have a non zero charge. Another one is $\mathcal{W}(2,7)$, that is consistent only for $c=-\frac{25}{2}$ and in that case the only primaries allowed have

$$w^2=-\frac{(2h+1)^2(16h+5)^2(16h+9)h^2}{4167450}\,.$$

There is not a finite number of primaries allowed but it is still a big constraint. The last example that we will give contains a fermionic operator. Indeed the Ramond sector of $\mathcal{W}(2,\frac{9}{2})$ with $c=-35$ has only 5 primaries. As can be seen most of the models found have a negative central charge so are not unitary.

Even if many $\mathcal{W}$-algebras have already been constrained to find rational CFTs, there are others that have not been studied a lot and this is what we will do for $\mathcal{W}(2,3)$ and $\mathcal{W}(2,4)$ next.

4.2 $\mathcal{W}(2,3)$ Algebra

In this section we will first describe the general idea that we will use to build $\mathcal{W}$-algebras and we will then apply it to derive $\mathcal{W}(2,3)$ and $\mathcal{W}(2,4)$. Following that we will find constraints that unitary CFTs with these symmetry algebras have to satisfy by studying the Gram matrix at the first few levels. The technique that will be used here to find the algebras is based on [27] and is called the Lie Bracket approach. We will use only one of the many possible ways of building the algebra. In some sense it is more complicated than other ways since we have to compute more structure constants and we have to use a complicated notion of normal ordering, but it is also simpler because we don't have to compute coefficients of derivative fields and there is only a few Jacobi identities to satisfy. Other methods are discussed in [23].

The strategy that we will use to build $\mathcal{W}$-algebras is mainly based on the expression (2.29) that we derived earlier for the commutation relations of modes of quasi-primary fields. Let's recall the formula since it will be very important:

$$\left[\phi_{(i)m},\phi_{(j)n}\right]=\sum_k C_{ij}^k p_{ijk}(m,n)\phi_{(k)m+n}+d_{ij}\delta_{m,-n}\binom{m+h_i-1}{2h_i-1}$$

with

$$p_{ijk}(m,n)=\sum_{\substack{r,s\in\mathbb{Z}_0^+\\ r+s=h_i+h_j-h_k-1}} C_{r,s}^{ijk}\binom{h_i-m-1}{r}\binom{h_j-n-1}{s}$$

where

$$C_{r,s}^{ijk}=\frac{(-1)^r(2h_k-1)!}{(h_i+h_j+h_k-2)!}\prod_{t=0}^{s-1}(2h_i-2-r-t)\prod_{u=0}^{r-1}(2h_j-2-s-u)\,.$$

In this expression $\phi_{(i)}$ and $\phi_{(j)}$ are quasi-primary fields of weights h_i and h_j and $\phi_{(k)}$ are other quasi-primary fields in the theory. This will include simple fields, but also derivatives and normal ordered products of fields that are combined to create primaries, as discussed in Chapter 2. Remember that due to the form of the coefficients $p_{ijk}(m,n)$ only fields with $h_k<h_i+h_j$ appear in the commutators. Given this formula, the strategy will be to

1. Decide the operator content of the algebra that we want to derive. In other words we need to decide what higher spin chiral primary operators we want to include in the commutation relations. This basically amounts to selecting the weight that we want for the operators. Of course the energy-momentum tensor must always be there.

2. Making use of the fact that we know the weights of all operators in the algebra we find every term allowed in (2.29), given that a symmetry algebra must close so we can only use the fields defined in the first part and their products. The authors have general formulas for the quasi-primary normal ordered products and recursive relations to compute some coefficients in the algebra but we will work more explicitly since we study simple cases.

3. Calculate the structure constants for the fields that are needed to complete the commutators that we found by using the general expression. We can use the formula in [27] or exploit the fact that they are related to correlators, like we will do later. In fact the constants involving only simple fields will not be determined that way.

4. To compute the last coefficients and to insure the validity of the Jacobi identity (or the associativity of the OPE) for the algebra we will compute specific cases of the identity. Actually it is possible to show that the only cases that are needed are the ones that include only simple fields other than the energy-momentum tensor so that the work is simplified a lot.

After following this procedure the algebra is completely fixed and is even possibly accompanied by come constraints that the structure constants need to satisfy.

4.2.1 Derivation of the algebra

Since we have all the tools to build any $\mathcal{W}$-algebra that we want, let's work out the simplest one, which includes only a weight 3 primary field $W(z)$ in addition to the energy-momentum tensor. The full algebra has to contain the Virasoro algebra (2.11) of course. It also contains the consequence of the fact that $W(z)$ is a primary on its modes, which comes directly from (2.25)

$$[L_m, W_n] = (2m - n)W_{m+n}\,. \tag{4.2}$$

The only operators that can appear in $[W_m, W_n]$ have weight smaller than 6 from what we already saw. Also, notice that the operators that appear in the commutator are the same that are in the OPE, but as for the TT OPE (2.23) the WW OPE has to be symmetric and thus cannot contain operators of odd weight since they would lead to odd powers of the coordinates. This can also be seen from the expression for the coefficients $p_{ijk}(m, n)$, that must be anti-symmetric under $m \leftrightarrow n$ just like the commutator. After all we conclude

that only operators of weight 2 and 4 can appear in the commutator of W with itself. In this context these operators are T and $\Lambda \equiv \mathcal{N}(TT)$. The commutator then looks like

$$[W_m, W_n] = d_{WW}\delta_{m,-n}\binom{m+2}{5} + C^T_{WW}p_{332}(m,n)L_{m+n} + C^\Lambda_{WW}p_{334}(m,n)\Lambda_{m+n}\,. \tag{4.3}$$

Note that we are free to choose the normalization of the spin 3 operator as we want and we choose it in such a way that $d_{WW} = c/3$, just like for the energy-momentum tensor which has $d_{TT} = c/2$. Also d_{TW} is basically the two-point function $\langle TW\rangle$ but the operators have different weights so it vanishes. The coefficients $p_{ijk}(m,n)$ can be computed directly with their defining formula (2.29) and the results are:

$$p_{332}(m,n) = \frac{(m-n)}{2}\left[\frac{(m+n+3)(m+n+2)}{15} - \frac{(m+2)(n+2)}{6}\right]$$

$$p_{334}(m,n) = \frac{m-n}{2}\,.$$

Furthermore $\binom{m+2}{5} = \frac{m(m^2-1)(m^2-4)}{120}$ so the only elements left to compute are the constants C^T_{WW} and C^Λ_{WW}. Remember from (2.28) that $C_{ijk} = \sum_l C^l_{ij}d_{lk}$ and d_{ij} is diagonal so we can write

$$C^T_{WW} = C_{WWT}(d_{TT})^{-1} = C^W_{TW}d_{WW}(d_{TT})^{-1} = C^W_{TW}\cdot\frac{c}{3}\cdot\frac{2}{c} = \frac{2C^W_{TW}}{3}\,.$$

We will use a trick to get the value of C^W_{TW} instead of computing it directly. This coefficient should appear in the commutator $[L_m, W_n]$ in the term involving W and since we already know what this commutator is we can read off the value of the constant. The actual coefficient in front of W_{m+n} in (4.2) is $C^W_{TW}p_{233}(m,n) = 2m - n$ and we can easily find that $p_{233}(m,n) = \frac{2m-n}{3}$ so we must conclude that $C^W_{TW} = 3$, which leads to $C^T_{WW} = 2$.

The only structure constant left to compute is therefore $C^\Lambda_{WW} = C_{WW\Lambda}(d_{\Lambda\Lambda})^{-1}$ but to find it we first need to know what the operator Λ is. Remember that it is defined as the combination of normal ordered products and derivatives of $T(z)$ that forms a quasi-primary operator. Of course all the terms must have the same weight so the only possibility is (up to a normalization) $\Lambda = N(TT) + \alpha\partial^2 T$. To fix α we simply ask that the modes of the operator satisfy the condition (2.25) for $m = -1, 0, 1$. This straightforward calculation gives $\alpha = -\frac{3}{10}$. In order to do the calculation we need to know the mode expansion of Λ. The modes of the normal ordered product are given by the usual formula (2.33) and the

modes of the derivative of an operator $O(z)$ of weight h are found by writing

$$\begin{aligned}\partial^m O = \partial^m \left(\sum_n z^{-n-h} O_n\right) &= \sum_n (-1)^m (n+h)(n+h+1)...(n+h+(m-1)) z^{-n-h-m} O_n \\ = \sum_n z^{-n-h-m} (\partial^m O)_n &\Longrightarrow (\partial^m O)_n = (-1)^m (n+h)...(n+h+(m-1)) O_n\end{aligned} \tag{4.4}$$

The modes of the operator Λ are then

$$\Lambda_n = \sum_{k>-2} L_{n-k} L_k + \sum_{k\leq -2} L_k L_{n-k} - \frac{3}{10}(n+2)(n+3) L_n \,.$$

We now have everything we need to compute the constants $C_{WW\Lambda}$ and $d_{\Lambda\Lambda}$ that will allow us to find C^{Λ}_{WW}. We will however have to take a step aside because we don't really know good ways of computing these numbers. There exist simple techniques and we will derive them.

First of all, let's consider a quasi-primary field $\phi^{(i)}(z)$ with weight h_i of which we want to know the two-point function coefficient $d_{\phi\phi}$. The way to do it is to compute the norm of the state $|\phi\rangle = \phi_{-h} |0\rangle$ by using (2.29)

$$\langle 0| \phi^{(i)}_h \phi^{(i)}_{-h} |0\rangle = \langle 0| \left[\phi^{(i)}_h, \phi^{(i)}_{-h}\right] |0\rangle = \sum_k C^k_{ii} p_{h_i h_i h_k}(h,-h) \langle 0| \phi^{(k)}_0 |0\rangle + d_{ii} \binom{2h-1}{2h-1} = d_{\phi\phi} \,.$$

To get this result we just computed the coefficient $p_{iik}(h,-h)$ and found that it vanishes. Secondly, we can rewrite the usual expression for the 3-point function (2.14) as

$$C_{ijk} = z_{ij}^{h_i+h_j-h_k} z_{jk}^{h_j+h_k-h_i} z_{ik}^{h_i+h_k-h_j} \langle \phi_i(z_i)\phi_j(z_j)\phi_k(z_k)\rangle$$

where $z_{ij} = z_i - z_j$. We can take the limits $z_i \to \infty$ and $z_k \to 0$ while keeping z_j fixed and take advantage of the way that field modes act on the vacuum to get

$$\begin{aligned} C_{ijk} &= \lim_{z_i\to\infty} \lim_{z_k\to 0} z_{ij}^{h_i+h_j-h_k} z_{jk}^{h_j+h_k-h_i} z_{ik}^{h_i+h_k-h_j} \langle \phi_i(z_i)\phi_j(z_j)\phi_k(z_k)\rangle \\ &= \lim_{z_i\to\infty} \lim_{z_k\to 0} z_i^{2h_i} z_j^{h_j+h_k-h_i} \langle \phi_i(z_i)\phi_j(z_j)\phi_k(z_k)\rangle \\ &= z_j^{h_j+h_k-h_i} \left\langle \phi_{(i)h_i}\phi_j(z_j)\phi_{(k)-h_k}\right\rangle \,. \end{aligned}$$

At this point we notice that the left-hand side is supposed to be a constant so the right-hand side must also be one and the only contribution can come from the z_j^0 term so that

$$C_{ijk} = \left\langle \phi_{(i)h_i} \phi_{(j)h_k - h_i} \phi_{(k)-h_k} \right\rangle .$$

After all this we can conclude that the only thing left to do is compute two simple correlation functions, which can be done by using the Virasoro algebra and (4.2). The straightforward calculation gives

$$d_{\Lambda\Lambda} = \langle 0 | \, \Lambda_4 \Lambda_{-4} \, | 0 \rangle = \frac{c(5c+22)}{10}$$

$$C_{WW\Lambda} = \langle 0 | \, W_3 W_1 \Lambda_{-4} \, | 0 \rangle = \frac{16c}{5}$$

where we used $\Lambda_{-4} \, |0\rangle = \left(L_{-2} L_{-2} - \frac{3}{5} L_{-4} \right) |0\rangle$. Combining these two together gives

$$C_{WW}^{\Lambda} = C_{WW\Lambda} (d_{\Lambda\Lambda})^{-1} = \frac{32}{5c+22} \equiv 2\beta$$

and putting together all of what we have found in this section into (4.3) leads to the final result

$$\begin{aligned} [W_m, W_n] = (m-n) \left[\frac{(m+n+3)(m+n+2)}{15} - \frac{(m+2)(n+2)}{6} \right] L_{m+n} \\ + \beta (m-n) \Lambda_{m+n} + \frac{c}{36} m (m^2-1)(m^2-4) \delta_{m,-n} . \end{aligned} \tag{4.5}$$

To follow the last step in our strategy we would need to compute the Jacobi identity

$$[[W_q, W_r], W_s] + [[W_s, W_q], W_r] + [[W_r, W_s], W_q] = 0$$

but it actually turns out to be trivially satisfied so there are no further constraints on the coefficients of the algebra.

4.2.2 Constraints

In this section we will derive some constraints that the primary states in the $\mathcal{W}(2,3)$ algebra have to satisfy in order for the representations built out of them to be unitary. This is actually different from what was already done to find RCFTs since they were mostly found for non-unitary values of the central charge. The condition that we will study is

that there is not supposed to be negative norm states in a unitary quantum theory. The work that we will do is basically the same as the one that lead to the minimal models by using the Kac determinant, but for $\mathcal{W}$-algebras. The simple tool that we will compute is the Gram matrix and we will ask that its eigenvalues be positive. As previously explained, the representations are built out of primary states $|h, w\rangle$ introduced in (4.1) and we will study the eigenvalues of the Gram matrix as a function of h, w, c.

The Kac determinant of this theory has already been computed in [28] to find minimal models with central charge

$$c = 2\left(1 - \frac{12}{(k+3)(k+4)}\right)$$

for $k \geq 1$. However these models cover only the range $0 < c < 2$ and it would be nice to understand better higher central charges, even if we don't expect to find minimal models. We could use the expression of the determinant from the paper but we will do it more directly.

We will start by studying the Gram matrix at level one. There are only two states, that we will call $|+\rangle = L_{-1}|h, w\rangle$ and $|-\rangle = W_{-1}|h, w\rangle$. The matrix that we want is then

$$M_1(h, c, w) = \begin{pmatrix} \langle +|+\rangle & \langle +|-\rangle \\ \langle -|+\rangle & \langle -|-\rangle \end{pmatrix}.$$

The elements can be computed using the explicit commutators (2.11), (4.2) and (4.5) of the $\mathcal{W}(2, 3)$ algebra. Here are the calculations (we already know that $\langle +|+\rangle = 2h$ since it uses only Virasoro):

$$\langle -|+\rangle = \langle +|-\rangle = \langle h, w| W_1 L_{-1}|h, w\rangle = \langle h, w| L_{-1} W_1|h, w\rangle + 3\langle h, w| W_0|h, w\rangle = 3w$$

$$\begin{aligned}\langle -|-\rangle &= \langle h, w| W_1 W_{-1}|h, w\rangle = \langle h, w| W_{-1} W_1|h, w\rangle + \langle h, w|[W_1, W_{-1}]|h, w\rangle \\ &= \langle h, w|\left(-\frac{L_0}{5} + 2\beta\Lambda_0\right)|h, w\rangle = -\frac{h}{5} + 2\beta\langle h, w|\left(L_1 L_{-1} + L_0 L_0 - \frac{9}{5}L_0\right)|h, w\rangle \\ &= -\frac{h}{5} + 2\beta\left(2h + h^2 - \frac{9h}{5}\right) = \frac{h(32h + 2 - c)}{22 + 5c}.\end{aligned}$$

The eigenvalues of the matrix are

$$e_\pm = \frac{1}{2(22+5c)}\left[h(9c + 32h + 46) \pm \sqrt{h^2(11c - 32h + 42)^2 + 36w^2(5c + 22)^2}\right].$$

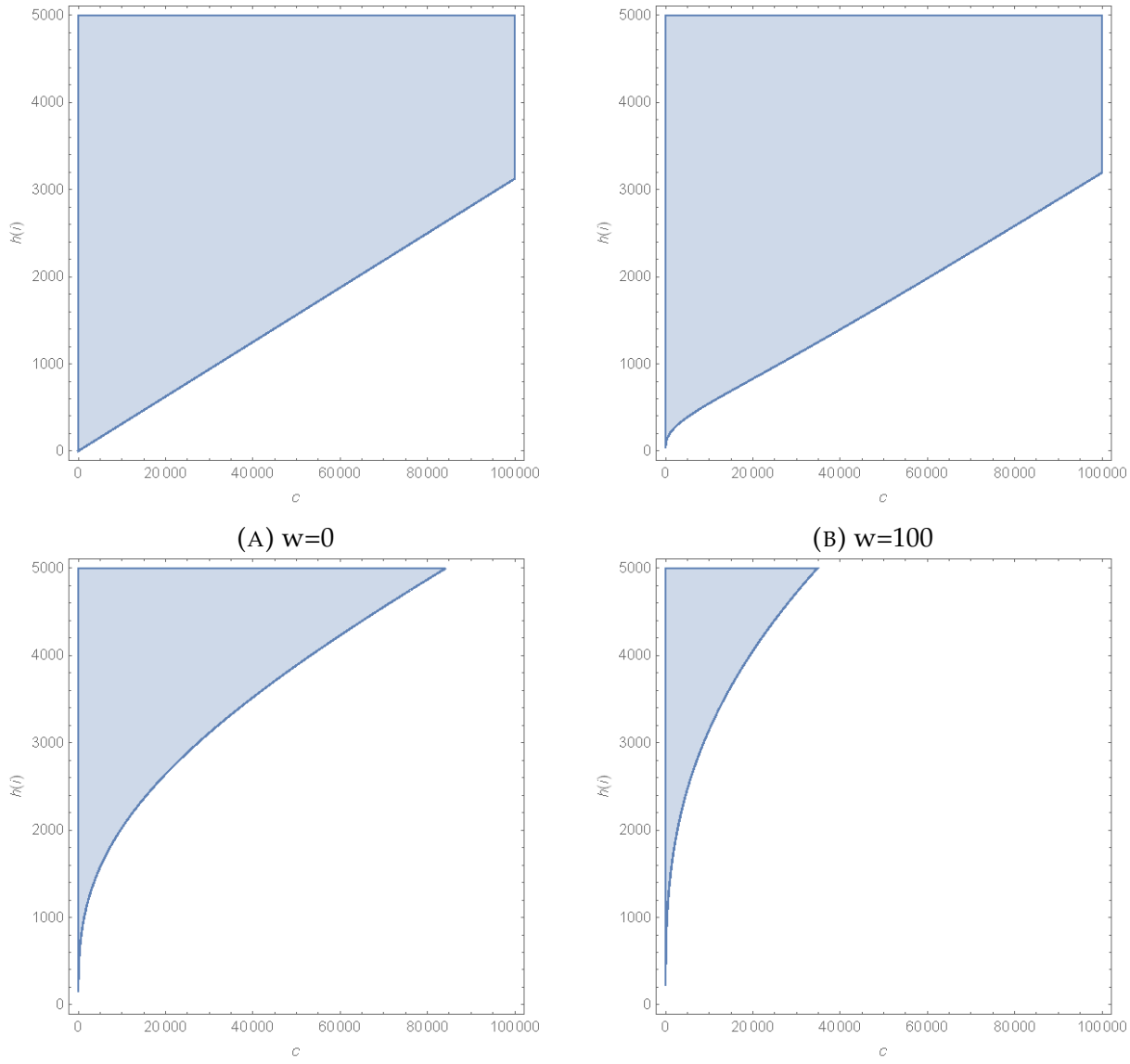
Since everything in the square root is positive the eigenvalues will all be real. However there can possibly be a negative one, which would mean negative norm states. The condition to have unitarity is then that the smallest eigenvalue e_- be positive, which happens only for

$$\frac{2h^2(32h+2-c)}{9(22+5c)} > w^2 \geq 0 \Longrightarrow h \geq \frac{c-2}{32}\,. \tag{4.6}$$

By using very simple arguments we have found a pretty strong lower bound on the weight of every primary operator that doesn't even depend on the charge. It's actually possible to find the same constraint by computing the eigenvalues of the operator W_0 at level 1 and asking that they are real since the operator is hermitian. It's tempting to compare the result to Hellerman's bound (3.9) but it's not clear at first that we can compare the constraints because Hellerman's bound applies to Virasoro primaries and our bound to primaries with respect to the full $\mathcal{W}(2,3)$ algebra. In other words the state that satisfies Hellerman's bound could in fact not be a primary in our context and not be restricted by our condition. This is after all not a problem because descendants have higher weight than their associated primary so if a $\mathcal{W}$-algebra descendant satisfies Hellerman's bound then its primary will too. Combining these two results then leaves only a small window of weight possible for the lightest primary. However since Hellerman's bound is about $\Delta = h + \bar{h}$ we must discuss theories that possess a field $\bar{W}(\bar{z})$ as well. Everything is the same for the bars and we find $\bar{h} \geq \frac{\bar{c}-2}{32}$, which when combined with the result for $W(z)$ gives $\Delta \geq \frac{c_{total}-4}{32}$.

To continue, the version of the constraint (4.6) that doesn't depend on the charge can of course be improved if we consider the inequality with w^2 instead. This leads to better bounds as the charge gets bigger, as shown in Figure 4.1. An interesting thing to look at is the large c limit. In a 2d CFT the central charge counts the number of degrees of freedom, so taking it large corresponds to a semi-classical theory. In that limit one separates states with the scaling of their weight with the central charge. States with $h \sim c$ are called heavy and $h \sim constant$ are called light. Taking $c \to \infty$ for light states in (4.6) results in asking the condition $w^2 < 0$, which is of course impossible. This means that in a semi-classical theory with $\mathcal{W}(2,3)$ symmetry there are no light states, apart obviously from the vacuum. (the bound $h \gtrsim c$ gives the same conclusion)

The next step would logically be to study the states at level two. It is analytically possible but it's simpler to do it numerically. If we do it we notice that no new constraints are obtained and it's natural to assume that nothing better will come from the higher levels. It's interesting to notice that the constraints obtained here are for any value of the



(A) w=0

(B) w=100

(C) w=1000

(D) w=2000

FIGURE 4.1: Constraints on the allowed values of the weight h and central charge c for different values of the charge w in a representation of the $\mathcal{W}(2,3)$-algebra, obtained by asking that the eigenvalues of the Gram matrix at level 1 are positive. The blue region is allowed and the white one is excluded. We observe that the bigger the charge is the tighter the constraint gets.

central charge, as opposed to what was obtained for Virasoro primaries by using the same kind of procedure, which gave something interesting only for $c < 1$.

4.3 $\mathcal{W}(2,4)$ Algebra

4.3.1 Derivation of the algebra

After having studied the simple $\mathcal{W}(2,3)$ for a while we will now start to study the more complicated $\mathcal{W}(2,4)$. The calculations are more involved but the idea is exactly the same so we will mostly just state the results. We will this time add a chiral primary operator of weight 4 to the Virasoro algebra, that we will call $W(z)$ once again. The fact that it is a primary leads to the commutation relation

$$[L_m, W_n] = (3m - n)W_{m+n}\,. \tag{4.7}$$

The commutator of two W together will be obtained from (2.29) like before. The operators that can appear this time have a weight of 2, 4 and 6. This means that at a first look we have

$$\begin{aligned}[W_m, W_n] = \frac{c}{4}\binom{m+3}{7}\delta_{m,-n} + C^T_{WW}p_{442}(m,n)L_{m+n} + C^W_{WW}p_{444}(m,n)W_{m+n} \\ + C^\Lambda_{WW}p_{444}(m,n)\Lambda_{m+n} + C^\Gamma_{WW}p_{446}(m,n)\Gamma_{m+n} + C^\Delta_{WW}p_{446}(m,n)\Delta_{m+n} \\ + C^\Omega_{WW}p_{446}(m,n)\Omega_{m+n}\,.\end{aligned} \tag{4.8}$$

The operator Λ is the same as in $\mathcal{W}(2,3)$ and the others can be found to be the following

$$\Gamma \equiv \mathcal{N}(T\partial^2 T) = N(T\partial^2 T) - \partial N(T\partial T) + \frac{2}{9}\partial^2 N(TT) - \frac{1}{42}\partial^4 T$$

$$\Delta \equiv \mathcal{N}(\Lambda T) = N(\Lambda T) - \frac{1}{6}\partial^2\Lambda$$

$$\Omega \equiv \mathcal{N}(WT) = N(WT) - \frac{1}{6}\partial^2 W\,.$$

Like we did before we normalize the spin 4 operator such that $d_{WW} = c/4$. We can again compute the functions $p(m,n)$ easily and the results are

$$p_{442}(m,n) = \frac{3m^4 - 2m^3 n - 2mn(n^2-10) + m^2(4n^2-39) + 3(n^4 - 13n^2 - 36)}{3360}$$

$$p_{444}(m,n) = \frac{m^2+n^2-mn-7}{36}$$
$$p_{446}(m,n) = \frac{m-n}{2}\,.$$

The structure constants can be calculated exactly the same way as for $\mathcal{W}(2,3)$. However there is a complication that arises when using the formula $C_{ijk} = \sum_l C_{ij}^l d_{lk}$ to obtain the structure constants since the matrix d_{ij} is no longer diagonal due to the presence of operators with the same dimension. This means that we need to invert the matrix to obtain the coefficients, which is still doable but more cumbersome. After some work we find

$$C_{WW}^T = 2\,, \qquad C_{WW}^\Delta = \frac{24(72c+13)}{(5c+22)(7c+68)(2c-1)}$$
$$C_{WW}^\Lambda = \frac{42}{5c+22}\,, \qquad C_{WW}^\Gamma = \frac{3(19c-524)}{10(7c+68)(2c-1)}$$
$$C_{WW}^\Omega = \frac{28}{3(c+24)} C_{WW}^W\,.$$

The coefficient C_{WW}^W is still not fixed by the regular means because to compute it using correlation functions we would actually need the WW commutator. This is where the Jacobi identity comes to the rescue. We need to ask that

$$[[W_q,W_r],W_s] + [[W_s,W_q],W_r] + [[W_r,W_s],W_q] = 0$$

and this leads to the constraint

$$(C_{WW}^W)^2 = \frac{54(c+24)(c^2-172c+196)}{(5c+22)(7c+68)(2c-1)}\,.$$

In obtaining this we have to assume that $c \neq -24$ so it's not valid for that value of the central charge. Also we see from the expression that when $c = -\frac{22}{5}, -\frac{68}{7}, \frac{1}{2}$ the coefficient blows up and there is a problem. Finally for $-\frac{22}{5} < c < \frac{1}{2}$ as well as $-24 < c < -\frac{68}{7}$ and $86 - 60\sqrt{2} < c < 86 + 60\sqrt{2}$ the coefficient C_{WW}^W would be imaginary and this is not allowed. At this point the algebra that we just derived is valid for any other value of the central charge. We would like to emphasize that again there is only one extra field so this is the only useful Jacobi identity and there are no more constraints that we can obtain that way.

To make everything clear, in order to perform any of the calculations that we skipped above we needed the expressions for the modes of the operators appearing in the algebra.

Using (2.33) and (4.4) as before we find

$$\Delta_n = \sum_{k\geq -1} \Lambda_{n-k} L_k + \sum_{k\leq -2} L_k \Lambda_{n-k} - \frac{(n+4)(n+5)}{6} \Lambda_n$$

$$\Omega_n = \sum_{k\geq -1} W_{n-k} L_k + \sum_{k\leq -2} L_k W_{n-k} - \frac{(n+4)(n+5)}{6} W_n$$

$$\begin{aligned}\Gamma_n = &\sum_{k\leq -2} \left[(k+2)(k+3) - (n+5)(k+2) + \frac{2(n+4)(n+5)}{9}\right] L_k L_{n-k} \\ &+ \sum_{k\geq -1} \left[(k+2)(k+3) - (n+5)(k+2) + \frac{2(n+4)(n+5)}{9}\right] L_{n-k} L_k \\ &- \frac{(n+2)(n+3)(n+4)(n+5)}{42} L_n \,.\end{aligned}$$

4.3.2 Constraints

We can now study the representations of $\mathcal{W}(2,4)$ the same way that we did before for $\mathcal{W}(2,3)$. However since the commutator (4.8) is very complicated it's not a good idea to do it by hand. This is why even at level one we do it numerically using Mathematica. The size of the expressions even forces us to stop using the eigenvalues to search for negative norm states so we look at the determinant instead. It still gives good constraints and it's possible to evaluate the eigenvalues explicitly at some specific set of (h,c,w) to maybe exclude other regions of parameter space if an even number of eigenvalues are negative. The states at level 1 are the same as before $|+\rangle = L_{-1}|h,w\rangle$ and $|-\rangle = W_{-1}|h,w\rangle$. Of course the operator $W(z)$ is not the same and its algebra is different. It turns out that this time level 2 does contribute with new constraints so we will include it directly in the study. The states are $|1\rangle = L_{-2}|h,w\rangle$, $|2\rangle = L_{-1}L_{-1}|h,w\rangle$, $|3\rangle = W_{-1}L_{-1}|h,w\rangle$, $|4\rangle = W_{-1}W_{-1}|h,w\rangle$ and $|5\rangle = W_{-2}|h,w\rangle$. This means that we evaluate the determinant of the 2×2 matrix at level one and of a 5×5 matrix at level two and we ask that each of them be positive. We also ask that it is real just in case. Since level 2 gives new constraints, it's natural to expect that level 3 would also contribute, but there are too many states and Mathematica is not strong enough so we haven't been able to study it.

The first result that we have to look at is Figure 4.2, which shows the constraints on h and c for $w=0$. We find that there is a lower bound on the allowed value for the weight. The value of the lowest limit if $h \geq \frac{c}{43} + \mathcal{O}(1)$ at large c, which is lower than in the $\mathcal{W}(2,3)$ case. The thin line close to $h=0$ is actually also excluded since there are two negative

eigenvalues for the corresponding parameters. The only state that is allowed under the boundary line is the identity $h = 0$. A simple check also shows that the determinant works with the eigenvalues for the rest of the plot.

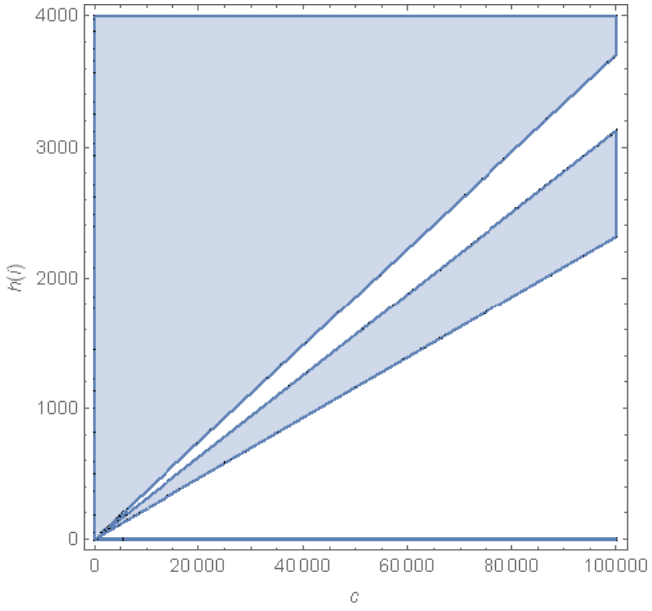


FIGURE 4.2: Constraint on h and c at w=0 coming from a negative determinant. The line close to $h = 0$ is also excluded because there are 2 negative eigenvalues.

Next, Figure 4.3 shows what happens to Figure 4.2 when we start varying the charge w. For small values the behavior stays the same and the boundary of the allowed region is a line. For high enough values of w the boundary becomes curved. For every one of the plots we see that the constraint $h \geq \frac{c}{43} + \mathcal{O}(1)$ is still satisfied so as in $\mathcal{W}(2,3)$ the weakest constraint comes from $w = 0$ and the following ones get higher as the charge increases. We can then conclude like before that there are no light states in the semi-classical limit. It might again be useful to compare the bound with Hellerman's result, but the conclusion will be weaker than what we already saw for $\mathcal{W}(2,3)$ since the bound is weaker. The threshold between allowed and not allowed states not only gets higher in the large c limit but also for small c. This means that small h are less and less allowed as we increase the charge. Note that the line close to $h = 0$ that is actually not allowed disappears as soon as we get away from $w = 0$. It's interesting to notice that there are less values of the weight allowed for states that have negative charge.

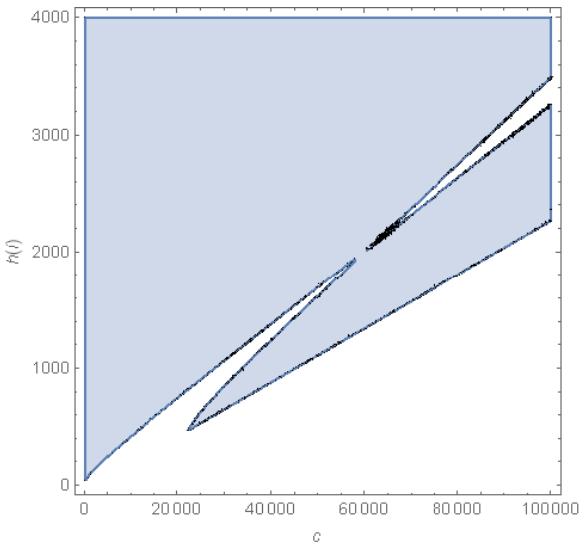
At this point the scale of every plot is very big and that's because we want to know the general behavior of the parameters, but Figure 4.4 shows what happens at fixed w for a

small central charge. We see that for the smallest values of the central charge every weight is excluded. This however doesn't come from a negative eigenvalue in the Gram matrix but from a complex one, which is due to the fact mentioned before that the algebra is not real in certain ranges of c. We can see from these plots that the constraint $h \geq \frac{c}{43} + \mathcal{O}(1)$ is valid for most of the parameters since the allowed values of c start close to 200, where the corrections for small c are already negligible. Of course there is a tiny region that is allowed close to $c = 1$ that doesn't satisfy the constraint. The appearance of small islands in the plot for very small c points in the direction of the existence of minimal models for small c, as in $\mathcal{W}(2, 3)$, but we would need to study higher levels to confirm this claim.

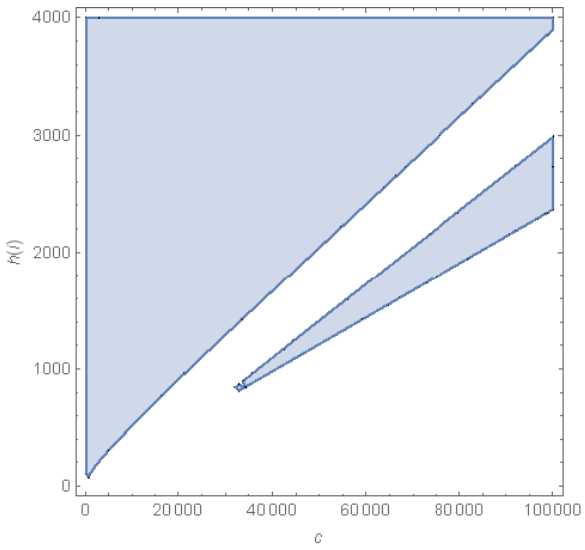
The next thing to do is study what happens at fixed central charge, which is done in Figure 4.5. The important thing to notice from these results is that for a fixed model (so fixed c) and a fixed weight h there will be a maximum value for $|w|$. This means that for a fixed energy the charge can't be arbitrarily high. Taken the other way it means that for a given charge there is a lower bound on the energy. As the central charge increases the constraint gets stronger and the upper bound on the charge of a state with given energy gets higher. The weaker bound is then obtained for the smallest central charge, which is $86 + 60\sqrt{2}$ if we exclude the small c region, but unfortunately we were unable to find an analytic expression. Finally there are values of the weight that are not allowed even for arbitrarily small charge and there are more and more of these excluded parameters as the central charge increases. This of course could have already been concluded from Figure 4.3.

The last thing that we want to show is what happens at fixed weight. This is redundant with the previous results since the behavior is somewhat clear already, but it's always good to have the full plots of Figure 4.6 available. These plots show that there is an upper bound on the charge of a fixed energy and the possible charge gets lower as the central charge increases. In the opposite way as before as we increase the fixed parameter (here the weight) the constraint gets weaker and more charges are allowed for a given weight. This was already clear from the study of Figure 4.5.

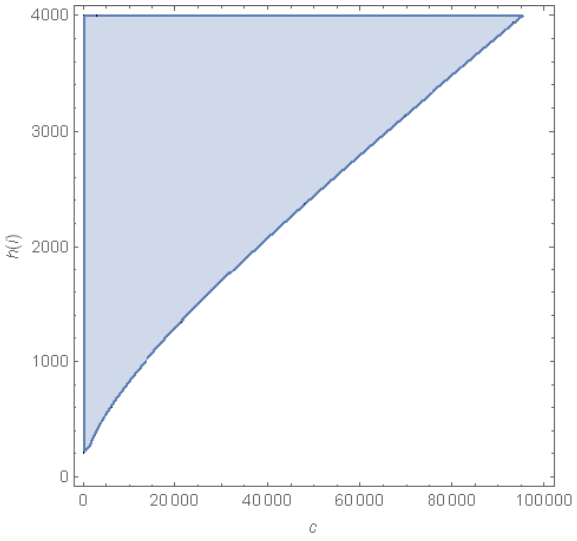
An interesting fact that we observed is that $h = 0$ is allowed only for the vacuum and as soon as we go away from $w = 0$ there are negative eigenvalues that arise. This means that it's impossible to have charged excitations with no energy, which sounds plausible.



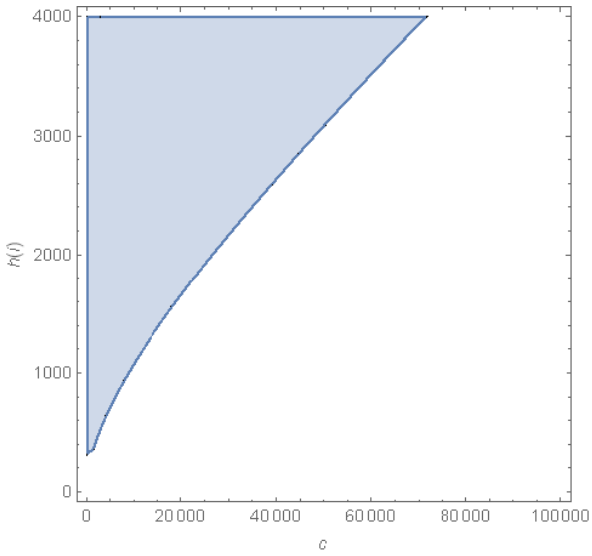
(A) w=10



(B) w=-10

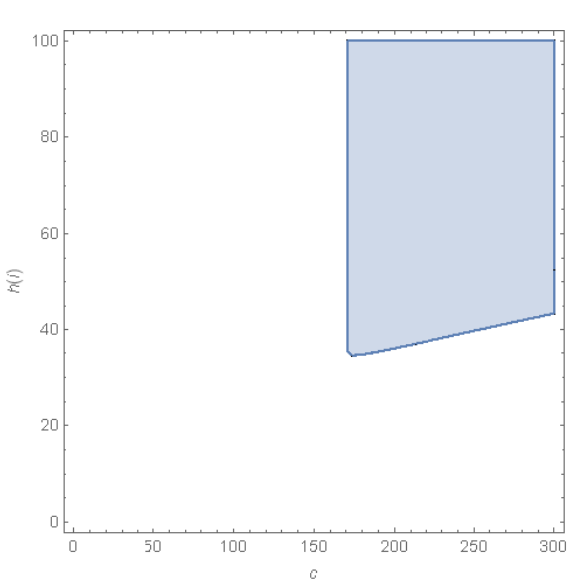


(C) w=100

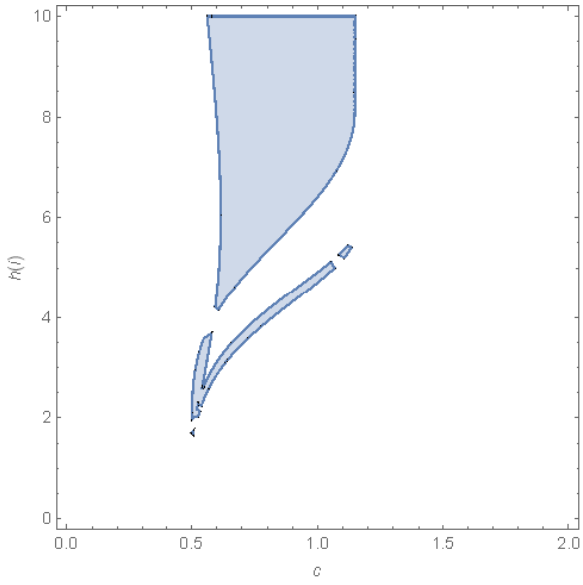


(D) w=-100

FIGURE 4.3: Constraints on the weight and the central charge for different values of the charge, obtained by asking for positive determinant. The blue region is excluded. A bigger charge leads to a tighter constraint and a negative charge constrains more than the same number but positive.

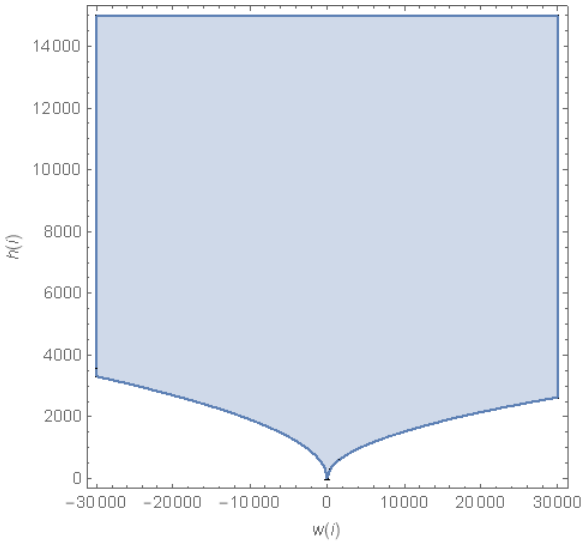


(A) The region $86-60\sqrt{2} \approx 1.2 < c < 86+60\sqrt{2} \approx 170.9$ is excluded

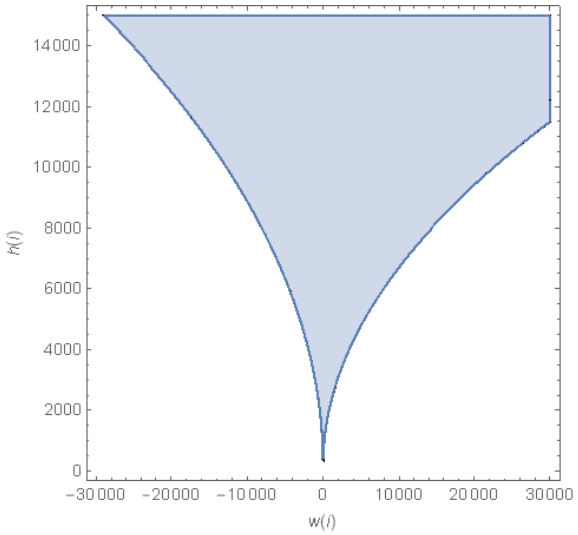


(B) The region $0 < c < \frac{1}{2}$ is excluded

FIGURE 4.4: Constraints for small central charge that are not coming from negative determinant but simply the fact that the constant C_{WW}^{W} in the algebra is complex. Example with $w = 10$.

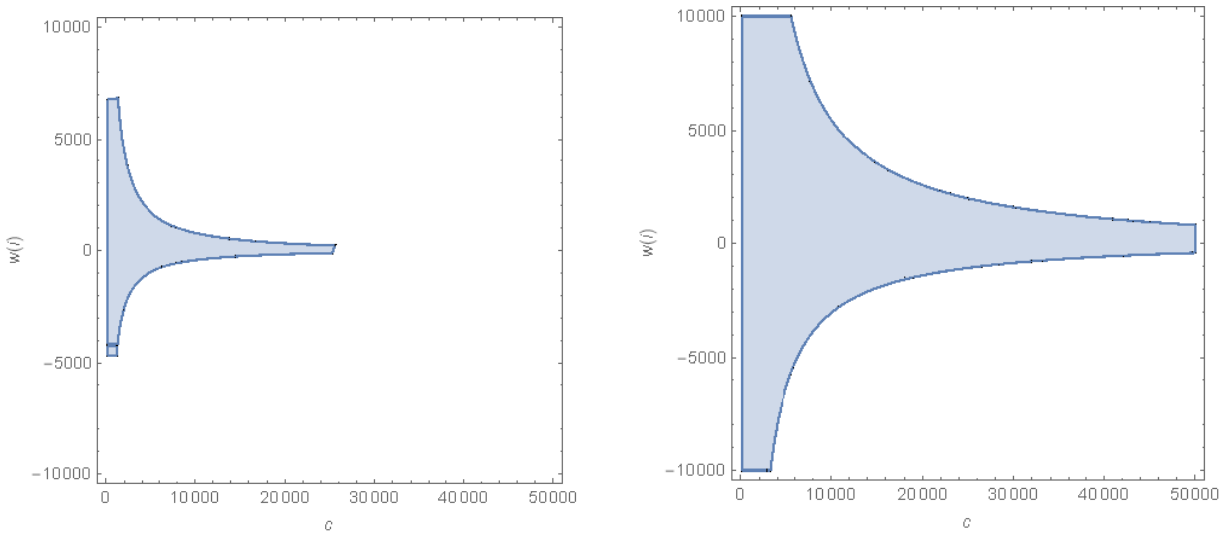


(A) c=500



(B) c=10000

FIGURE 4.5: Constraints on the weight and the charge at different values of the central charge. The allowed region gets smaller and higher as c increases. For fixed value of h there is an upper bound on w.



(A) h=2000 (B) h=5000

FIGURE 4.6: Constraints on the charge and the central charge for different values of the weight. The constraint gets weaker for higher values of h. This shows an upper bound on the charge at fixed c.

Chapter 5

Applications to Holography

Up until now everything that we discussed concerned conformal field theories in 2 dimensions. However it is well known that one of the most important applications of CFTs is through the gauge/gravity duality to study theories of quantum gravity in Anti de Sitter space (AdS). In this section we will introduce gravity in AdS_3 and we will motivate the AdS/CFT correspondence, without however going into much details. We will discuss only the results that are needed to interpret in the gravity context what we obtained so far in this thesis. We will also have to discuss higher spin gravity so we can talk about applications of $\mathcal{W}$-algebras and to do that we will introduce the Chern-Simons formulation of 3d gravity.

5.1 The AdS/CFT Correspondence

One of the biggest indicator that there should be a way of relating a 2d CFT to AdS_3 spacetime is because of the asymptotic symmetries of the vacuum solution to Einstein's equations with a negative cosmological constant. Anti de Sitter is a maximally symmetric spacetime that has a constant negative curvature. The metric in $2+1$ dimensions with global coordinates is

$$ds^2 = -\left(1+\frac{r^2}{\ell^2}\right)dt^2 + \left(1+\frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\phi^2 \tag{5.1}$$

with ℓ being the AdS length, which characterizes the curvature of the space. The cosmological constant that gives rise to this solution is $\Lambda = -\frac{1}{\ell^2}$ so the flat space limit is $\ell \to \infty$. The asymptotic symmetries of that metric have been studied a while ago by Brown and Henneaux but we will follow the more recent work [29] that puts it in the context of holography. If we want to quantize a theory on this background, we need to define boundary

conditions. We need them to be strong enough to have a well defined action of the diffeomorphism group but not too strong so that massive excitations are still allowed. It turns out that such boundary conditions are

$$g_{tt} = -\frac{r^2}{\ell^2} + \mathcal{O}(1)\,, \qquad g_{t\phi} = \mathcal{O}(1)\,, \qquad g_{tr} = \mathcal{O}\left(\frac{1}{r^3}\right)\,,$$

$$g_{rr} = \frac{\ell^2}{r^2} + \mathcal{O}\left(\frac{1}{r^4}\right)\,, \qquad g_{r\phi} = \mathcal{O}\left(\frac{1}{r^3}\right)\,, \qquad g_{\phi\phi} = r^2 + \mathcal{O}(1)\,.$$

These basically define what we mean when we say that a space is asymptotically AdS_3. Diffeomorphisms that preserve this structure are generated by vector fields $\zeta^\mu(r,t,\phi)$ of the form

$$\zeta^t = \ell(T^+ + T^-) + \frac{\ell^3}{2r^2}(\partial_+^2 T^+ + \partial_-^2 T^-) + \mathcal{O}\left(\frac{1}{r^4}\right)$$

$$\zeta^\phi = T^+ - T^- - \frac{\ell^2}{2r^2}(\partial_+^2 T^+ - \partial_-^2 T^-) + \mathcal{O}\left(\frac{1}{r^4}\right)$$

$$\zeta^r = -r(\partial_+ T^+ + \partial_- T^-) + \mathcal{O}\left(\frac{1}{r}\right)$$

where $2\partial_\pm = \ell\partial_t \pm \partial_\phi$ and $T^\pm(r,t,\phi) = T^\pm\left(\frac{t}{\ell} \pm \phi\right)$ such that $\partial_\pm T^\mp = 0$. Actually the diffeomorphisms with $T^\pm = 0$ fall off so fast at infinity that there is no global charge associated with them. In other words they don't generate a new state and should not be considered in the asymptotic symmetry algebra. With this assumption we denote the generators of transformations with $T^+ = e^{in\left(\frac{t}{\ell}+\phi\right)}$ and $T^- = e^{in\left(\frac{t}{\ell}-\phi\right)}$ respectively by L_n and $\bar{L}_n$ and after some calculations we find that the algebra that they satisfy is exactly two copies of the Virasoro algebra (2.11) with central charge $\bar{c} = c = \frac{3\ell}{2G}$ where G is Newton's gravitational constant. This shows that the states in a theory of quantum gravity on an asymptotically AdS_3 background have to form a representation of the conformal group in 2d, so the theory is a CFT. It's important to notice that in a quantum theory of gravity the semi-classical limit corresponds to a small G, which in turns corresponds to large central charge, so it fits with what we know from CFT.

It will be useful for the future interpretation to mention that black holes exist in AdS_3 and an expression for their metric is

$$ds^2 = -\ell^2(r^2 - 8GM)dt^2 + \ell^2(r^2 - 8GM)^{-1}dr^2 + r^2 d\phi^2\,.$$

We consider only non-rotating solutions because they are simpler and this is all we will

need. These spacetimes are called BTZ black holes and their mass is of course M. Using the famous Bekenstein-Hawking formula for the entropy of the black hole, we can compute the entropy of a BTZ black hole

$$S = \frac{area}{4G} = \pi\ell\sqrt{\frac{2M}{G}}\,.$$

It's important to notice that pure AdS_3 corresponds to this metric with $M = -\frac{1}{8G}$ and that the lightest BTZ black hole has mass $M = 0$ because between these two values there is a naked singularity that appears.

From what we just described it's clear that something can be done involving quantum gravity in AdS_3 and 2d conformal field theories. This is called the AdS/CFT correspondence and it has been discovered by Maldacena in 1997 and widely studied ever since. A way of stating the correspondence is that a theory of quantum gravity in asymptotically AdS_3 spacetime is dual to a 2d conformal field theory in the sense that every element in one theory can be associated with one in the other and every result in one can be found using the other. The most important point in the correspondence for us is that the Hilbert space of both theories is the same. In fact non-holomorphic primary states in a CFT correspond to massive states at rest with respect to the AdS global time because their energy can't be lowered by using the momentum generators L_{-1} and $\bar{L}_{-1}$. Descendants are then considered as excitations of the primaries located at the boundary of the spacetime and that are called boundary gravitons.

5.2 Interpretation of the Modular Invariance Bounds

Now that we know a little bit about the AdS/CFT correspondence we can use it to interpret our results from Chapter 3. Modular invariance can be understood in the gravity context but we will focus on the results and not the derivations. The first result that will be useful is Cardy's formula for the asymptotic growth of the density of states in a 2d CFT, which is $\rho(E) = \exp\left(2\pi\sqrt{\frac{c}{3}E}\right)$. A state with high energy in the gravity theory is a black hole and because of the difference in dimensions with the CFT we have to introduce a factor of the characteristic length to write $E = \ell M$, with M being the mass of the black hole. If we know the number of states we can compute the entropy by taking the log so making use of $c = \frac{3\ell}{2G}$ we can see easily that Cardy's formula gives a corresponding

entropy of

$$S = \log \rho(E) = 2\pi\sqrt{\frac{c}{3}E} = 2\pi\sqrt{\frac{\ell}{2G}\ell M}$$

which is exactly the Bekenstein-Hawking entropy of the BTZ black holes. This result is very important because it provides an explanation to what the microstates of the BTZ black holes are. Indeed by using Hawking's formula we can know the entropy of a black hole but this number is supposed to count microstates, which are hard to identify. However now we know that they really correspond to quantum states in a 2d CFT.

The next result to interpret is Hellerman's bound. Actually the reasoning will be the same to study Qualls' bound, except that this one applies uniquely to gravity theories with only even spin primary operators. We already saw that the energy of a state in the CFT, which is in fact the scaling dimension, is related to the energy in AdS by $\Delta = \ell E$. For a primary state we also know that the state is at rest and the energy is then the rest mass such that $M = \frac{\Delta}{\ell}$. This means that the reinterpretation of the bounds on the lowest dimensions for primary operators is that they become upper bounds on the mass of states with no boundary excitations. Using again the expression for the holographic central charge, but for $\bar{c}$ too, we can rewrite Hellerman's bound as

$$\Delta \leq \frac{c_{tot}}{12} + \mathcal{O}(1) \Longrightarrow M \leq \frac{1}{4G} + \mathcal{O}\left(\frac{1}{\ell}\right) .$$

This bound was valid only for $c_{tot} > 2$, which corresponds to gravity theories with curvature less than the Planck size. If we look at the flat space limit $\ell \to \infty$ we find the suggestive bound $M \leq \frac{1}{4G}$. This is twice the difference between the lightest BTZ black hole and pure AdS_3, which is actually the right number to compare with. Black hole states are always present in a theory of quantum gravity with matter and this bound means that even when quantum corrections are accounted for there is always a state with energy at the scale of the lightest BTZ. This is again an important result because there is no reason from the pure gravity theory to expect such a bound to hold at the quantum level since quantum corrections can contribute by a big factor to the tree level value of the mass. Finally like stated earlier Qualls' bound can be understood exactly the same way and it gives the same result but with exactly the mass of the lightest BTZ black hole and only for theories with even spin operators.

5.3 Higher Spin Gravity and $\mathcal{W}$-algebras

We have done everything we could to describe our results in a gravity theory by using the general framework of AdS/CFT but now to interpret the results on $\mathcal{W}$-algebras we need to introduce higher spin gravity. In order to do that we will need to present a way of reformulating 3d Einstein gravity with a negative cosmological constant. This discussion and the one for the higher spin generalization follow [30]. Unfortunately we will not be able to present a clear interpretation but at least the context to use will be clear. First of all let's review the expression for the action of the spacetime metric that is relevant here:

$$S_{grav} = \frac{1}{16\pi G}\int \mathrm{d}^3x\sqrt{|g|}\left(R + \frac{2}{\ell^2}\right) .$$

At this point we will need to use the language of differential forms but we will not introduce it since it's only used briefly and the results are more important. If we consider the Cartan formulation of General Relativity, it's possible to combine the dual of the spin connection $\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{bc}$ and the dreibein e^a into connections $A = \omega + \frac{e}{\ell}$ and $\bar{A} = \omega - \frac{e}{\ell}$ ($a, b, c, ...$ are Lorentz indices, not spacetime indices). It's important to note that the bar doesn't have anything to do with conjugation. The generators defined by $A = A^a J_a$ then satisfy the $SL(2,\mathbb{R})$ algebra $[J_a, J_b] = \epsilon_{abc}\eta^{cd}J_d$ and A is a gauge connection for $SL(2,\mathbb{R})$. We can use these definitions to rewrite the gravitational action as

$$S_{grav} = \frac{k}{4\pi}\int \mathrm{Tr}\big[CS(A) - CS(\bar{A})\big]$$

where $CS(A) = A\wedge dA + \frac{2}{3}A\wedge A\wedge A$ is the Chern-Simons action, which describes a topological gauge theory. The number k is the level of the theory and can be written as $k = \ell/4G$. This reformulation of 3d gravity with a negative cosmological constant as a topological theory was possible because the metric has no local degrees of freedom in 3 dimensions. It is important that even in that formulation it's possible to recover the fact that the asymptotic symmetry algebra is the Virasoro algebra. More can be said about 3d gravity by studying this version of the action, but we will jump right ahead to a generalization.

Now that we have rephrased AdS_3 Einstein gravity as an $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ Chern-Simons gauge theory we can use it to define what we mean by higher spin gravity. In fact it is just the extension of the gauge group to $SL(N,\mathbb{R})\times SL(N,\mathbb{R})$, which means that there are fields with spin $3, 4, ..., N$ in the theory in addition to the metric. It is possible to do

the analysis of the asymptotic symmetries and we find that the symmetry algebra is two copies of $\mathcal{W}_N$, which points towards a realization of the gauge/gravity duality involving the extensions of the Virasoro algebra. Actually people have studied the correspondence between higher spin gravity and CFTs with $\mathcal{W}_N$ symmetry and it has been established for some theories, for example in [31].

Since we know what $\mathcal{W}$ algebras mean in the context of gravity the results in Chapter 4 can now be interpreted simply as constraints on the energy and the charge of primary states in the gravity theory. However it is not that clear what these states are physically because the formulation of the theory is not intuitive. There is then a little bit more work to be done in that direction, even if we know in what framework we should do the interpretation.

Chapter 6

Conclusion

6.1 Wrap-up

We have now come to the final point of this thesis. While doing this work we have found that modular invariance is very powerful in 2d conformal field theories and we have exploited this fact to put constraints on the spectrum of operators. We have in addition found that unitarity puts also strong constraints on CFTs and we have used this to study the representations of theories with extended symmetries called $\mathcal{W}$-algebras. We have finally used the gauge/gravity duality to interpret our results in the context of gravitational theories.

First, we started by introducing the subject of conformal field theory on its own. We derived the conformal group and conformal algebra in various dimensions and then focused on $d = 2$. We quantized the theory radially and talked about operator product expansions for different kinds of operators. We then discussed the construction of the Hilbert space of such a theory with highest weight representations and we introduced a way of checking their unitarity. The last thing that was discussed about general CFTs is modular invariance of the torus partition function.

Second, we exploited modular invariance to constrain the spectrum of operators in 2d CFTs with different properties. We used invariance under the S transformation to derive an expression for the asymptotic density of states in a 2d CFT $\rho(E) = \exp\left(2\pi\sqrt{\frac{c}{3}E}\right)$. In order to use even more modular invariance we derived a general expression for the partition function in terms of primary fields. We then used it along with S invariance to find the bound $\Delta_1 \leq \frac{c+\bar{c}}{12} + \mathcal{O}(1)$ on the lightest operator in a unitary 2d CFT with $c, \bar{c} > 1$ and no extra chiral operator. The next step has been to use S and ST invariance to derive the bound $\Delta_1 \leq \frac{c+\bar{c}}{24} + \mathcal{O}(1)$ that is applicable only for CFTs that contain exclusively even

spin operators. After all that we derived a constraint coming from a general modular transformation and suggested a way of using it to find further constraints.

Third, we used the technique introduced before to impose unitarity on representations of $\mathcal{W}$-algebras and their spectrum. We started by describing what these objects are and we then derived the $\mathcal{W}(2,3)$ and $\mathcal{W}(2,4)$ algebras. We continued by computing the Gram matrix at level 1 for $\mathcal{W}(2,3)$ and asking that it is positive definite gave us a bound $h \geq \frac{c-2}{32}$ along with some graphic constraints. We computed the same thing for $\mathcal{W}(2,4)$ at level 1 and level 2 and asked that the determinant be positive to find $h \geq \frac{c}{43} + \mathcal{O}(1)$ along with many graphic constraints on the charge, the weight and the central charge.

Finally, we introduced the AdS_3/CFT_2 correspondence to interpret the previous results in the context of AdS gravity. We found that the asymptotic symmetries of AdS_3 spacetime satisfy the Virasoro algebra and we were able to show that Cardy's formula corresponds to black hole entropy. The bounds on operator dimensions are interpreted as bounds on the center-of-mass energy of massive states. We also introduced the Chern-Simons formulation of 3d gravity to generalize it to higher spin gravity and motivate the relation with $\mathcal{W}$-algebras.

Before finishing we would like to thank the creators of the LaTeX template that has been used to write this thesis. The files can be found at http://www.latextemplates.com/template/masters-doctoral-thesis.

6.2 Future Work

Many interesting questions emerged from the work done in this thesis. It is possible to think of many generalizations or applications of the arguments discussed in the text and it is now time to mention some of the ideas that we would like to work on in the future. Of course a deeper understanding of the relation between higher spin gravity and $\mathcal{W}_N$ algebras is the first thing that comes to mind but we would like to discuss other directions.

First of all we studied in Chapter 4 the representations of the $\mathcal{W}(2,3)$ and $\mathcal{W}(2,4)$ algebras so it would be nice to study $\mathcal{W}(2,3,4)$ in order to see what properties it has in common with the others. It should behave as a combination of both of them since it contains the same fields as them, but the new commutation relation of the spin 3 and the spin 4 operators together could add interesting properties. Other $\mathcal{W}$-algebras with higher spin

operators would also be interesting to study.

Second, to continue on what we did in Chapter 3 we could consider a 2d CFT that has a conserved current and write its partition function in terms of the primaries of the full symmetry algebra to exploit modular invariance and constraint the spectrum of weights and charges. Actually this has been done recently in [32] for the case where the current is coming from a $U(1)$ symmetry and it would be nice to generalize that to currents coming from a $SU(2)$ symmetry.

Finally, a cool way of continuing on with the work in this thesis would be to combine the two subjects that we covered. Writing the partition function of a CFT that has $\mathcal{W}$-symmetry in terms of full primaries and use modular invariance to constrain the spectrum of weights and charges would allow us to improve the results of Chapter 4 by using techniques from Chapter 3. However to do this one needs to know the modular transformation properties of the new partition function and this is not known in general. The case of $\mathcal{W}(2,3)$ has been studied in [33] and [34] so this could possibly be exploited to find something new.

Appendix A

Properties of the Dedekind η Function

In this appendix we will introduce the transformation rule of the eta function under the action of the modular group as found in Chapter 3.4 of [9]. We will then use this general formula to look at the specific cases that are needed in Chapter 3. As a reminder, the eta function is defined on the upper-half plane $H = \{\tau : \mathrm{Im}(\tau) > 0\}$ as

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n \tau}\right) . \tag{A.1}$$

It is a very useful tool in number theory and as it turns out also in conformal field theory. An element of the modular group acts as $\tau \to \frac{a\tau+b}{c\tau+d}$ with integer coefficients satisfying $ad - bc = 1$. Under these conditions the following transformation rule applies for $c > 0$:

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \epsilon(a, b, c, d)\sqrt{-i(c\tau + d)}\eta(\tau) \tag{A.2}$$

with

$$\epsilon(a, b, c, d) = \exp\left[\pi i \left(\frac{a + d}{12c} + s(-d, c)\right)\right] \tag{A.3}$$

and

$$s(h, k) = \sum_{r=0}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left[\frac{hr}{k}\right] - \frac{1}{2}\right) . \tag{A.4}$$

The actual modular transformations that we need are easy to get from (A.2).

- In Section 3.2 we study the S transformation, which corresponds to $b = -1$, $a = d = 0$ and $c = 1$. We get directly the transformation rule from equation (A.2)

$$\eta\left(\frac{-1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau) . \tag{A.5}$$

- In Section 3.3 we study the ST transformation, with $a = 0$, $b = -1$ and $c = d = 1$. The transformation rule is then

$$\eta\left(\frac{-1}{\tau+1}\right) = e^{\frac{\pi i}{12}}\sqrt{-i(\tau+1)}\eta(\tau) = e^{-\frac{\pi i}{6}}\sqrt{\tau+1}\eta(\tau)\,. \tag{A.6}$$

Now that we have the transformation properties of the eta function, we can derive the properties of its derivatives that are used in the derivations of Chapter 3.

First of all, we can take the derivative of (A.5) with respect to τ to get

$$\frac{1}{\tau^2}\eta'\left(\frac{-1}{\tau}\right) = \sqrt{-i\tau}\eta'(\tau) + \frac{1}{2}\sqrt{\frac{-i}{\tau}}\eta(\tau)\,.$$

Evaluating this at the fixed point of the S transformation $\tau = i$, we find

$$\eta'(i) = \frac{i}{4}\eta(i)\,. \tag{A.7}$$

Taking the second derivative gives us nothing more than the first one so we can't obtain a relation between $\eta''(i)$ and $\eta(i)$. That's why we use another way to compute the ratio in the text. If we take the third derivative of the transformation property we get:

$$\frac{1}{\tau^6}\eta'''\left(\frac{-1}{\tau}\right) - \frac{6}{\tau^5}\eta''\left(\frac{-1}{\tau}\right) + \frac{6}{\tau^4}\eta'\left(\frac{-1}{\tau}\right) = \sqrt{-i\tau}\eta'''(\tau) + \frac{3}{2}\sqrt{\frac{-i}{\tau}}\eta''(\tau) - \frac{3}{4}\sqrt{\frac{-i}{\tau^3}}\eta'(\tau) + \frac{3}{8}\sqrt{\frac{-i}{\tau^5}}\eta(\tau)\,,$$

which, when evaluated at the fixed point and with the use of the previous result (A.7), gives

$$\eta'''(i) = \frac{15i}{32}\left(\eta(i) + 8\eta''(i)\right)\,. \tag{A.8}$$

We can do the same thing with (A.6). The first derivative of the transformation rule gives

$$\frac{1}{(\tau+1)^2}\eta'\left(\frac{-1}{\tau+1}\right) = \frac{e^{-\frac{\pi i}{6}}}{2}\sqrt{\frac{1}{(\tau+1)}}\eta(\tau) + e^{-\frac{\pi i}{6}}\sqrt{\tau+1}\eta'(\tau)\,.$$

Evaluating it at the fixed point of the ST transformation $w \equiv e^{\frac{2\pi i}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ gives us

$$\eta'(w) = \frac{i\sqrt{3}}{6}\eta(w)\,. \tag{A.9}$$

Appendix B

Mathematical Reminders

In this appendix we will review some mathematical notions that are useful to remember in order to follow the flow of the text, especially Chapter 2.

B.1 Generators

The notion of generator of a coordinate transformation is in general pretty clear but we will review here the conventions that are used in physics, more specifically in [3]. Under an infinitesimal coordinate transformation $x^\mu \to x'^\mu = x^\mu + \epsilon_a \frac{\delta x^\mu}{\delta \epsilon_a}$, a scalar function $\phi(x)$ will transform such that $\phi'(x') = \phi(x)$. In other words the new function evaluated at the new point gives the same result as the old function evaluated at the old point, which is obvious. We then define the generator of a transformation as

$$\phi'(x) - \phi(x) = -i\epsilon_a G_a \phi(x)$$

and G_a basically describes the effect of the transformation on the function. Inverting the coordinate transformation, Taylor expanding and using the transformation rule for ϕ gives

$$\phi'(x) = \phi'\left(x' - \epsilon_a \frac{\delta x^\mu}{\delta \epsilon_a}\right) = \phi'(x') - \epsilon_a \frac{\delta x^\mu}{\delta \epsilon_a}\partial_\mu \phi'(x') = \phi(x) - \epsilon_a \frac{\delta x^\mu}{\delta \epsilon_a}\partial_\mu \phi(x)$$

Using this leads directly to the operator

$$G_a = -i\frac{\delta x^\mu}{\delta \epsilon_a}\partial_\mu \,. \tag{B.1}$$

As an example, consider an infinitesimal dilation $x' = x + \epsilon x$ (with one parameter). In that case $\frac{\delta x^\mu}{\delta \epsilon} = x^\mu$ so we find $D = -ix^\mu \partial_\mu$, like stated in the text.

B.2 Cauchy Integrals

Complex analysis is a very useful tool in studying 2d CFT so we will review some important results from the mathematical theory here, more specifically about complex integrals. This is based on [13]. First of all a function $f(z)$ of the complex variable $z = x+iy$ is analytic in a region if it is differentiable in that region (holomorphic if everywhere). The contour integral in the complex plane of an analytic function vanishes

$$\oint_C f(z)\mathrm{d}z = 0\,. \tag{B.2}$$

This is called Cauchy's theorem. This follows from the fact that the real and imaginary parts of a differentiable function $f = u + iv$ must satisfy the Cauchy-Riemann equations

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\,. \end{aligned} \tag{B.3}$$

Conversely if the Cauchy-Riemann equations are satisfied by continuous derivatives, then $f(z)$ is differentiable. Another important result is Cauchy's formula, which says that

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^n}\mathrm{d}z = \frac{1}{(n-1)!}f^{(n-1)}(a)\,. \tag{B.4}$$

This formula can be used with $f(z) = 1$ and $a = 0$ to find a result that is used a lot in this thesis

$$\frac{1}{2\pi i}\oint \mathrm{d}z\, z^{-n} = \delta_{n,1}\,.$$

Finally we can expand a function in a Taylor series around a point where it is analytic just like in real analysis but around a point $x = a$ where it is singular we can expand it in Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \tag{B.5}$$

where the coefficients can be found by using Cauchy's formula to be

$$c_n = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-a)^{n+1}}\mathrm{d}z\,.$$

The contour γ is a path that encloses the point a and on which the function is analytic.

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